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*Research Article*

# On the Set of Fixed Points and Periodic Points of Continuously Differentiable Functions

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In recent years, researchers have studied the size of different sets related to the dynamics of self-maps of an interval. In this note we investigate the sets of fixed points and periodic points of continuously differentiable functions and show that typically such functions have a finite set of fixed points and a countable set of periodic points.

## 1. Introduction and Notation

The set of periodic points of self-maps of intervals has been studied for different reasons. The functions with smaller sets of periodic points are more likely not to share a periodic point. Of course, one has to decide what "big" or "small" means and how to describe this notion. In this direction one would be interested in studying the size of the sets of periodic points of self-maps of an interval, in particular, and other sets arising in dynamical systems in general (see [1–5]). For example, typically continuous functions have a first category set of periodic points (see [1, 5]). This result was generalized in [2] for the set of chain recurrent points. At times, even the smallness of these sets in some sense could be useful. For example, in [6] we showed that two commuting continuous self-maps of an interval share a periodic point if one has a countable set of periodic points. Schwartz (see [7]) was able to show that if one of the two commuting continuous functions is also continuously differentiable, then it would necessarily follow that the functions share a periodic point. Schwartz's result along with the results given in [6] may suggest that continuously differentiable functions have a countable set of periodic points. This is not true in general. However, in this note we show that typically such functions have a finite set of fixed points and a countable set of periodic points. Here $F_1(f) = \{x : f(x) = x\}$ denotes the set of fixed points of $f$.

For $f$ and $n \in \mathbb{N}$ we define $f^n(x)$ by induction:

$$f^n(x) = f\left(f^{n-1}(x)\right) \quad \text{with } f^0(x) = x, \qquad f^1(x) = f(x). \tag{1}$$

The orbit of $x$ under $f$ is given by the sequence $\text{orb}(x, f) = \{f^i(x)\}_{i=0}^{\infty}$. For $n \in \mathbb{N}$, let $F_n(f) = \{x : f^n(x) = x\}$ and $P_f^n$ be the set of periodic points of order $n$; that is,

$$P_f^n = F_n(f) \setminus \cup_{k<n} F_k(f). \tag{2}$$

Two functions on a given interval are said to be of the same monotone type if both are either strictly increasing or strictly decreasing on that interval. Here, for a partition $T$ of the interval $[a, b]$, $\|T\|$ is the length of the largest subinterval of $T$, $B(f, \epsilon)$ is the open ball about $f$ with radius $\epsilon$, $A^\circ$ denotes the set of interior points of $A$, and $\lambda(I)$ is the length of the interval $I$.

## 2. Continuously Differentiable Functions

For $I = [a, b]$, consider $C(I, I)$ and $C^1(I, I)$ to be the family of all continuous maps and continuously differentiable maps from $I$ into itself, respectively. Recall that the usual metrics $\rho_0$ and $\rho_1$ on $C(I,I)$ and $C^1(I,I)$, respectively, are given by

$$\rho_0(f,g)=\sup_{x\in I}|f(x)-g(x)| \quad \text{for } f,g\in C(I,I),$$
$$\rho_1(f,g)=\rho_0(f,g)+\rho_0(f',g') \quad \text{for } f,g\in C^1(I,I). \tag{3}$$

It is well known that the metric spaces $(C(I,I);\rho_0)$ and $(C^1(I,I);\rho_1)$ are complete and hence Baire's category theorem holds in these spaces. We say that a typical function in $C(I,I)$ or $C^1(I,I)$ has a certain property if the set of those functions which does not have this property is of first category in $C(I,I)$ or in $C^1(I,I)$. (Some authors prefer using the term generic instead of typical.) It is known that typically continuous self-maps of an interval have $\sigma$-perfect, measure zero sets of periodic points (see [2]). Here we show that typically members of $(C^1(I,I),\rho_1)$ have a finite set of fixed points and a countable set of periodic points.

