Research Article

Lag, Anticipated, and Complete Synchronization and Cascade Control in the Dynamical Systems

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We obtain the lag, anticipated, and complete hybrid projective synchronization control (LACHPS) of dynamical systems to study the chaotic attractors and control problem of the chaotic systems. For illustration, we take the Colpitts oscillators as an example to achieve the analytical expressions of control functions. Numerical simulations are used to show the effectiveness of the proposed technique.

1. Introduction

Chaotic control and the structure of chaotic systems have attracted much attention in nonlinear sciences, especially in physics, chemistry, and biology. Different types of the structure and control method have been found in a variety of chaotic systems, such as constructing method [1–10], adaptive method [11], projective-Lag synchronization method [12–15], backstepping method [16], Q-S synchronization method [17–20], and many others.

At the same time, many different types of synchronization in chaotic (hyperchaotic) systems were presented, for example, complete synchronization, generalized synchronization, phase synchronization, antisynchronization, general projective synchronization, lag synchronization, and anticipate synchronization, and so on.

To two dynamical systems, consider a full state hybrid projective synchronization (FSHPS) method [21], where the responses of the synchronized dynamical states synchronize up to a constant scaling matrix. In this paper, based on the Lyapunov stability theory, we propose a scheme of lag, anticipated, and complete hybrid projective synchronization control (LACHPS). In this method, every state variable of master system synchronizes other incompatible state variables of slave system; particularly, for oscillators, two different designs are shown.

When $x(t)$ and $y(t)$ are the state vectors of two n-dimensional chaotic systems. These two systems are completely synchronized [22] if the synchronization error $\|y(t) - x(t)\| \to 0$ as $t \to \infty$. AS [23] is defined when the error $\|y(t) + x(t)\| \to 0$ as $t \to \infty$. PS [24] is a situation in which the state vectors $x(t)$ and $y(t)$ synchronize up to a constant factor α (i.e. $\|y(t) - \alpha x(t)\| \to 0$ as $t \to \infty$. MPS [25] is defined if the state vectors of two systems synchronize up to a constant scaling matrix which means that $\|y(t) - Mx(t)\| \to 0$ as $t \to \infty$. LS [13] implies that the state variables of the two coupled chaotic systems become synchronized but with a time lag with respect to each other; that is, $\|y(t) - x(t - \tau)\| \to 0$ as $t \to \infty$, where τ is the positive time lag. PLS has been introduced recently in [15, 26–28] as $\|y(t) - \beta x(t - \tau)\| \to 0$ as $t \to \infty$, where β is a constant scaling factor. Synchronization can be addressed as a stabilization problem. This means that the trajectories of the synchronization error have to be stabilized at the origin.

In realistic and engineering applications, LS and PLS always affect the dynamical behaviors of chaotic systems. For example, in the telephone communication system, the voice one hears on the receiver side at time t is the voice from the transmitter side at time t. LS and PLS have been recently studied on systems described in [15, 28–30]. For more details about chaotic control see [31–38] and for the elements of the cyclicity theory of planar systems see [39–41]. Our goal in

TABLE 1

Setting the matrix M	Setting the matrix N and the value of τ	Type of synchronization
$M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$	$N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n), \tau = 0$	HPS
$M = \text{diag}(1, 1, \ldots, 1)$	$N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n), \tau > 0$	PLS
$M = \text{diag}(1, 1, \ldots, 1)$	$N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n), \tau = 0$	PS
$M = \text{diag}(1, 1, \ldots, 1)$	$N = \text{diag}(1, 1, 1, \ldots, 1), \tau > 0$	LS
$M = \text{diag}(1, 1, \ldots, 1)$	$N = \text{diag}(1, 1, 1, \ldots, 1), \tau = 0$	CS
$M = \text{diag}(1, 1, \ldots, 1)$	$N = \text{diag}(f_1, f_2, \ldots, f_n), \tau = 0$	FS

this paper is to introduce and investigate the lag, anticipated, and complete hybrid projective synchronization control (LACHPS) of two n-dimensional nonlinear systems.

Definition 1. For n-dimensional master and slave nonlinear systems as $\dot{x} = F(x,t)$, $\dot{y} = G(y,t) + U$, where $x = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $y = (y_1(t), y_2(t), \ldots, y_n(t))^T$, $U = (u_1(x, y), u_2(x, y), \ldots, u_n(x, y))$ is a controller to be determined later. The LACHPS is defined if the synchronization error $\lim_{t \to \infty} \|My(t) - Nx(t - \tau)\| \to 0$, where $x(t)$ and $y(t)$ are the state vectors of two systems, the matrixes M and N are defined as $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ and $N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, $\delta_i(\epsilon_i) = $ constants or a "scaling function matrix" $(i = 1, 2, \ldots, n)$, and τ is the time lag or anticipated. It is said that the master system and slave system are globally (i) lag hybrid projective synchronization control ($\tau > 0$, τ is called the synchronization lag); (ii) hybrid complete projective synchronization control ($\tau = 0$); and (iii) anticipated hybrid projective synchronization control ($\tau < 0$, $-\tau > 0$ is called the synchronization anticipation).

We remark that the above-mentioned types of synchronization are special cases of our definition. Table 1 illustrates these types of synchronization.

In order to show the results of LACHPS of two nonlinear systems, we choose the chaotic Colpitts oscillators as an example.

This paper is organized as follows. In Section 2, we show the general scheme description and theorem. In Sections 3 and 4, the Colpitts oscillator as a example is shown via applications of the LACHPS control method and cascade method. And numerical simulations are used to show the effectiveness. Finally, conclusions are drawn.

2. The Extended Control Method and the Main Results

In this section, the extended hybrid projective control method is designed to achieve synchronization control based on [42–49] method. Consider the master system in the form of

$$\dot{x}(t) = \Phi x(t) + F(x(t), t), \tag{1}$$

where $x(t) \in R^n$, Φ is an $m \times m$ constant matrix, and $F : R^n \to R^n$ is a nonlinear function. Assume that the slave system coupled with (1) is as follows:

$$\dot{y}(t) = \Phi y(t) + G(y(t), t) + U, \tag{2}$$

where $y(t) \in R^n$, and U is a controller to be determined later. Denote $e_i = \delta_i x_i(t) - \epsilon_i y_i(t - \tau)(i = 1, 2, \ldots, n)$ and $\delta_i(\epsilon_i) = $ constants or a scaling function matrix. If $\lim_{t \to \infty} \|e\| = 0$, $e = (e_1, e_2, \ldots, e_n)$, these two chaotic systems can be controlled via the LACHPS.

Proposition 2. *When the matrices $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ and $N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ are two invertible diagonal function matrices; lag, anticipated, and complete hybrid projective synchronization between the two systems* (1) *and* (2) *will occur, if the following conditions are satisfied:*

(i)

$$U = N^{-1}\left(gx(t) - hy(t - \tau)\right) + N^{-1}MF(x(t), t)$$
$$- G(y(t - \tau), t) + N^{-1}K\left[Mx(t) - Ny(t - \tau)\right], \tag{3}$$

where $g = \text{diag}(\dot{\delta}_1, \dot{\delta}_2, \ldots, \dot{\delta}_n)$, $h = \text{diag}(\dot{\epsilon}_1, \dot{\epsilon}_2, \ldots, \dot{\epsilon}_n)$ and $K \in R^{n \times n}$;

(ii) *the real parts of all the eigenvalues of $(\Phi + K)$ are negative.*

Proof. According to $e = Mx(t) - Ny(t - \tau)$ in definition of LACHPS, one can get

$$\dot{e}(t) = \dot{M}x(t) + M\dot{x}(t) - \dot{N}y(t - \tau) - N\dot{y}(t - \tau)$$

$$= gx(t) + M\left[\Phi x(t) + F(x(t), t)\right] - hy(t - \tau)$$

$$\quad - N\left(\Phi y(t - \tau) + G(y(t - \tau), t) + U\right)$$

$$= \Phi\left(Mx(t) - Ny(t - \tau)\right) + gx(t) + MF(x(t), t)$$

$$\quad - hy(t - \tau) - NG(y(t - \tau), t)$$

$$\quad - N\left[N^{-1}\left(gx(t) - hy(t - \tau)\right) + N^{-1}MF(x(t), t)\right.$$

$$\quad\quad - G(y(t - \tau), t)$$

$$\quad\quad + \left. N^{-1}K\left(Mx(t) - Ny(t - \tau)\right)\right]$$

$$= (\Phi + K)e(t). \tag{4}$$

We solve the above equation $\dot{e}(t) = (\Phi + K)e(t)$, and

$$\|e(t)\| = \left\|e^{(\Phi + K)t}e(0)\right\|. \tag{5}$$

Because the real parts of all the eigenvalues of $(\Phi + K)$ are negative, $\|e(t)\| \to 0$ if $t \to 0$. Namely, $\lim_{t \to 0}\|e(t)\| = 0$.

