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*Research Article*

# Sufficient and Necessary Conditions of Complete Convergence for Weighted Sums of $\tilde{\rho}$-Mixing Random Variables

**Chongfeng Lan**[1,2]

[1] *School of Economics and Management, Fuyang Normal College, Fuyang 236037, China*
[2] *School of Economics and Management, Beijing University of Posts and Telecommunications, Beijing 100876, China*

Correspondence should be addressed to Chongfeng Lan; lchfym@sina.com

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The equivalent conditions of complete convergence are established for weighted sums of $\tilde{\rho}$-mixing random variables with different distributions. Our results extend and improve the Baum and Katz complete convergence theorem. As an application, the Marcinkiewicz-Zygmund type strong law of large numbers for sequence of $\tilde{\rho}$-mixing random variables is obtained.

## 1. Introduction

Let $(\Omega, F, P)$ be a probability space. The random variables we deal with are all defined on $(\Omega, F, P)$. Let $\{X_n, n \geq 1\}$ be a sequence of random variables. For each nonempty set $S \subset N$, write $F_s = \sigma(X_i, i \in S)$. Given $\sigma$-algebras $B$, $R$ in $F$, let

$$\rho(B, R) = \sup\{|\operatorname{corr}(X, Y)|; X \in L_2(B), Y \in L_2(R)\}, \quad (1)$$

where $\operatorname{corr}(X, Y) = (EXY - EXEY)/(\operatorname{Var} X \operatorname{Var} Y)^{1/2}$. Define the $\tilde{\rho}$-mixing coefficients by

$$\tilde{\rho}(n) = \sup \rho(F_S, F_T), \quad (2)$$

where (for a given positive integer $N$) this sup is taken over all pairs of nonempty finite subsets $S$, $T$ of $N$ such that dist $(S, T) \geq n$.

Obviously $0 \leq \tilde{\rho}(n+1) \leq \tilde{\rho}(n) \leq 1$, $n \geq 0$, and $\tilde{\rho}(0) = 1$ except in the trivial case where all of the random variables $X_i$ are degenerate.

*Definition 1.* A sequence of random variables is said to be a $\tilde{\rho}$-mixing sequence of random variables if there exists $k \in N$ such that $\tilde{\rho}(k) < 1$.

Note that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables, then $\tilde{\rho}(n) = 0$ for all $n \geq 1$. $\tilde{\rho}$-mixing is similar to $\rho$-mixing, but both are quite different. $\rho(k)$ is defined by (2) with index sets restricted to subsets $S$ of $[1, n]$ and subsets of $T$ of $[n + k, \infty)$, $n, k \in N$. On the other hand, $\rho$-mixing sequence assumes the condition $\rho(k) \to 0$, but $\tilde{\rho}$-mixing sequence assumes the condition that there exists $k \in N$ such that $\rho(k) < 1$; from this point of view, $\tilde{\rho}$-mixing is weaker than $\rho$-mixing.

The concept of $\tilde{\rho}$-mixing random variables was introduced by Bradley [1] and a number of limit theories for $\tilde{\rho}$-mixing sequences of random variables have been established by many authors. We refer to Bradley [1] for the central limit theorem, Bryc and Smoleński [2], Peligrad and Gut [3], and Utev and Peligrad [4] for moment inequalities, Gan [5], Kuczmaszewska [6], and Wu and Jiang [7] for almost sure convergence, and Cai [8], Zhu [9], An and Yuan [10], Zhou et al. [11], Shen and Hu [12], Guo and Zhu [13], Wang et al. [14], and Sung [15, 16] for complete convergence.

A sequence $\{X_n, n \geq 1\}$ of random variables *converges completely* to the constant $C$ if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty \quad \text{for any } \epsilon > 0. \quad (3)$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to C$ almost surely. Hence, complete convergence is one of the most important problems in probability theory. Since the concept of complete convergence was introduced by Hsu and Robbins [17], there have been many authors who devoted the study to complete convergence for independent and identically distributed random variables. One of the most important results is Baum and Katz theorem [18]. The theorem was further generalized and extended in different ways. Katz [19] and Chow [20] formed the following generalization with a normalization of Marcinkiewicz-Zygmund type theorem for the strong law of large numbers.