**Lemma 1.** *Let $n\ge 1$ and $f\in C^1(I,I)$ so that $F_n(f)$ is not finite. Then there exists $x_0\in I$ so that $f^n(x_0)=x_0$ and $(f^n)'(x_0)=1$.*

*Proof.* Suppose $F_n(f)$ is not finite, and then we can choose a strictly monotone sequence in $F_n(f)$. Without loss of generality, assume $\{x_i\}_{i=1}^{\infty}\subset F_n(f)$ so that $x_i<x_{i+1}$ for each $i\ge 1$. Let $h(x)=f^n(x)-x$, and then, for each $i$, $h(x_i)=h(x_{i+1})=0$. Thus by Rolle's Theorem we have $y_i$, $x_i<y_i<x_{i+1}$ so that $h'(y_i)=0$. Let $x_0=\lim_{i\to\infty}x_i$, then $x_0\in I$, and $\lim_{i\to\infty}y_i=x_0$, and from the continuity of $f^n$ and $(f^n)'$ it follows that $f^n(x_0)=x_0$ and $(f^n)'(x_0)=1$. □

**Lemma 2.** *For each $f\in C^1(I,I)$ and $0<\epsilon<1$ there is a polynomial $p:I\to I^o$ with $\rho_1(f,p)<\epsilon$.*

*Proof.* Let $M=\sup_{x\in I}\{|f'(x)|,|a|,|b|,1\}$ where $I=[a,b]$. Take $\beta=\epsilon/8M$, $\alpha=\beta(a+b)/2$, and $g(x)=\alpha+(1-\beta)f(x)$. It is easy to see that $0<\beta<1$, $g(I)\subset I^o$, and $\rho_1(f,g)<\epsilon/4$. Let

$$\epsilon_1=\frac{1}{2}\inf_{x\in[a,b]}\left\{g(x)-a,b-g(x),\frac{\epsilon}{8(b-a)},\frac{\epsilon}{8}\right\} \tag{4}$$

and $s(t)$ be a polynomial with $\rho_0(g',s)<\epsilon_1/(b-a+1)$. Then $p(x)=g(a)+\int_a^x s(t)dt$ is a polynomial and $g(x)-p(x)=\int_a^x(g'(t)-s(t))dt$. Thus we have

$$\rho_1(g,p)=\rho_0(g',s)+\rho_0(g,p)\le\frac{\epsilon_1}{(b-a)+1}+\frac{\epsilon_1(b-a)}{(b-a)+1}$$
$$=\epsilon_1<\frac{\epsilon}{4}, \tag{5}$$

hence $\rho_1(f,p)\le\rho_1(f,g)+\rho_1(g,p)<\epsilon/2$. From $g(I)\subset[a+2\epsilon_1,b-2\epsilon_1]$ and $\rho_1(g,p)\le\epsilon_1$ it follows that $p(I)\subset I^o$. □

**Lemma 3.** *Let $n>2$ be a positive integer and $f\in C^1(I,I)$. If $x_0$ is a periodic point of order $n$ with $(f^n)'(x_0)=1$, then $orb(x_0,f)\cap\{a,b\}=\emptyset$.*

*Proof.* We show that $a\notin \operatorname{orb}(x_0,f)$. The case $b\notin\operatorname{orb}(x_0,f)$ follows similarly. Let $\operatorname{orb}(x_0,f)=\{a=x_0,x_1,\dots,x_n=x_0\}$ with $x_i=f(x_{i-1})$ for $1\le i\le n$. Since $n>2$, there exist $x_i\in\operatorname{orb}(x_0,f)\cap(a,b)$ and $1\le s\le n-1$ so that $f^s(x_i)=a$. Since $f^s\in C^1(I,I)$ and $(f^s)'(x_0)\neq 0$, there exists $\delta_1>0$ so that $a<x_i-\delta_1<x_i<x_i+\delta_1<b$ and $f^s$ is strictly increasing or strictly decreasing on $J=[x_i-\delta_1,x_i+\delta_1]$. Let $f^s$ be strictly increasing on $J$, and then for $x_i-\delta_1\le x\le x_i$ we have $f^s(x)\le f^s(x_i)=a$; hence, $f^s|_{[x_i-\delta_1,x_i]}=a$, implying that $(f^s)'(x_i)=0$, a contradiction to $(f^n)'(x_0)=1$. On the other hand when $f^s$ is strictly decreasing on $J$, for $x_i\le x\le x_i+\delta_1$ we have $f^s(x)\le f^s(x_i)=a$; hence, $f^s|_{[x_i,x_i+\delta_1]}=a$ implying that $(f^s)'(x_i)=0$, a contradiction to $(f^n)'(x_0)=1$. □