For a feasible control, the feedback K must be selected such that all the eigenvalues of $(\Phi + K)$, if any, have negative real parts. Thus, if the matrix $(\Phi + K)$ is in full rank, the system $\dot{e}$ is asymptotically stable at the origin, which implies that (1) and (2) are in the state of LACHPS control. □

Proposition 3. *Let a constant matrix M and a diagonal function matrix $N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$; lag, anticipated, and complete hybrid projective synchronization between the two systems* (1) *and* (2) *will occur, if the following conditions are satisfied:*

   (i)

$$U = -N^{-1}hy(t-\tau) + N^{-1}MF(x(t),t)$$
$$- G(y(t-\tau),t) - N^{-1}K[Mx(t) - Ny(t-\tau)], \tag{6}$$

   where $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$, $h = \text{diag}(\dot{\epsilon}_1, \dot{\epsilon}_2, \ldots, \dot{\epsilon}_n)$, and $K \in R^{n \times n}$;

   (ii) *the real parts of all the eigenvalues of $(\Phi + K)$ are negative.*

Similar to the way of Theorem 5, the proof of Proposition 3 is straightforward in Appendix.

Proposition 4. *Let a diagonal function matrix $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ and a constant matrix N; the extended hybrid projective synchronization control between the two systems* (1) *and* (2) *will occur, if the following conditions are satisfied:*

   (i)

$$U = -N^{-1}gx(t) + N^{-1}MF(x(t),t)$$
$$- G(y(t-\tau),t) - N^{-1}K[Mx(t) - Ny(t-\tau)], \tag{7}$$

   where $g = \text{diag}(\dot{\delta}_1, \dot{\delta}_2, \ldots, \dot{\delta}_n)$, $h = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, and $K \in R^{n \times n}$;

   (ii) *The real parts of all the eigenvalues of $(\Phi + K)$ are negative.*

Similar to the way of the Proposition 2, the proof of Proposition 4 is straightforward in Appendix.

In order to choose a suitable control law U or a vector function K, and $\dot{e}(t) = (\Phi + K)e(t)$ is asymptotically stable, we give the following theorem such that systems (1) and (2) are in the state of LACHPS control.

Theorem 5. *If the conditions are satisfied $P(\Phi + K) + (\Phi + K)^H P = -Q$, lag, anticipated, and complete hybrid projective synchronization between the two systems* (1) *and* (2) *can be achieved, where P, Q are real symmetric positive definite matrix, $K \in R^{n \times n}$, H stands for conjugate transpose of a matrix.*

Proof. According to $e = Mx(t) - Ny(t - \tau)$ in definition of LACHPS, one can get

$$\dot{e}(t) = (\Phi + K)e(t). \tag{8}$$

If λ is one of the eigenvalues of matrix $\Phi + K$ and the corresponding nonzero eigenvector is β,

$$(\Phi + K)\beta = \lambda\beta. \tag{9}$$

Multiplying the above equation left by $\beta^H P$, we obtain

$$\beta^H P(\Phi + K)\beta = \beta^H P\lambda\beta. \tag{10}$$

Similarly, we also can derive that

$$\left[\beta^H(\Phi + K)^H\right]P\beta = \overline{\lambda}\beta^H P\beta. \tag{11}$$

From the above two equations, we can obtain

$$\lambda + \overline{\lambda} = \frac{\beta^H\left[P(\Phi + K) + (\Phi + K)^H P\right]\beta}{\beta^H P\beta}. \tag{12}$$

Since $\beta^H[P(\Phi + K) + (\Phi + K)^H P]\beta = -Q$, and P and Q are real symmetric positive definite matrix,

$$\beta^H P\beta > 0, \qquad \beta^H Q\beta > 0,$$
$$\lambda + \overline{\lambda} = -\frac{\beta^H Q\beta}{\beta^H P\beta} < 0. \tag{13}$$

According to the stability theory, the system $\dot{e}$ is asymptotically stable at the origin. □

Remark 6. If we rewrite (2) as

$$\dot{y}(t) = \theta y(t) + G(y(t),t) + U, \tag{14}$$

we can obtain the following results.

   (1) Let two invertible diagonal function matrix $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ and $N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$; lag, anticipated, and complete hybrid projective synchronization between the two systems (1) and (2) will occur, if the following conditions are satisfied:

   (i)

$$U = N^{-1}\left(gx(t) - hy(t-\tau)\right) + N^{-1}MF(x(t),t)$$
$$- G(y(t-\tau),t) + (\phi + K)y(t-\tau) \tag{15}$$
$$- N^{-1}(\theta + K)Mx(t),$$

   where $g = \text{diag}(\dot{\delta}_1, \dot{\delta}_2, \ldots, \dot{\delta}_n)$, $h = \text{diag}(\dot{\epsilon}_1, \dot{\epsilon}_2, \ldots, \dot{\epsilon}_n)$, $\theta \in R^{n \times n}$, and $K \in R^{n \times n}$.

   (2) Let a constant matrix M and a diagonal function matrix $N = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$; lag, anticipated, and complete hybrid projective synchronization between the two systems (1) and (2) will occur, if the following conditions are satisfied:

   (ii)

$$U = N^{-1}\left[MF(x(t),t) - hy(t-\tau)\right] - G(y(t-\tau),t)$$
$$+ N^{-1}(\theta + K)\left[Mx(t) - Ny(t-\tau)\right], \tag{16}$$

   where $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$, $h = \text{diag}(\dot{\epsilon}_1, \dot{\epsilon}_2, \ldots, \dot{\epsilon}_n)$, $\theta \in R^{n \times n}$, and $K \in R^{n \times n}$.

   (3) Let a diagonal function matrix $M = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ and a constant matrix N; lag, anticipated, and complete hybrid projective synchronization between the two systems (1) and (2) will occur, if the following conditions are satisfied:

(iii)

$$U = N^{-1}MF(x(t),t) + N^{-1}gx(t) - G(y(t-\tau),t)$$
$$- N^{-1}(\theta+K)Mx(t) + (\theta+K)y(t-\tau), \tag{17}$$

where $g = \text{diag}(\dot{\delta}_1, \dot{\delta}_2, \ldots, \dot{\delta}_n)$, $\theta \in R^{n\times n}$, and $K \in R^{n\times n}$.

(iv) The real parts of all the eigenvalues of $(\phi+\theta+K)$ are negative with Cases 1–3.

3. Applications of the LACHPS Control Method

Now, we introduce the following nonlinear system:

$$\dot{x}(t) = az(t) - c|y(t)| + d,$$
$$\dot{y}(t) = az(t), \tag{18}$$
$$\dot{z}(t) = -\frac{x(t)}{2a} - \frac{y(t)}{2a} - bz(t),$$

where a, b, c, and d are real constants, if $a = -2$, $b = 0.4$, $c = 1.62$, and $d = 3$, the simulation results of system (18) with the initial conditions $(0, 0, 0)$. System (18) has chaotic attractor as shown in Figures 1 and 2. System (18) temporal evolution of the state variables is shown in Figure 3. For more detailed dynamical properties of system (2), the reader should refer to [50].

In the following, we rewrite the chaotic system (18) as a master system:

$$\dot{x}_m(t) = az_m(t) - c|y_m(t)| + d,$$
$$\dot{y}_m(t) = az_m(t), \tag{19}$$
$$\dot{z}_m(t) = -\frac{x_m(t)}{2a} - \frac{y_m(t)}{2a} - bz_m(t)$$

and the system related to (20), given by

$$\dot{x}_s(t) = az_s(t) - c|y_s(t)| + d + u_1,$$
$$\dot{y}_s(t) = az_s(t) + u_2, \tag{20}$$
$$\dot{z}_s(t) = -\frac{x_s(t)}{2a} - \frac{y_s(t)}{2a} - bz_s(t) + u_3$$

as a slave system, where the subscripts "m" and "s" stand for the master system and slave system, respectively. Let the error state be

$$e(t) = (e_1, e_2, e_3)^T$$
$$= [\delta_1 x_m(t) - \epsilon_1 x_s(t-\tau),$$
$$\delta_2 y_m(t) - \epsilon_2 y_s(t-\tau), \tag{21}$$
$$\delta_3 z_m(t) - \epsilon_3 z_s(t-\tau)]^T.$$

Then from (18) and (19), we obtain the error system

$$\dot{e}_1 = \delta_1 [az_m(t) - c|y_m(t)| + d]$$
$$- \epsilon_1 [az_s(t-\tau) - c|y_s(t-\tau)| + d + u_1],$$

$$\dot{e}_2 = \delta_2 az_m(t) - \epsilon_2 az_s(t-\tau) - \epsilon_2 u_2,$$

$$\dot{e}_3 = \delta_3 \left[-\frac{x_m(t)}{2a} - \frac{y_m(t)}{2a} - bz_m(t) \right]$$
$$- \epsilon_3 \left[-\frac{x_s(t-\tau)}{2a} - \frac{y_s(t-\tau)}{2a} - bz_s(t-\tau) + u_3 \right]. \tag{22}$$

To the LACHPS synchronization control between systems (18) and (19), we have the following theorem.