**Theorem 2** (see [21]). *Let $\alpha p \geq 1$, $\alpha > 1/2$, and let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. If $p \geq 1$, assume that $EX_1 = 0$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i| > \epsilon n^{\alpha}) < \infty$ *for all* $\epsilon > 0$.

In many stochastic models, the assumption of independence among random variables is not plausible. So it is necessary to extend the concept of independence to dependence cases. Peligrad and Gut [3] extended this result from independent and identically distributed case to the case of $\tilde{\rho}$-mixing random variables with identical distribution. But they did not prove whether the result of Theorem 2 of the case $\alpha p = 1$ holds for $\tilde{\rho}$-mixing sequence. In practical applications it is difficult to check the independence of a sample or the samples are not independent observations. Therefore, in recent investigations limit theorems are very often considered for sequences of dependent random variables. Recently, a number of limit theorems for dependent random variables have been established by many authors. We can refer to Sung [22], Wu and Jiang [23], Wu [24], and Shen [25].

Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of constants. The strong convergence results for weighted sums $\sum_{k=1}^{n} a_{nk} X_k$ have been studied by many authors; see, for example, Cuzick [26], Choi and Sung [27], Bai and Cheng [28], Chen and Gan [29], and so forth. Many useful linear statistics are weighted sums. Examples include least squares estimators, nonparametric regression function estimators, and jackknife estimates.

Inspired by Theorem 2.1 of Kuczmaszewska [30], our main purpose in this work is to extend the complete convergence for weighted sums $\sum_{k=1}^{n} a_{nk} X_k$ of independent and identically distributed random variables to the case of $\tilde{\rho}$-mixing random variables. However, our proven methods are different from the ones by Kuczmaszewska [30]; by applying inequality (13) of Lemma 10 our proof is much simpler than the one by Kuczmaszewska. Our proof of necessary condition (using Lemma 10) is original. We provide sufficient and necessary conditions of complete convergence for weighted sums of $\tilde{\rho}$-mixing random variables with different distributions. As applications, the Baum and Katz type result and the Marcinkiewicz-Zygmund type strong law of large numbers for sequences of $\tilde{\rho}$-mixing random variables are obtained. In addition, our main results extend and improve the corresponding results of Peligrad and Gut [3].

Throughout this paper, the symbol $C$ denotes a positive constant which is not necessarily the same one in each appearance, $a_n = O(b_n)$ will mean $a_n \leq C(b_n)$ for sufficiently large $n$, $a_n \ll b_n$ will mean $a_n = O(b_n)$, and $I(A)$ is the indicator function of event $A$.

## 2. Main Results

Now we state our main results of this paper. The proofs will be given in Section 3.

**Theorem 3.** *Let $X$ be a random variable and let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables satisfying the condition*

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_k| > x) = CP(|X| > x) \tag{4}$$

*for all $x > 0$, all $n \geq 1$, and some positive constant $C$. Let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a sequence of real numbers such that*

$$|a_{nk}| \asymp n^{-\alpha}, \quad \forall 1 \leq k \leq n, \; n \geq 1, \tag{5}$$

*where $a \asymp b$ means $a = O(b)$ and $b = O(a)$. Let $\alpha p \geq 1$, $\alpha > 1/2$, and if $\alpha \leq 1$, assume that $EX_n = 0$, $n \geq 1$. Then the following statements are equivalent:*

(i) $E|X|^p < \infty$,

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_{ni} X_i| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

*Remark 4.* When proving the limit theorem of $\tilde{\rho}$-mixing random variables with different distributions, many authors apply the condition of $\{X_n, n \geq 1\}$ being stochastically dominated by $X$, that is, for some constant $C > 0$, $P(|X_n| \geq x) \leq CP(|X| \geq x)$, for all $x \geq 0$, $n \geq 1$, which implies that $(1/n)\sum_{k=1}^{n} P(|X_k| > x) \leq CP(|X| > x)$, but the converse is not true. Hence our condition of (4) is weaker than the condition of stochastic dominance.