**Lemma 4.** *For $k\ge 1$, the set*

$$\mathbb{E}_k=\left\{f\in C^1(I,I):F_k(f)\cap\left\{x:\left(f^k\right)'(x)=1\right\}\neq\emptyset\right\} \tag{6}$$

*is closed in $C^1(I,I)$.*

*Proof.* Let $g_n\in\mathbb{E}_k$ and $\rho_1(g_n,g)\to 0$. Then there exists $\{t_n\}_{n=1}^{\infty}\subset I$ such that $g_n^k(t_n)=t_n$ and $(g_n^k)'(t_n)=1$. Without loss of generality, we may assume that $\lim_{n\to\infty}t_n=t_0$, then $t_0\in I$. Let $\epsilon>0$ be arbitrary. Due to the uniform continuity of $g^k$ on $I$ there exists a positive integer $n_1$ such that, for $n\ge n_1$, $|g^k(t_n)-g^k(t_0)|<\epsilon$. Since $\rho_0(g_n,g)\to 0$, there exists a positive integer $n_2$ such that, for $n\ge n_2$, $\rho_0(g_n^k,g^k)<\epsilon$. Choose the positive integer $n_3$ so that $|t_n-t_0|<\epsilon$ for $n\ge n_3$, and let $m_1=\max(n_1,n_2,n_3)$. Then for $m\ge m_1$ we have

$$\begin{aligned}&\left|g^k(t_0)-t_0\right|\\&\quad\le\left|g^k(t_0)-g^k(t_m)\right|+\left|g^k(t_m)-g_m^k(t_m)\right|\\&\qquad+\left|g_m^k(t_m)-t_0\right|\\&\quad\le\epsilon+\rho_0\left(g^k,g_m^k\right)+\left|t_m-t_0\right|\le 3\epsilon,\end{aligned} \tag{7}$$

Implying that $g^k(t_0)=t_0$. We also have

$$\left|\left(g_n^k\right)'(t_n)-\left(g^k\right)'(t_n)\right|\le\lim_{n\to\infty}\rho_1\left(g_n^k,g^k\right)=0. \tag{8}$$

Thus we have $\lim_{n\to\infty}|1-(g^k)'(t_n)|=|1-(g^k)'(t_0)|=0$ and $g\in\mathbb{E}_k$. □

**Theorem 5.** *There exists a residual subset $M_1$ of $C^1(I,I)$ such that, for every $f\in M_1$, $F_1(f)$ is finite.*

*Proof.* If $F_1(f)$ has infinitely many elements, then from Lemma 1 it follows that there exists a point $x_0\in F_1(f)$ so that $f'(x_0)=1$. Let

$$\mathbb{E}_1=\left\{f\in C^1(I,I):\exists x_0\in I\cap F_1(f),\ f'(x_0)=1\right\}. \tag{9}$$

From Lemma 4 the set $\mathbb{E}_1$ is closed in $C^1(I,I)$. To show that $\mathbb{E}_1$ has no interior point, let $f\in\mathbb{E}_1$ and $\epsilon>0$, and then, by