Proposition 7. *For the chaotic Colpitts oscillator* (18)*, if one of the following feedback controllers* $u_i(i = 1, 2, 3)$ *is chosen for the slave system* (19)

$$u_1 = (\delta_1 az_m(t) - \delta_1 c|y_m(t)| + \delta_1 d - \epsilon_1 az_s(t-\tau)$$
$$+ \epsilon_1 c|y_s(t-\tau)| - \epsilon_1 d - ae_3) \times \epsilon_1^{-1},$$
$$u_2 = \frac{\delta_2 az_m(t) - \epsilon_2 az_s(t-\tau) - ae_3 + e_2}{\epsilon_2},$$
$$u_3 = (-\delta_3 x_m(t) - \delta_3 y_m(t) - 2\delta_3 bz_m(t)a \tag{23}$$
$$+ \epsilon_3 x_s(t-\tau) + \epsilon_3 y_s(t-\tau)) \times (2\epsilon_3 a)^{-1}$$
$$+ \frac{2\epsilon_3 bz_s(t-\tau)a + e_1 + e_2 + 2be_3 a}{2\epsilon_3 a},$$

$$u_1 = \frac{(\epsilon_3 a - \epsilon_1 a)z_s(t-\tau)}{\epsilon_1} + \frac{(\epsilon_1 c - c\epsilon_2)|y_s(t-\tau)|}{\epsilon_1}$$
$$+ \frac{(\delta_1 a - a\delta_3)z_m(t)}{\epsilon_1}$$
$$+ \frac{(c\delta_2 - \delta_1 c)|y_m(t)|}{\epsilon_1} + \frac{-\epsilon_1 d + \delta_1 d}{\epsilon_1},$$

$$u_2 = \frac{(-\epsilon_2 a + a\epsilon_3)z_s(t-\tau)}{\epsilon_2} + \frac{(-a\delta_3 + \delta_2 a)z_m(t)}{\epsilon_2}$$
$$+ (-\delta_1 x_m(t) - \delta_2 |y_m(t)|$$
$$+ \epsilon_2 |y_s(t-\tau)| + \epsilon_1 x_s(t-\tau)) \times \epsilon_2^{-1},$$

$$u_3 = \frac{(-2\epsilon_3 a + 2\epsilon_3 ba)z_s(t-\tau)}{2\epsilon_3 a}$$
$$+ \frac{(\epsilon_3 - 2\epsilon_2 a)|y_s(t-\tau)|}{2\epsilon_3 a} + \frac{(\epsilon_3 - 2\epsilon_1 a)x_s(t-\tau)}{2\epsilon_3 a}$$
$$+ \frac{(-2\delta_3 ba + 2\delta_3 a)z_m(t)}{2\epsilon_3 a} + \frac{(-\delta_3 + 2\delta_2 a)|y_m(t)|}{2\epsilon_3 a}$$
$$+ \frac{(2\delta_1 a - \delta_3)x_m(t)}{2\epsilon_3 a}, \tag{24}$$

where $a < 0$, $b > 0$, $c > 0$, $d > 0$, and ϵ_i and δ_i are real, then the zero solution of the error system (21) is globally stable, and
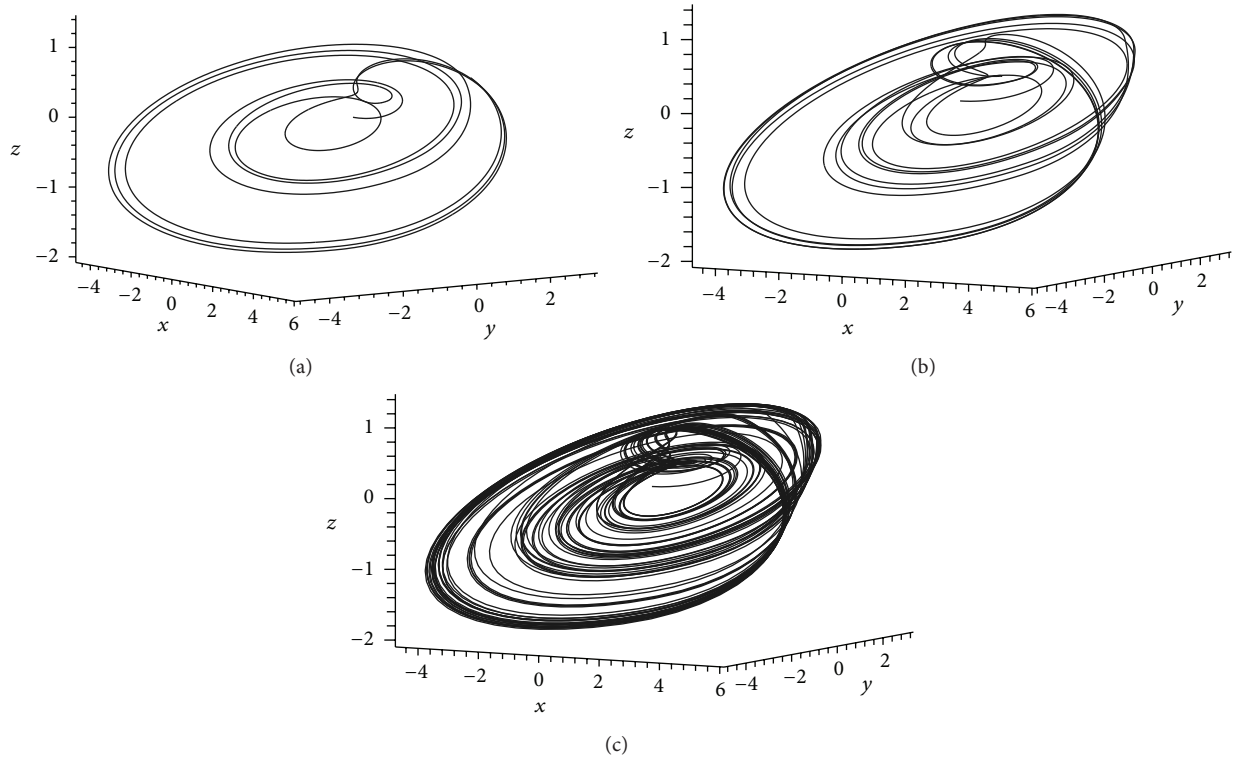
(a)

(b)

(c)

Figure 1: Chaotic attractors for the Colpitts system with temporal evolution in different 3D spaces.
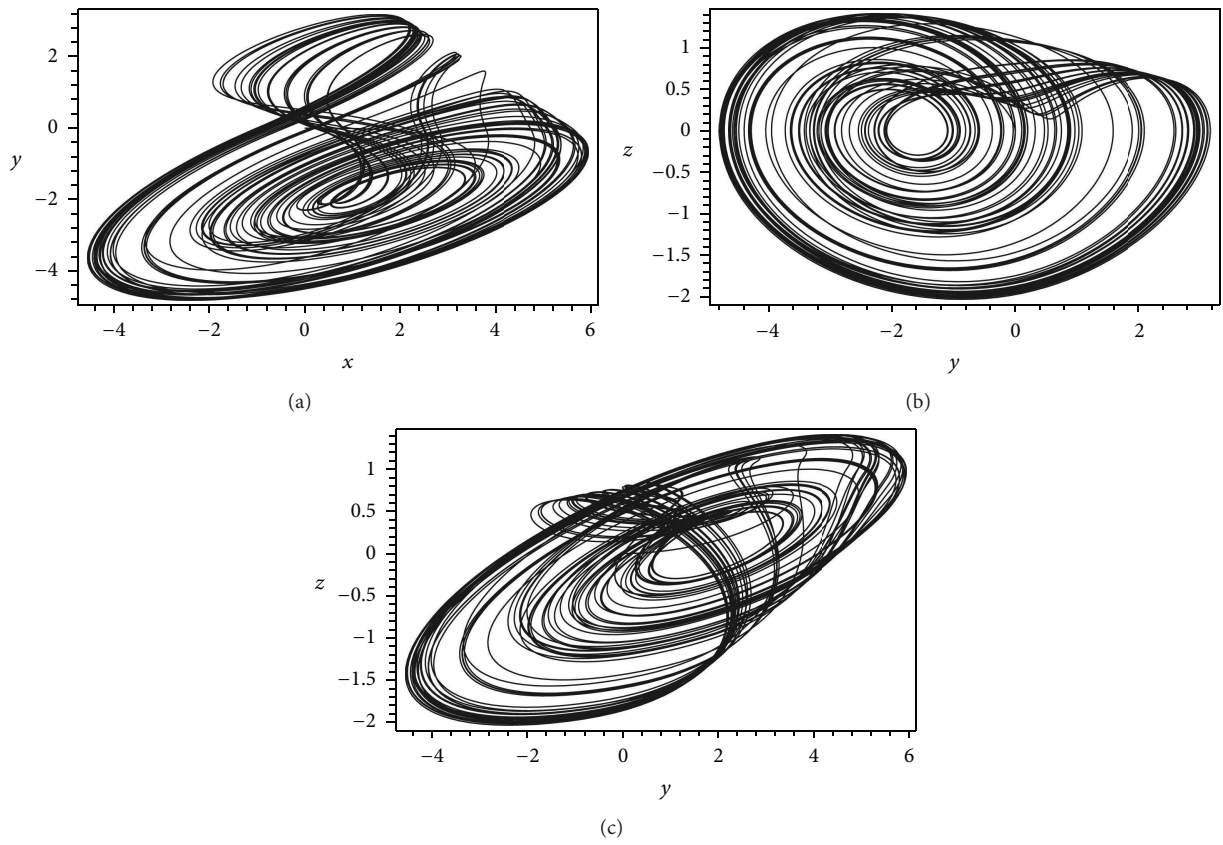

(a)

(b)

(c)

Figure 2: The phase figure for the Colpitts system with temporal evolution in different plane.
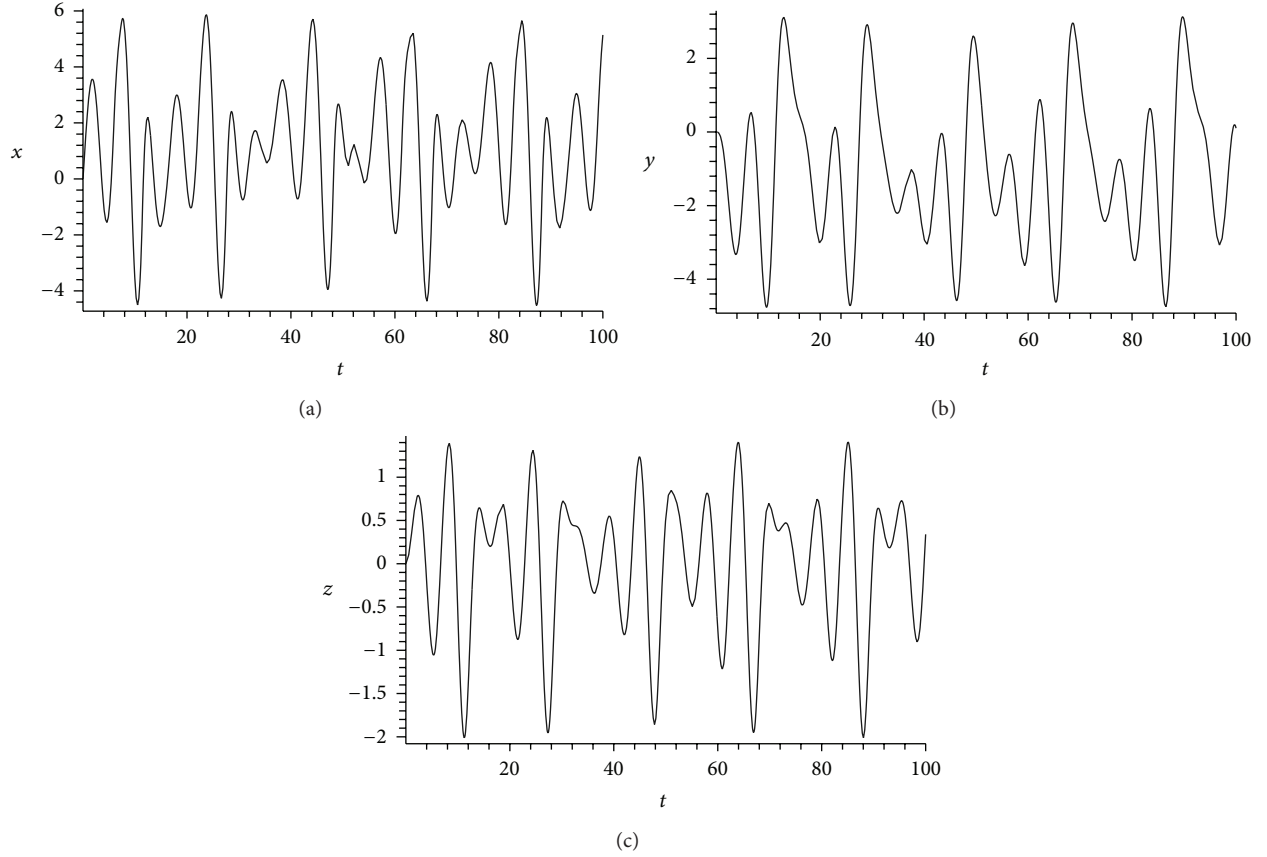
(a)

(b)

(c)

Figure 3: The temporal evolution of the state variables.

thus (i) globally lag synchronization for $\tau < 0$, (ii) anticipated synchronization for $\tau > 0$, and (iii) complete synchronization for $\tau = 0$ occur between the master system (18) and the slave system (19).

Proof. Similar to the way of Propositions 2–4, the proof of Proposition 7 is straightforward and we omit the detail steps. We give another proof method via Lyapunov function in the following.

Consider the controller (22) and choose the following quadratic form, positive definite of Lyapunov function:

$$V(e_1, e_2, e_3) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right), \qquad (25)$$

which implies that $P = \mathrm{diag}(1/2, 1/2, 1/2)$ and thus $\lambda_{\min}(P) = 1/2$, $\lambda_{\max}(P) = 1/2$. Differentiating $V(t)$ along the trajectory of system (21) yields

$$\left.\frac{dV(t)}{dt}\right|_{(21)} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3$$

$$= e_1\left[\delta_1\left(az_m(t) - c\left|y_m(t)\right| + d\right) - \epsilon_1\left(az_s(t-\tau) - c\left|y_s(t-\tau)\right| + d + u_1\right)\right]$$

$$+ e_2\left[\delta_2 az_m(t) - \epsilon_2 az_s(t-\tau) - u_2\right]$$

$$+ e_3\left[\delta_3\left(-\frac{x_m(t)}{2a} - \frac{y_m(t)}{2a} - bz_m(t)\right)\right]$$

$$- e_3\left[\epsilon_3\left(-\frac{x_s(t-\tau)}{2a} - \frac{y_s(t-\tau)}{2a}\right.\right.$$

$$\left.\left. -bz_s(t-\tau) + u_3\right)\right].$$

$$(26)$$

We put u_1, u_2, and u_3 into (25) and then simplify and yield

$$\left.\frac{dV(t)}{dt}\right|_{(21)} = ae_1e_3 + e_2\left(ae_3 - e_2\right)$$

$$+ e_3\left(\frac{-e_1}{2a} - \frac{-1}{2a}e_2 - be_3\right)$$

$$= -e_2^2 - be_3^2 + \left(a - \frac{1}{2a}\right)e_1e_3 + \left(a - \frac{1}{2a}\right)e_2e_3$$

$$\leq \left(\frac{a}{2} - \frac{1}{4a}\right)e_1^2 + \left(\frac{a}{2} - \frac{1}{4a} - 1\right)e_2^2$$

$$+ \left(a - \frac{1}{2a} - b\right)e_3^2,$$

$$(27)$$

where $Q = \mathrm{diag}(a/2 - 1/4a, a/2 - 1/4a - 1, a - 1/2a - b)$. Then using Lemma 1 [20], we have the estimation

$$e_1^2 + e_2^2 + e_3^2$$

$$\leq \frac{\lambda_{\max}}{\lambda_{\min}}\left[e_1^2(0) + e_2^2(0) + e_3^2(0)\right]e^{-(\lambda_{\min}(Q)/\lambda_{\max}(P))(t-t_0)}$$
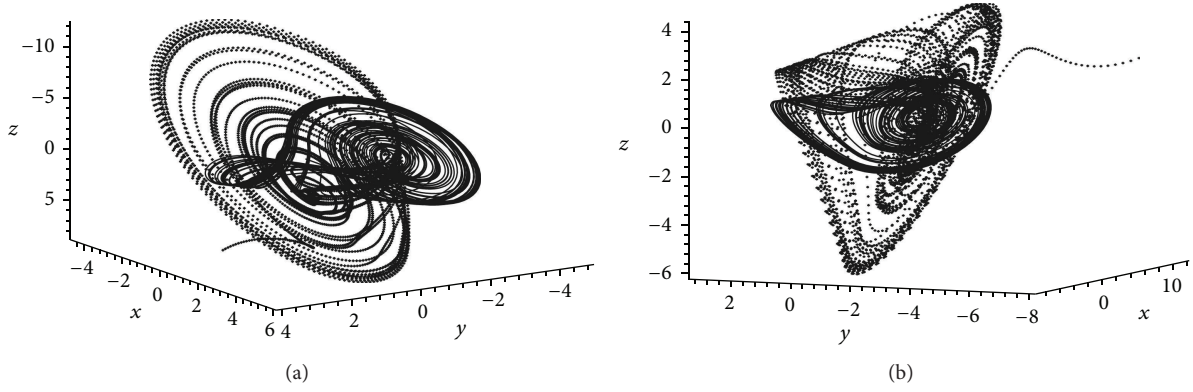
(a)

(b)