Lemma 2, there exists a polynomial $p$ such that $\rho_1(f,p) < \epsilon/2$ and $p(I) \subset [a+\epsilon_1, b-\epsilon_1]$ for some $0 < \epsilon_1 < \epsilon/2$. Let

$$M(p) = \{x : p'(x) = 1\} = \{t_1, t_2, \ldots, t_m\}. \quad (10)$$

Choose $0 < \epsilon_2 < \epsilon_1$ so that $p(x) - x \neq \epsilon_2$ for all $x \in M(p)$ and take $h = p - \epsilon_2$. Then $h \in C^1(I,I)$, $\rho_1(h,f) < \epsilon$, $M(h) = \{x : h'(x) = 1\} = M(p)$, and $h(x) \neq x$ on $M(h)$, implying that $\mathbb{E}_1$ is of first category and $M_1 = C^1(I,I) \setminus \mathbb{E}_1$ is residual. □

**Theorem 6.** *The set of functions $f \in C^1(I,I)$ with $f(a) = b, f(b) = a$, and $(f^2)'(a) = 1$ is a nowhere dense subset of $C^1(I,I)$.*

*Proof.* Let $A = \{f \in C^1(I,I) : f(a) = b,\ f(b) = a,$ and $(f^2)'(a) = 1\}$. It is easy to see that $A$ is a closed subset of $C^1(I,I)$. To show that $A$ is nowhere dense, let $f \in C^1(I,I)$ and $\epsilon > 0$. Choose $0 < \gamma < \min\{\epsilon/(4(b-a)+4), 1\}$ and $a < c < b$ such that $|f(t) - f(a)| < (b-a)[1-\epsilon/4]$ for $t \in [a,c]$. Let $h(x) = f(x) + \int_a^x k(t)dt$, where

$$k(t) = \begin{cases} -\gamma, & t = a, \\ 0, & t = a + \dfrac{c-a}{4}, \\ \dfrac{\gamma}{3}, & t = a + \dfrac{(c-a)}{2}, \\ 0, & t = c,\ t = b, \\ \text{linear}, & \text{otherwise}. \end{cases} \quad (11)$$

It is clear that $h(a) = f(a) = b$, $h(b) = f(b) = a$, $h'(a) = f'(a) - \gamma$, and $h'(b) = f'(b)$. It is easy to see that $h \in C^1(I,I)$, $(h^2)'(a) = (h^2)'(b) \neq 1$, and $\rho_1(f,h) < \epsilon/2$. □

**Theorem 7.** *Let $\epsilon > 0$, $f \in C^1(I,I)$, and $A_n = \{x : (f^n)'(x) = 1\} \cap F_n(f)$ be a finite set, and $orb(x,f) \subset (a,b)$ for $x \in A_n$. Then there exists a function $g \in C^1(I,I)$ with $\rho_1(f,g) < \epsilon/4$ such that $\{x : (g^n)'(x) = 1\} \cap F_n(g) = \emptyset$.*

*Proof.* Let $A_n = \{x : (f^n)'(x) = 1\} \cap F_n(f) = \{z_j\}_{j=1}^l$. Let $B = \{t_j\}_{j=1}^k$ be the elements of $A_n$ with distinct orbits, that is, $\mathrm{orb}(t_i,f) \cap \mathrm{orb}(t_j,f) = \emptyset$ for $i \neq j$ and $\cup_{j=1}^k \mathrm{orb}(t_j,f) \supseteq A_n$. It is clear that each $t_j \in B$ is a periodic point of $f$ with some period $m$ where $m$ is a factor of $n$. This suggests that if one can construct a function $g$ that is sufficiently close to $f$, and either $t_j \notin P_f^m$ or $(f^m)'(t_j) \neq 1$, then

$$\{x : (g^n)'(x) = 1\} \cap F_n(g) \subseteq A_n \setminus \mathrm{orb}(t_j, f). \quad (12)$$

By repeating this process for each $t_j \in B$ in a finite number of steps we can construct the desired function $g$. Thus for convenience, we assume that, for $1 \leq j \leq k$, $z_j \in P_f^n$. Consider the partition $T$ of $[a,b]$ obtained by $\{f^i(t_j), 1 \leq i \leq n, 1 \leq j \leq k\}$, $\epsilon_1 = (1/4)\min\{\epsilon, \| T \|\}$, and choose a positive $\delta$ less than $\epsilon_1$ such that, for each $i$, $1 \leq i \leq n$ and each $j$, $1 \leq j \leq k$, and if $|t - t_j| < \delta$, then $|f^i(t) - f^i(t_j)| < \epsilon_1$. Given that, for $x \in A_n$, $\mathrm{orb}(x,f) \subset (a,b)$, $(f^n)'$ is continuous, and $(f^n)'(x) = 1$ on $B$, we may choose the nondegenerate closed intervals $H_j$, $1 \leq j \leq k$ of length less than $\delta$ and the positive numbers $\gamma_j$ such that