FIGURE 4: (a) The synchronized attractors in (x, y, z) space and (b) the synchronized attractors with scaling factor in (x, y, z) space, "$- -$" denotes for the master system, "$\cdots$" denotes for the slave system synchronized.
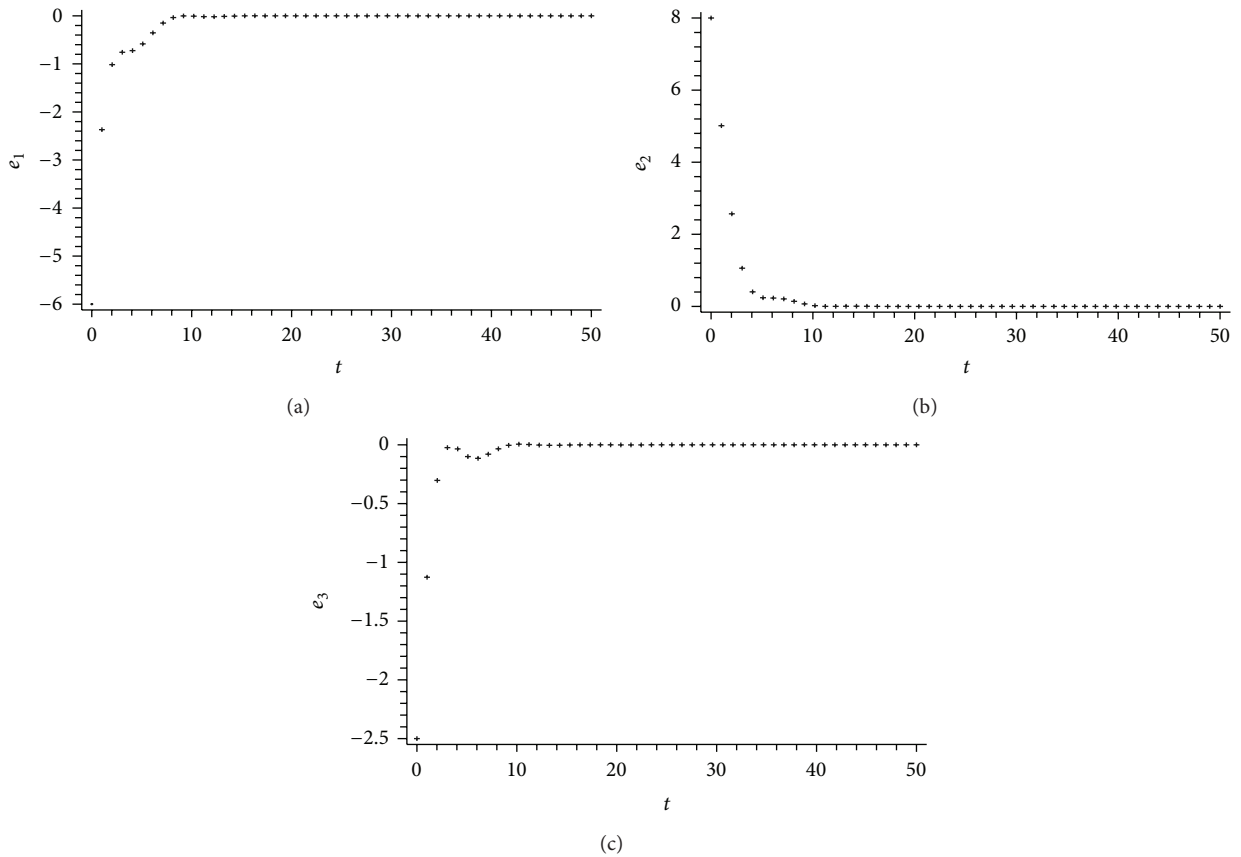

(a)

(b)

(c)

FIGURE 5: The orbits of error states: (a) $e_1 = \delta_1 x_m(t) - \epsilon_1 x_s(t - \tau)$, (b) $e_2 = \delta_2 y_m(t) - \epsilon_2 y_s(t - \tau)$, and (c) $e_3 = \delta_3 z_m(t) - \epsilon_3 z_s(t - \tau)$.

$$= \left[e_1^2(0) + e_2^2(0) + e_3^2(0) \right]$$
$$\times e^{-2(\lambda_{\min}(a/2 - 1/4a, a/2 - 1/4a - 1, a - 1/2a - b))(t - t_0)}. \tag{28}$$

Namely, if $a/2 - 1/4a < 0$, $a/2 - 1/4a - 1 < 0$, and $a - 1/2a - b < 0$, which implies that the conclusion is true. Similarly, for the controllers (23), we can still use the method to obtain the estimation. □

Remark 8. (1) The nonlinear feedback controllers can be used to simultaneously obtain (i) hybrid lag synchronization for $\tau > 0$, (ii) hybrid anticipated synchronization for $\tau < 0$, and (iii) hybrid complete synchronization for $\tau = 0$ between the master system (19) and the slave system (20).

(2) Although the above-obtained feedback controllers are nonlinear, they are simpler than those of the so-called natural control controllers, which are derived by using with a simple stable matrix M and N for the master system (19) and the slave system (20).
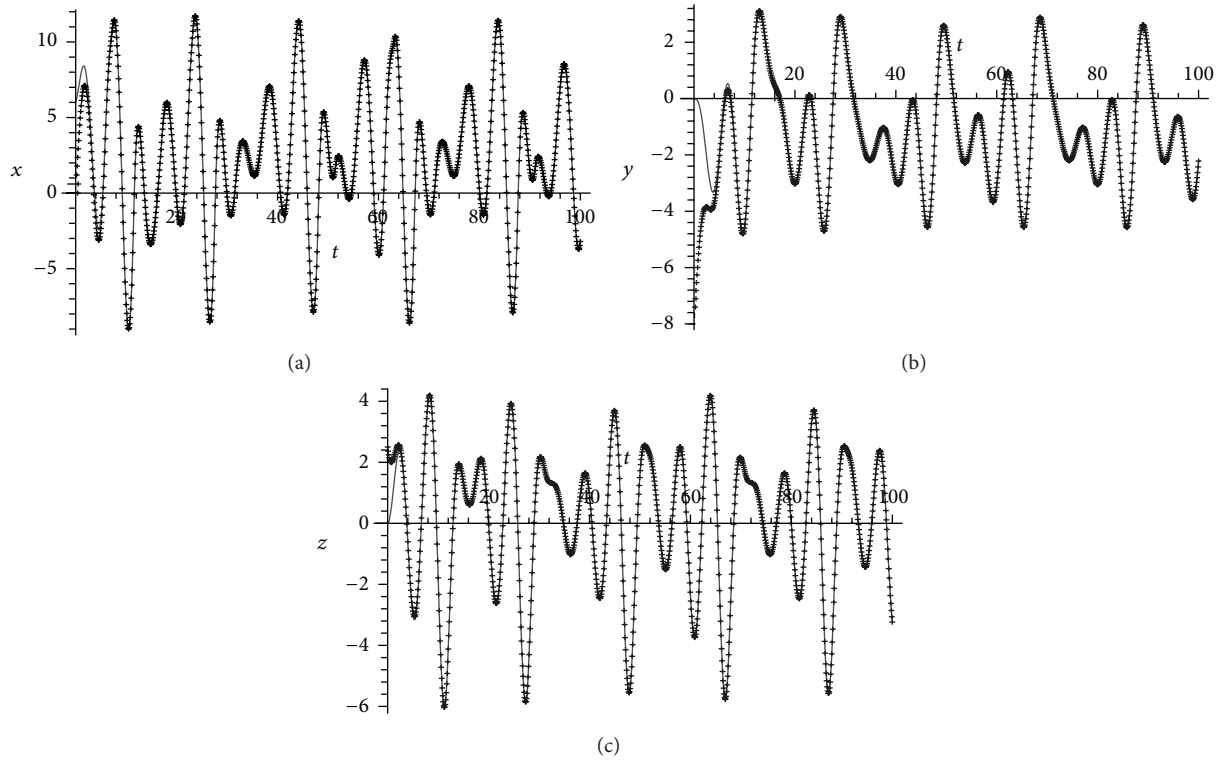
(a)

(b)

(c)

FIGURE 6: The solutions of the master and slave systems with control law. (a) Signals x_m (the dashed line) and x_s (the solid line). (b) Signals y_m (the dashed line) and y_s (the solid line). (c) Signals z_m (the dashed line) and z_s (the solid line).
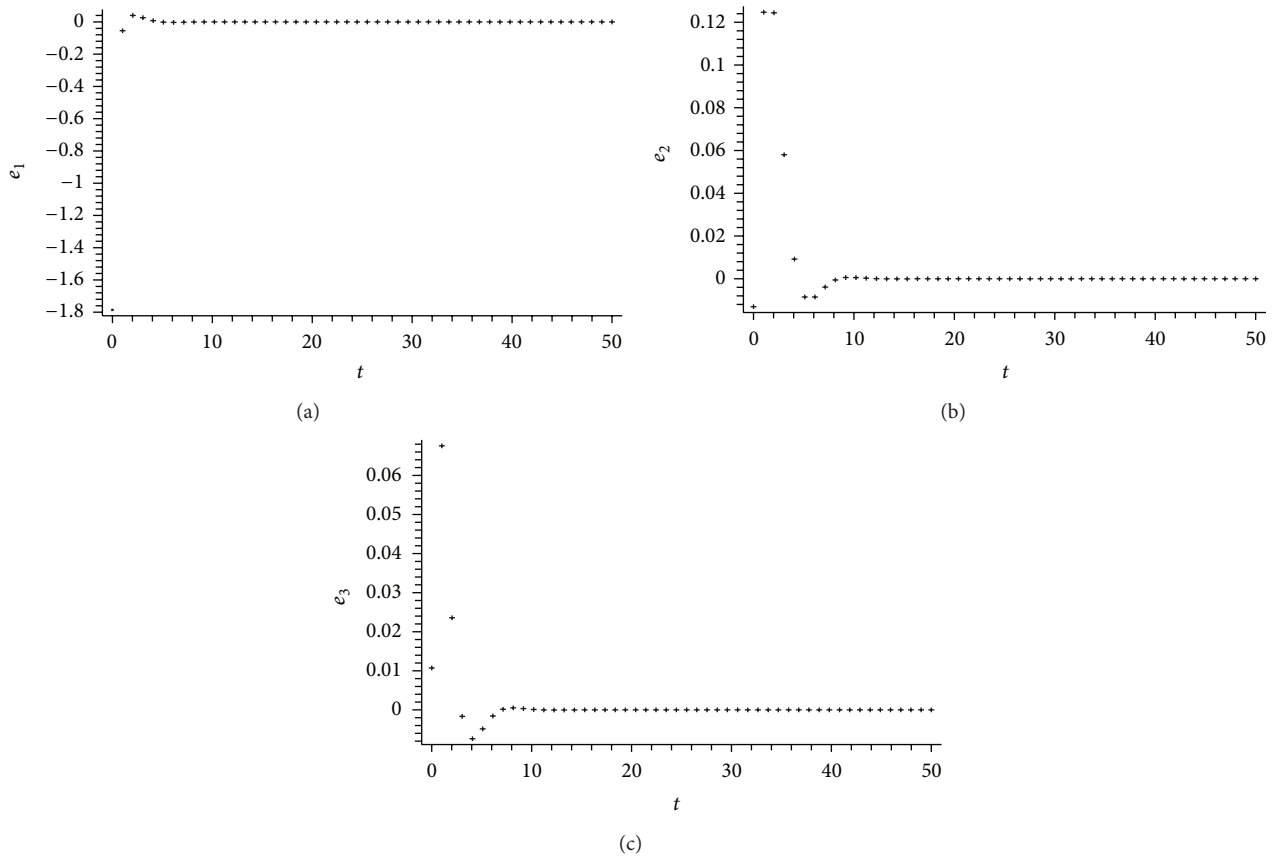

(a)

(b)

(c)

FIGURE 7: The orbits of error states: (a) $e_1 = \delta_1 x_m(t) - \epsilon_1 x_s(t - \tau)$, (b) $e_2 = \delta_2 y_m(t) - \epsilon_2 y_s(t - \tau)$, and (c) $e_3 = \delta_3 z_m(t) - \epsilon_3 z_s(t - \tau)$.
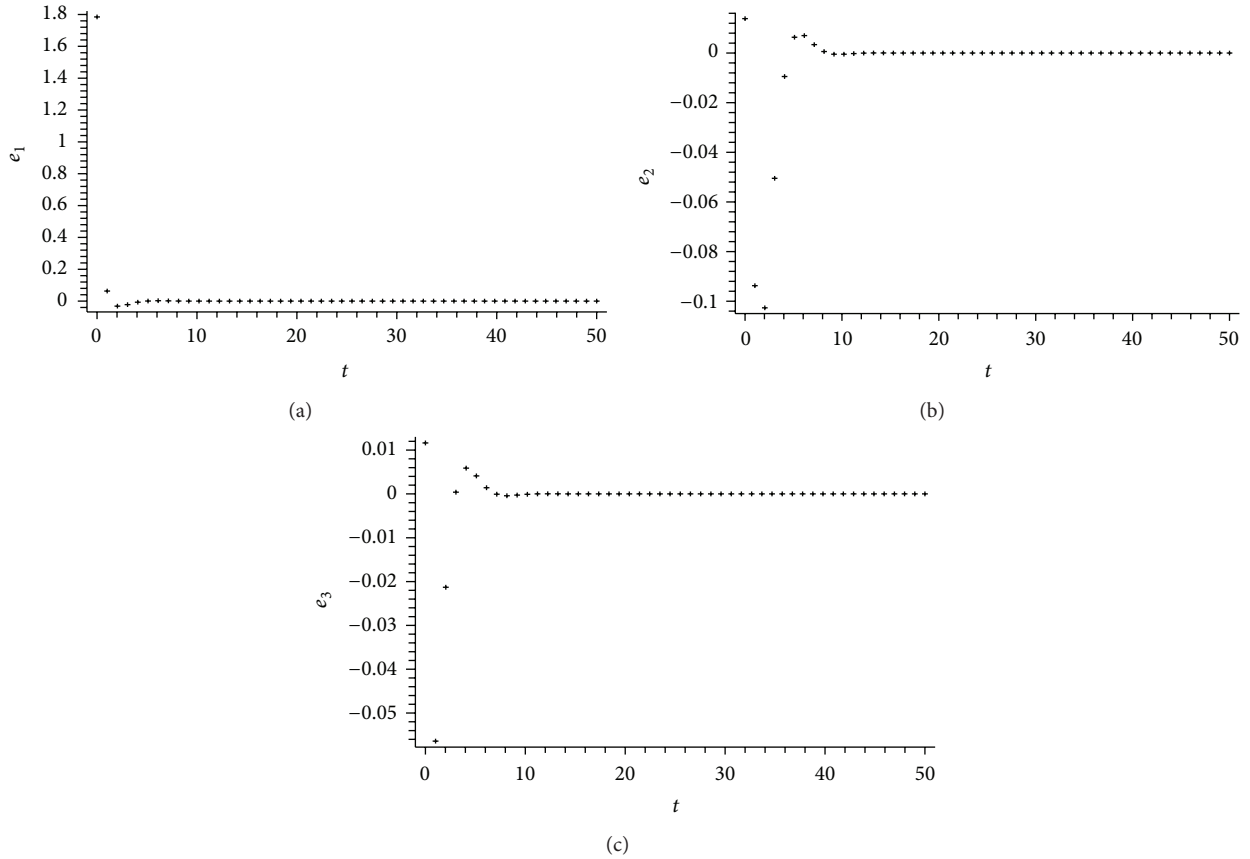
(a)

(b)

(c)

FIGURE 8: The orbits of error states: (a) $e_1 = \delta_1 x_m(t) - \epsilon_1 x_s(t - \tau)$, (b) $e_2 = \delta_2 y_m(t) - \epsilon_2 y_s(t - \tau)$, and (c) $e_3 = \delta_3 z_m(t) - \epsilon_3 z_s(t - \tau)$.

In the following, we obtain the numerical simulations results to prove the effective control. Numerical simulations results are presented to demonstrate the effectiveness of the proposed synchronization methods. The parameters are chosen to be $[a, b, c, d] = [-2, 0.4, 1.62, 3]$ in all simulations so that the chaotic system exhibits a chaotic behavior if no control is applied. The initial value $[x_m(0), y_m(0), z_m(0)]$ is taken as the random number $[0, 0, 0]$ and $[x_s(0), y_s(0), z_s(0)] = [3, 4, 5]$. The parameters $[\delta_1, \delta_2, \delta_3, \epsilon_1, \epsilon_2, \epsilon_3] = [2, -2, 0.5, 2, 1, 3]$.

Case 1. Hybrid complete projective control: in the case $\tau = 0$, without loss of generality, the initial values of the error dynamical system (21) are $e_1(0) = \delta_1 x_m(0) - \epsilon_1 x_s(0) = -6$, $e_2(0) = \delta_2 y_m(0) - \epsilon_2 y_s(0) = 8$, and $e_3(0) = \delta_3 z_m(0) - \epsilon_3 z_s(0) = -2.5$. The dynamics of hybrid complete control errors for the master system (19) and the slave system (20) is displayed in Figures 4, 5, and 6. Figure 4 shows the chaotic attractors of the master and slave systems with different initial values in the same coordinate. Figures 5(a)–5(c) show the evolutions of the error functions e_1, e_2, and e_3. Figures 6(a)–6(c) the solutions of the master and slave systems with control law.

Case 2. Hybrid lag projective control: in the case $\tau > 0$, without loss of generality, we set $\tau = 0.3$. Thus the initial values of the error dynamical system (21) are $e_1(0) = \delta_1 x_m(0) -$

$\epsilon_1 x_s(-0.3) = 1.784475527$, $e_2(0) = \delta_2 y_m(0) - \epsilon_2 y_s(-0.3) = 0.01384929027$, and $e_3(0) = \delta_3 z_m(0) - \epsilon_3 z_s(-0.3) = 0.01162129660$. For simplification, we only give the dynamics of the evolutions of hybrid lag control errors for the master system (19) and the slave system (20) displayed in Figure 7.