(i) $t_j \in (H_j)^\circ$ is the midpoint of $H_j$ and $f^i(H_j) \cap \mathrm{orb}(t_j,f) = \{f^i(t_j)\}$ for $0 \leq i \leq n$,

(ii) $f$ is strictly monotone on each $H_{j,i} = f^i(H_j)$ for $0 \leq i \leq n-1$, $1 \leq j \leq k$, where $H_{j,0} = H_j$,

(iii) the intervals $H_{j,i} = f^i(H_j)$ are mutually disjoint for $1 \leq i \leq n$, $1 \leq j \leq k$,

(iv) $a < \min(H_{j,i}) - \gamma_j\lambda(H_{j,i}) < \max(H_{j,i}) + \gamma_j\lambda(H_{j,i}) < b$, for $0 \leq i \leq n-1$,

(v) $\max_{1\leq j\leq k}\{\gamma_j, (b-a)\gamma_j\} < (1/8)[\min_{0\leq i\leq n,\, 1\leq j\leq k}\{\epsilon/k, \lambda(f^i(H_j)), \beta_{j,i}\}]$, where $\beta_{j,i} = \inf\{|f'(x)| : x \in f^i(H_j)\}$.

Let $x_0 = t_i$ for some $i$, $\gamma$, and $J = [c,d]$ be the associated $\gamma_i$ and $H_i$, respectively. Define

$$g_{J,x_0,\gamma}(x) = \begin{cases} -\dfrac{\gamma}{3}, & \text{if } x = c + \dfrac{x_0 - c}{2}, \\ 0, & \text{if } x = c + \dfrac{3(x_0 - c)}{4}, \\ \gamma, & \text{if } x = x_0, \\ 0, & \text{if } x = x_0 + \dfrac{d - x_0}{4}, \\ -\dfrac{\gamma}{3}, & \text{if } x = x_0 + \dfrac{d - x_0}{2}, \\ 0, & \text{if } x \in [a,b] \setminus (c,d), \\ \text{linear}, & \text{otherwise}, \end{cases}$$

$$r_{J,x_0,\gamma}(x) = \begin{cases} \gamma, & \text{if } x = c + \dfrac{x_0 - c}{4}, \\ 0, & \text{if } x = x_0, \\ \dfrac{-\gamma(x_0 - c)}{d - x_0} & \text{if } x = x_0 + \dfrac{3(d - x_0)}{4}, \\ 0 & \text{if } x \in [a,b] \setminus (c,d), \\ \text{linear}, & \text{otherwise}, \end{cases} \quad (13)$$

$$h_1(x) = \int_a^x g_{J,x_0,\gamma}(t)\,dt,$$

$$h_2(x) = \int_a^x r_{J,x_0,\gamma}(t)\,dt.$$

We have $h_m(c) = h_m(d) = h'_m(c) = h'_m(d) = 0$ for $m = 1, 2$, $h_1(x_0) = 0$, $h'_1(x_0) = \gamma$, $h_2(x_0) = (x_0 - c)\gamma/2$, and $h'_2(x_0) = 0$. By considering four different cases, we construct a function $h \in C^1(I,I) \cap B(f,\epsilon)$ so that, for Cases A and B, $F_n(h) \cap \{x : (h^n)'(x) = 1\} \cap J = \emptyset$ and $F_n(h) \cap \{x : (h^n)'(x) = 1\} \cap J = \{x_0\}$ for Cases C and D. From conditions (iv) and (v) we have $h(I) \subset I$, and from condition (v) it follows that $h$ is of the same monotone type as $f$ on each $f^s(J)$, $0 \leq s < n$.