Case 3. Hybrid anticipated projective control: in the case $\tau < 0$, without loss of generality, we set $\tau = -0.3$. Thus the initial values of the error dynamical system (21) are $e_1(0) = \delta_1 x_m(0) - \epsilon_1 x_s(-0.3) = -1.785359335$, $e_2(0) = \delta_2 y_m(0) - \epsilon_2 y_s(-0.3) = -0.01304389936$, and $e_3(0) = \delta_3 z_m(0) - \epsilon_3 z_s(-0.3) = 0.01072905365$. For simplification, we only give the dynamics of the evolutions of hybrid lag control errors for the master system (19) and the slave system (20) as displayed in Figure 8.

4. Applications of the LACHPS Control Method via Cascade Control Idea

In the section, based on the idea of cascade approach [42, 50, 51], we achieve the effectiveness control idea.

Firstly, we take the system (18) as master system. The slave system is given by

$$\dot{X}(t - \tau) = az(t - \tau) - c|Q(t - \tau)| + d + u_1,$$

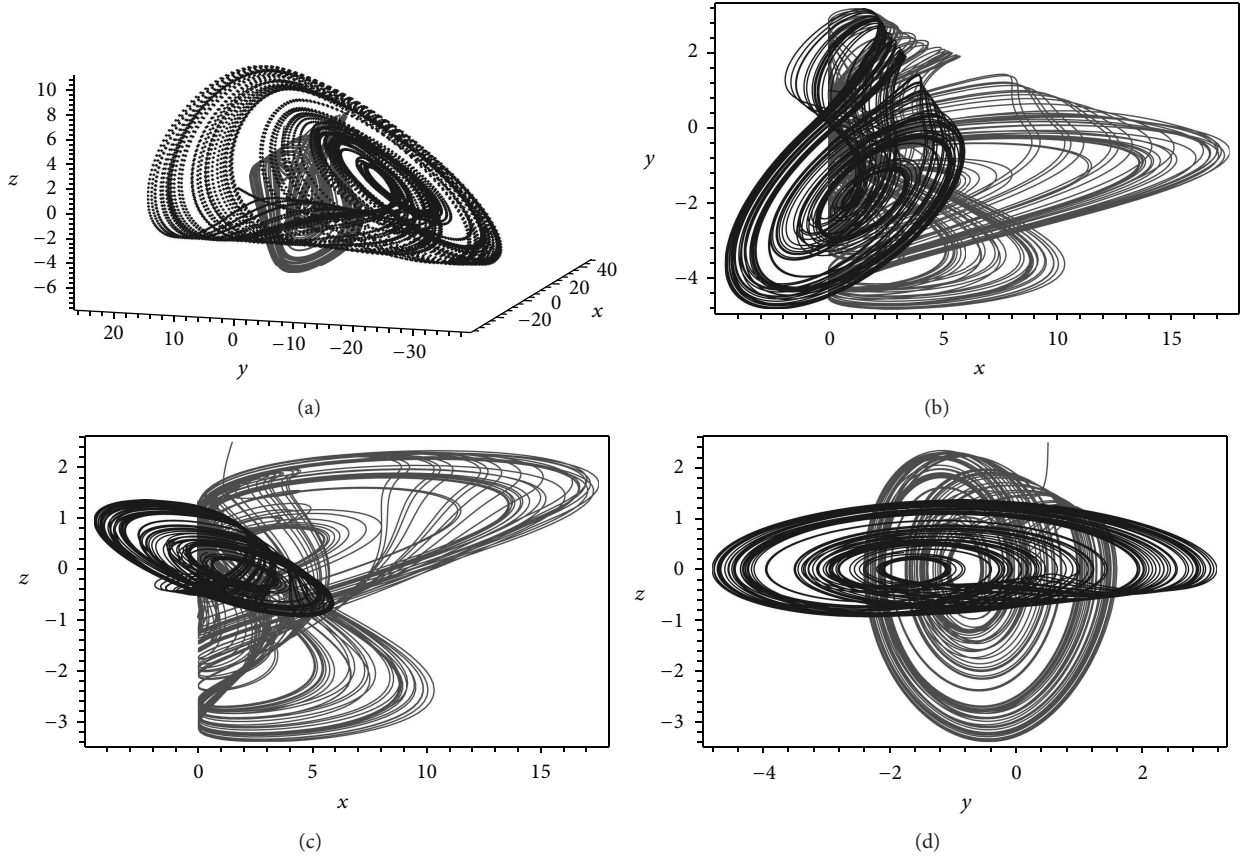$$\dot{Q}(t - \tau) = az(t - \tau) + u_2,$$

(29)

(a)

(b)

(c)

(d)

FIGURE 9: (a) The synchronized attractors in (x, y, z) space and the phase figure in the different space (b) (x, y) space, (c) (x, z) space, (d) (y, z) space "– –" denotes for the master system, "$\cdots$" denotes for the slave system synchronized.

where $(u_1, u_2)^T$ is external control functions that is to be designed below.

Let the error states functions of systems (18) and (29) as follows:

$$e_1 = X(t - \tau) - \epsilon_1 x(t), \qquad e_2 = Q(t - \tau) - \epsilon_2 y(t), \quad (30)$$

where $\epsilon_1 = a_{11}x(t) + a_{12}$, $\epsilon_2 = a_{21}y(t) + a_{22}$, and τ is the time lag or anticipated. The goal of the control is to find a controller $(u_1, u_2)^T$ such that the states of the master system (18) and the states of the slave system (29) are globally synchronized asymptotically; that is,

$$\lim_{t \to \infty} \|X(t - \tau) - \epsilon_1 x(t)\| = 0,$$
$$\lim_{t \to \infty} \|Q(t - \tau) - \epsilon_2 y(t)\| = 0. \tag{31}$$

Let us define the Lyapunov functions as

$$V_1 = \frac{1}{2}\left(e_1^2 + e_2^2\right). \tag{32}$$

If the Lyapunov function (32) satisfies the conditions

$$V_1 > 0 \quad \text{if } (e_1, e_2) \neq (0, 0),$$
$$V_1 = 0 \quad \text{if } (e_1, e_2) = (0, 0),$$

$$\dot{V}_1 < 0 \quad \text{if } (e_1, e_2) \neq (0, 0),$$
$$\dot{V}_1 = 0 \quad \text{if } (e_1, e_2) = (0, 0). \tag{33}$$

then e_i $(i = 1, 2)$ will asymptotically tend to zero and

$$\lim_{t \to \infty} \|X(t - \tau) - \epsilon_1 x(t)\| = 0,$$
$$\lim_{t \to \infty} \|Q(t - \tau) - \epsilon_2 y(t)\| = 0. \tag{34}$$

With the aid of Maple and we omit the details by the aid of Maple soft, we choose

$$u_1 = -e_1(t) - az(t - \tau) + c|Q(t - \tau)| - d$$
$$+ 2a_{11}x(t)az(t) - 2a_{11}x(t)c|y(t)| + 2a_{11}x(t)d$$
$$+ a_{12}az(t) - a_{12}c|y(t)| + a_{12}d + k_1, \tag{35}$$
$$u_2 = -e_2(t) - az(t - \tau)$$
$$+ 2a_{21}az(t)y(t) + az(t)a_{22} + k_2;$$

then (18) and (29) will be satisfied. Next we take (29) as the master system, and the slave one is as follows:

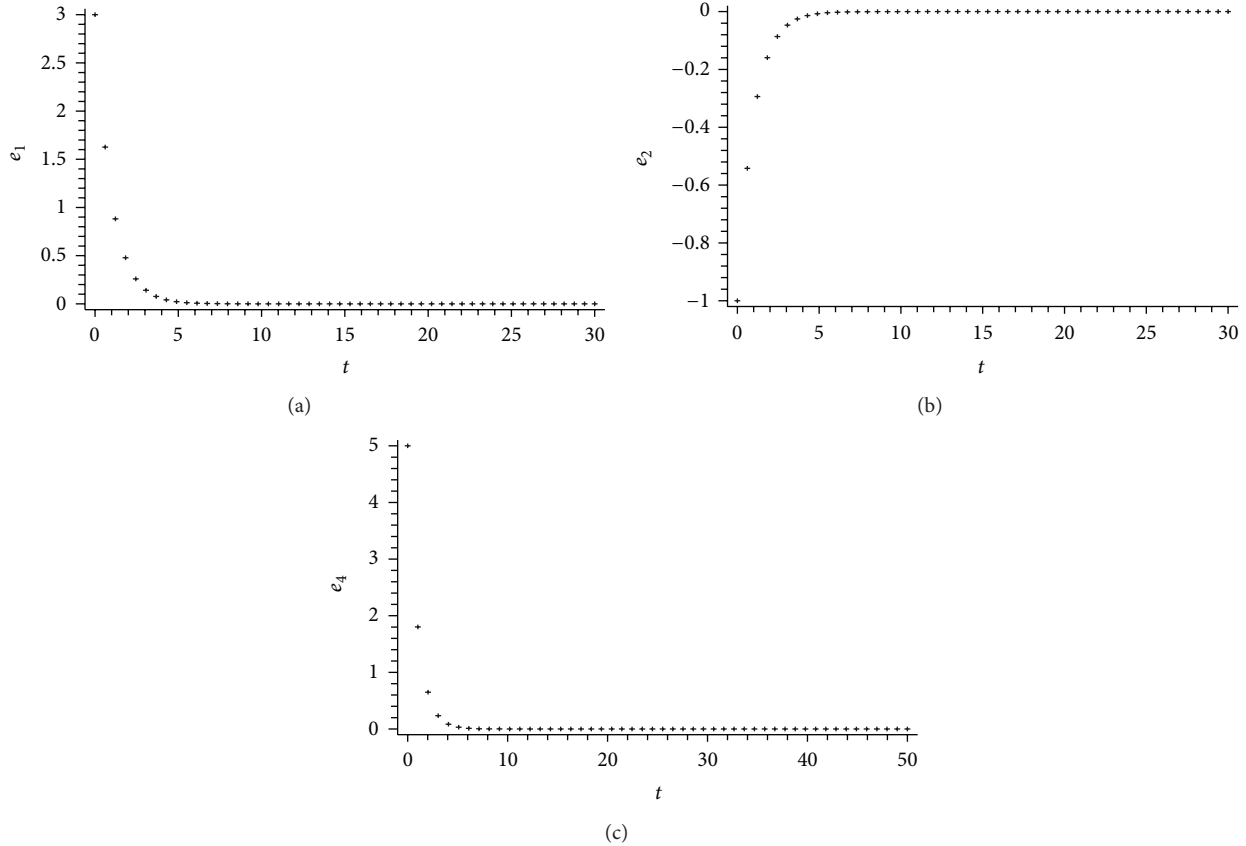$$\dot{Y}(t - \tau) = aZ(t - \tau) + u_3,$$

FIGURE 10: The orbits of error states: (a) $e_1 = X(t-\tau) - \epsilon_1 x(t)$, (b) $e_3 = Y(t-\tau) - \epsilon_3 y(t)$, and (c) $e_4 = Z(t-\tau) - \epsilon_4 z(t)$.

$$\dot{Z}(t-\tau) = -\frac{x(t-\tau)}{2a} - \frac{y(t-\tau)}{2a} - bz(t-\tau) + u_4, \tag{36}$$

where $(u_3, u_4)^T$ is a desired controller.

The relevant Lyapunov function can be chosen as

$$V_2 = \frac{1}{2}\left(e_3^2 + e_4^2\right), \tag{37}$$

where $e_3 = Y(t-\tau) - \epsilon_3 y(t)$, $e_4 = Z(t-\tau) - \epsilon_4 z(t)$, $\epsilon_3 = a_{31}y(t) + a_{32}$, $\epsilon_4 = a_{41}z(t) + a_{42}$, and τ is the time lag or anticipated. We take u_3, u_4 as

$$
\begin{aligned}
u_3 &= -Y(t-\tau) + a_{31}y(t)^2 + y(t)a_{32} - aZ(t-\tau) \\
&\quad + 2a_{31}az(t)y(t) + az(t)a_{32} + k_3,
\end{aligned}
$$

$$
\begin{aligned}
u_4 &= -\frac{1}{2a}\Big[2aZ(t-\tau) - 2aa_{41}z(t)^2 \\
&\quad - 2aa_{42}z(t) - Y(t-\tau) + a_{42}x(t) \\
&\quad + a_{42}y(t) + 2a_{42}bz(t)a + 2a_{41}z(t)y(t) \\
&\quad - X(t-\tau) - 2bZ(t-\tau)a \\
&\quad + 2a_{41}z(t)x(t) + 4a_{41}z(t)^2 ba - 2k_4 a\Big],
\end{aligned}
\tag{38}
$$

which make e_3 and e_4 approach to zero when $t \to +\infty$. Therefor the LACHPS is achieved for the systems (29) and (36) via cascade method.

For simplicity and illustration, the parameters $k_1 = 2$, $k_2 = 3$, $k_3 = 4$, $k_4 = 2$, we consider $[a_{11}, a_{12}, a_{21}, a_{21}, a_{31}, a_{32}, a_{41}, a_{42}] = [1, 0, 0, 3, 0, 2, 0, -5]$ and the initial values $[x(0), y(0), z(0), X(0), Y(0), Z(0), Q(0)] = [0, 1, 1, 3, 1, 5, -5]$. We may only choose $\tau = 0$; other cases are similar. Figure 9 shows the LACHPS via cascade method for the system (18). And Figures 10(a)–10(c) show the numeric simulations of the error functions e_1, e_3, and e_4.

5. Conclusion

In this paper, based on the stability theory and an active control technique, we investigate the lag, anticipated, and complete hybrid projective synchronization control (LACHPS) for nonlinear chaotic systems. A nonlinear controller has been proposed to achieve lag, anticipated, and complete projective synchronization of chaotic systems. The proposed synchronization is simple and theoretically rigorous. Colpitts oscillators are used to illustrate the effectiveness of the proposed synchronization scheme. It should be note that lag synchronization control, anticipated synchronization control, and complete synchronization control. Therefore, the results of this paper are more applicable and representative.

Appendix

Proof of Proposition 3. According to $e = Mx(t) - Ny(t - \tau)$ in definition of LACHPS, one can get

$$
\begin{aligned}
\dot{e}(t) &= M\dot{x}(t) - \dot{N}y(t-\tau) - N\dot{y}(t-\tau) \\
&= M\left[\phi x(t) + F(x(t), t)\right] - hy(t-\tau) \\
&\quad - N\left(\phi y(t-\tau) + G(y(t-\tau), t) + U\right) \\
&= \phi\left(Mx(t) - Ny(t-\tau)\right) + MF(x(t), t) \\
&\quad - hy(t-\tau) - NG(y(t-\tau), t) \\
&\quad - N\left[-N^{-1}hy(t-\tau)\right. \\
&\qquad + N^{-1}MF(x(t), t) - G(y(t-\tau), t) \\
&\qquad \left. - N^{-1}K(Mx(t) - Ny(t-\tau))\right] \\
&= (\phi + K)e(t).
\end{aligned}
\tag{A.1}
$$

We solve the above equation $\dot{e}(t) = (\phi + K)e(t)$, and

$$
\|e(t)\| = \left\|e^{(\phi+K)t}e(0)\right\|.
\tag{A.2}
$$

Because the real parts of all the eigenvalues of $(\phi + K)$ are negative, $\|e(t)\| \to 0$ if $t \to 0$. Namely, $\lim_{t \to 0}\|e(t)\| = 0$. For a feasible control, the feedback K must be selected such that all the eigenvalues of $(\phi + K)$, if any, have negative real parts. Thus, if the matrix $(\phi + K)$ is in full rank, the system $\dot{e}$ is asymptotically stable at the origin, which implies that (1) and (2) are in the state of LACHPS control. □

Proof of Proposition 4. From $e = Mx(t) - Ny(t - \tau)$ in definition of LACHPS, one can get

$$
\begin{aligned}
\dot{e}(t) &= \dot{M}x(t) + M\dot{x}(t) - N\dot{y}(t-\tau) \\
&= M\left[\phi x(t) + F(x(t), t)\right] + gx(t) \\
&\quad - N\left(\phi y(t-\tau) + G(y(t-\tau), t) + U\right) \\
&= \phi\left(Mx(t) - Ny(t-\tau)\right) \\
&\quad + MF(x(t), t) + gx(t) - NG(y(t-\tau), t) \\
&\quad - N\left[-N^{-1}gx(t) + N^{-1}MF(x(t), t)\right. \\
&\qquad - G(y(t-\tau), t) \\
&\qquad \left. - N^{-1}K\left[Mx(t) - Ny(t-\tau)\right]\right] \\
&= (\phi + K)e(t).
\end{aligned}
\tag{A.3}
$$

We solve the above equation $\dot{e}(t) = (\phi + K)e(t)$, and

$$
\|e(t)\| = \left\|e^{(\phi+K)t}e(0)\right\|.
\tag{A.4}
$$

Because the real parts of all the eigenvalues of $(\phi + K)$ are negative, $\|e(t)\| \to 0$ if $t \to 0$. Namely, $\lim_{t \to 0}\|e(t)\| = 0$.

For a feasible control, the feedback K must be selected such that all the eigenvalues of $(\phi + K)$, if any, have negative real parts. Thus, if the matrix $(\phi + K)$ is in full rank, the system $\dot{e}$ is asymptotically stable at the origin, which implies that (1) and (2) are in the state of LACHPS control.

In this case, the active control method [22] is usually adopted to obtain the gain matrix K for any specified eigenvalues of $(\phi + K)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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