*Case A.* Let $f^n(x) < x$ for $J \setminus \{x_0\}$ and $f^n(x_0) = x_0$.

(i) If $f$ is strictly increasing on $J$, then by taking $h(x) = f(x) - h_2(x)$ we have $h(x) < f(x)$ on $J^o$

and $h(x) = f(x)$ on $I \setminus J^o$ and the function $h^{n-1}$ is also strictly increasing on $f(J)$, and as a result $h^n(x) < f^n(x) \leq x$ on $J^o$. Thus $F_n(h) \cap J = \emptyset$.

(ii) If $f$ is strictly decreasing on $J$, then by taking $h(x) = f(x) + h_2(x)$ we have $h(x) > f(x)$ on $J^o$ and $h(x) = f(x)$ on $I \setminus J^o$ and the function $h^{n-1}$ is also strictly decreasing on $f(J)$ and as a result, $h^n(x) < f^n(x) \leq x$ on $J^o$. Thus $F_n(h) \cap J = \emptyset$.

*Case B.* Let $f^n(x) > x$ for $J \setminus \{x_0\}$ and $f^n(x_0) = x_0$.

If $f$ is strictly increasing on $J$, take $h(x) = f(x) + h_2(x)$, and if $f$ is strictly decreasing on $J$, take $h(x) = f(x) - h_2(x)$. Then similar to Case A, we may show that $h^n(x) > f^n(x) \geq x$ for $x \in J^o$; hence, $F_n(h) \cap J = \emptyset$.

*Case C.* Let $f^n(x) < x$ for $c < x < x_0$, $f^n(x) > x$ for $x_0 < x < d$, and $f^n(x_0) = x_0$.

(i) If $f$ is strictly increasing on $J$, then by taking $h(x) = f(x) + h_1(x)$ we have $h(x) < f(x)$ for $c < x < x_0$ and $h(x) > f(x)$ for $x_0 < x < d$, and the function $h^{n-1}$ is also strictly increasing on $f(J)$. Thus $h^n(x) < f^n(x) < x$ for $c < x < x_0$ and $h^n(x) > f^n(x) > x$ for $x_0 < x < d$; hence, $F_n(h) \cap J = \{x_0\}$.

(ii) If $f$ is strictly decreasing on $J$, then by taking $h(x) = f(x) - h_1(x)$ we have $h(x) > f(x)$ for $c < x < x_0$ and $h(x) < f(x)$ for $x_0 < x < d$, and the function $h^{n-1}$ is also decreasing on $f(J)$. Thus $h^n(x) < f^n(x) < x$ for $c < x < x_0$ and $h^n(x) > f^n(x) > x$ for $x_0 < x < d$; hence, $F_n(h) \cap J = \{x_0\}$.

*Case D.* Let $f^n(x) > x$ for $c < x < x_0$, $f^n(x) < x$ for $x_0 < x < d$, and $f^n(x_0) = x_0$.

If $f$ is strictly increasing on $J$, take $h(x) = f(x) - h_1(x)$, and if $f$ is strictly decreasing on $J$, take $h(x) = f(x) + h_1(x)$. Then similar to Case C, we may show that $h^n(x) > f^n(x) > x$ for $c < x < x_0$ and $h^n(x) < f^n(x) < x$ for $x_0 < x < d$. Thus $F_n(h) \cap J = \{x_0\}$.

Note that $h^n$ is strictly increasing on $J \cup f^n(J)$, so we have

$$F_n(h) \cap [(J \setminus f^n(J)) \cup (f^n(J) \setminus J)] = \emptyset. \quad (14)$$

Thus for $x \in I$ either $\operatorname{orb}(x, h) = \operatorname{orb}(x, f)$ or $\operatorname{orb}(x, h) \cap J \neq \emptyset$, of which in such case $F_n(h) \cap J = \{x_0\}$, $h'(x_0) = f'(x_0) \pm \gamma$, and $h'(h^i(x_0)) = f'(f^i(x_0))$ for $1 \leq i \leq n-1$. Thus

$$\begin{aligned}(h^n)'(x_0) &= h'\left(h^{n-1}(x_0)\right) h'\left(h^{n-2}(x_0)\right) \cdots h'(h(x_0)) h'(x_0) \\ &= f'\left(f^{n-1}(x_0)\right) f'\left(f^{n-2}(x_0)\right) \\ &\quad \times f'\left(f^{n-3}(x_0)\right) \cdots f'(f(x_0)) h'(x_0) \\ &= \frac{(f^n)'(x_0) h'(x_0)}{f'(x_0)} = \frac{f'(x_0) \pm \gamma}{f'(x_0)} \\ &= \left[1 \pm \frac{\gamma}{f'(x_0)}\right] \neq 1.\end{aligned} \quad (15)$$

We have

$$\begin{aligned}F_n(h) &\cap \left\{x : (h^n)'(x) = 1\right\} \\ &= \left[F_n(f) \cap \left\{x : (f^n)'(x) = 1\right\}\right] \setminus \{x_0\}.\end{aligned} \quad (16)$$

Also

$$\begin{aligned}\rho_0(f', h') &\leq \sup_{x \in J} \{|g(x)|, |r(x)|\} \leq \frac{\epsilon}{8k}, \\ \rho_0(f, h) &\leq \sup_{x \in J} \{|h_1(x)|, |h_2(x)|\} \leq \frac{\epsilon}{8k}.\end{aligned} \quad (17)$$

Hence $\rho_1(f, h) \leq \rho_0(f, h) + \rho_0(f', h') \leq \epsilon/4k$.

Doing this recursively for each $t_j$, $1 \leq j \leq k$, we get a function $g \in B(f, \epsilon/4)$ such that $F_n(g) \cap \{x : (g^n)'(x) = 1\} = \emptyset$. □

**Theorem 8.** *Typically continuously differentiable self-maps of intervals have a countable set of periodic points.*

*Proof.* Take

$$\begin{aligned}H_n &= \left\{f \in C^1(I, I) : F_n(f) \text{ is not finite}\right\}, \\ B_1 &= \left\{f \in C^1(I, I) : a \in P_f^2 \cap \left\{x : (f^2)'(x) = 1\right\}\right. \\ &\quad \left.\text{with } f(a) = b\right\}, \\ F_1 &= \left\{f \in C^1(I, I) : \exists x_0 \in I \text{ such that}\right. \\ &\quad \left. x_0 \in F_1(f) \text{ and } f'(x_0) = 1\right\}, \\ \mathbb{E}_n &= \left\{f \in C^1(I, I) : F_n(f) \cap \left\{x : (f^n)'(x) = 1\right\} \neq \emptyset\right\}.\end{aligned} \quad (18)$$

The sets $F_1$ and $B_1$ are first category sets (see Theorems 5 and 6). For $n \geq 1$ from Lemmas 4 and 1 we have that $\mathbb{E}_n$ is closed and $H_n \subseteq \mathbb{E}_n$. Without loss of generality we may assume that $f \in C^1(I, I)$ is neither a constant function nor a polynomial of first degree, and then by Lemma 2 we may choose a polynomial $g \in C^1(I, I)$ of degree $n \geq 2$ such that $g(I) \subset I^o$ and $\rho_1(f, g) < \epsilon/4$. Using Theorem 7 we can construct $h \in C^1(I, I)$ so that $\rho_1(g, h) < \epsilon/4$ and the set $A_n = F_n(h) \cap \{x : (h^n)'(x) = 1\} = \emptyset$, thus $\rho_1(f, h) < \epsilon$ and $h \notin \mathbb{E}_n$, hence $\mathbb{E}_n^o = \emptyset$. This implies that for each $k \geq 2$ the set $\mathbb{E}_k$ is nowhere dense. Thus $F = (\cup_{k=2}^{\infty} \mathbb{E}_k) \cup F_1 \cup B_1$ is a first category set, so $M = C^1(I, I) \setminus F$ is a residual set, and, for $f \in M$, $P(f) = \cup_{n=1}^{\infty} F_n(f)$ is countable. □

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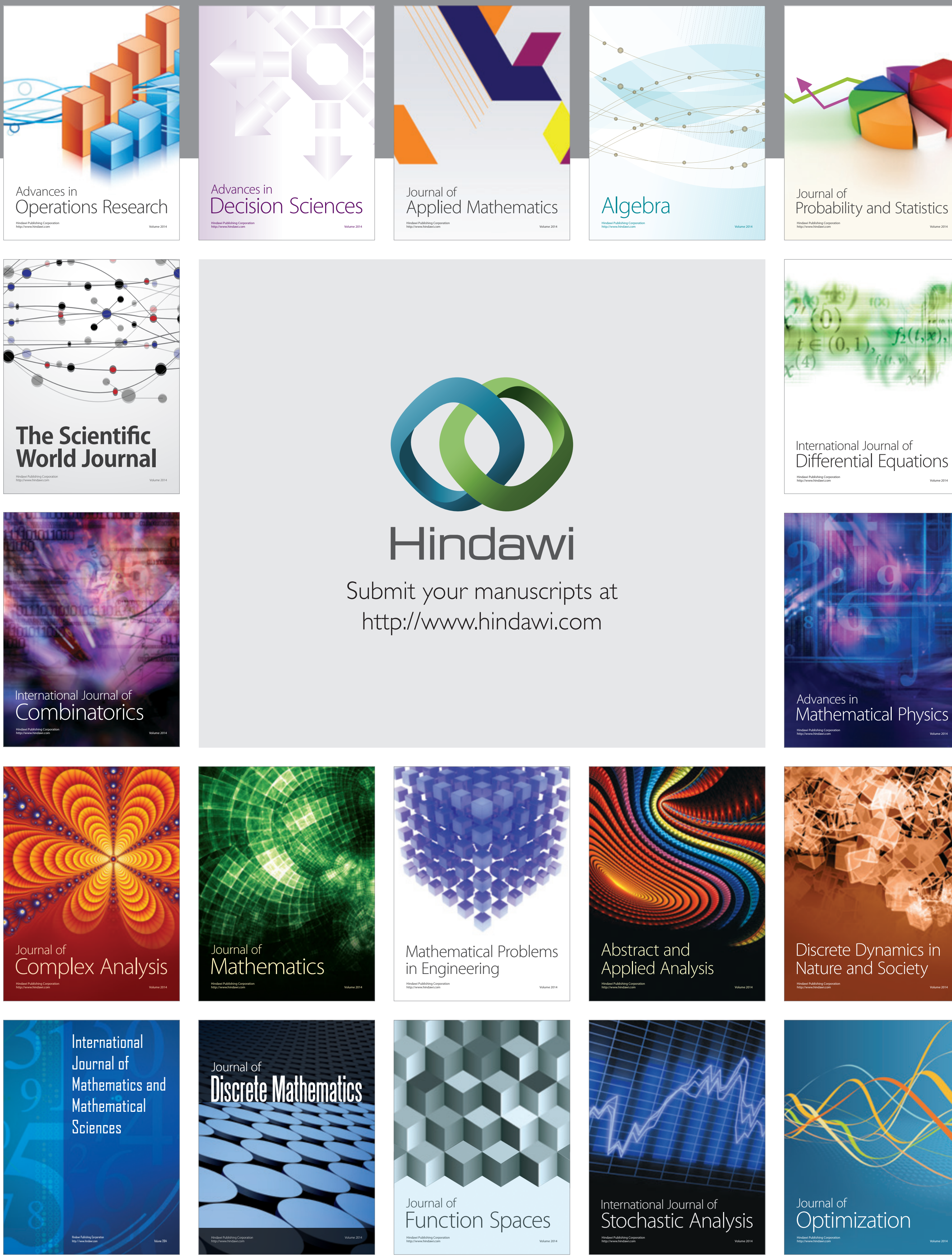
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