When $\{X_n, n \geq 1\}$ is a sequence of $\tilde{\rho}$-mixing identically distributed random variables and $a_{ni} = n^{-\alpha}$, for all $1 \leq i \leq n$, $n \geq 1$, then Theorem 3 becomes Baum and Katz complete convergence theorem as follows.

**Corollary 5.** *Let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing identically distributed random variables. Let $\alpha p \geq 1$, $\alpha > 1/2$, and if $\alpha \leq 1$, assume that $EX_n = 0$, $n \geq 1$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i| > \epsilon n^{\alpha}) < \infty$ *for all* $\epsilon > 0$.

*Remark 6.* Corollary 5 not only generalizes Theorem 2 to $\tilde{\rho}$-mixing case, but also extends Theorem 2 of Peligrad and Gut [3] to the case $\alpha p = 1$. Therefore, Corollary 5 improves and extends the well-known Baum and Katz theorem.

An and Yuan [10, Theorem 2] presented a Marcinkiewicz-Zygmund type strong law of large numbers for $\tilde{\rho}$-mixing sequence. We find that the proof of their Theorem 2 is wrong because the theorem is based on Theorem 1 [10]. However, the author thinks that their proofs of Theorem 1 have a little problem, since condition (1.2) does not hold for the array with $\{a_{ni}, 1 \le i \le n\}$. An and Yuan [10, Theorem 1] proved the implication (i) $\Rightarrow$ (ii) under condition (1.3) and proved the converse under conditions (1.2) and (1.3). However, the array satisfying both (1.2) and (1.3) does not exist. Noting that $\sharp A_{nk}/(k+1) \le \sum_{i=1}^{n} |a_{ni}|^p \le O(n^\delta)$, we have that $ne^{-1/k} \le \sharp A_{nk} \le (k+1)O(n^\delta)$. But this does not hold when $k$ is fixed and $n$ is large enough. In this paper, we obtain a new complete convergence result for weighted sums of $\tilde{\rho}$-mixing random variables without assumption of identical distribution. Our result generalizes and sharpens the result of An and Yuan [10]. The following corollary provides the Marcinkiewicz-Zygmund type strong law of large numbers of $\tilde{\rho}$-mixing random variables without assumption of identical distribution.

**Corollary 7.** *Let $X$ be a random variable and let $\{X_n, n \ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables satisfying the condition*

$$\frac{1}{n}\sum_{k=1}^{n} P\left(|X_k| > x\right) = CP\left(|X| > x\right) \tag{6}$$

*for all $x > 0$, all $n \ge 1$, and some positive constant $C$. $E|X|^p < \infty$ for some $0 < p < 2$ and if $1 \le p < 2$, assume that $EX_n = 0$, $n \ge 1$. Then, for any $\epsilon > 0$,*

$$\begin{gathered}\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right| > \epsilon n^{1/p}\right) < \infty, \\ \frac{1}{n^{1/p}}\left|\sum_{i=1}^{n} X_i\right| \longrightarrow 0 \quad a.s.,\ n \longrightarrow \infty.\end{gathered} \tag{7}$$

## 3. Proof of Main Results

The following lemmas are useful for the proof of the main results.

**Lemma 8** (see [4]). *Suppose $K$ is a positive integer, $0 \le r < 1$, and $q \ge 2$. Then there exists a constant $D = D(K, r, q)$ such that the following statement holds.*

*If $\{X_i, i \ge 1\}$ is a sequence of random variables such that $\tilde{\rho}(K) \le r$ and $EX_n = 0$ and $E|X_i|^q < \infty$ for all $i \ge 1$, then, for every $n \ge 1$,*

$$E\left(\max_{1\le i\le n}\left|\sum_{j=1}^{i} X_j\right|^q\right) \le D\left[\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right]. \tag{8}$$

**Lemma 9** (see [30]). *Let $\{X_n, n \ge 1\}$ be a sequence of random variables which is weakly mean dominated by a random variable $X$; that is, for all $x \ge 0$ and some positive constant $C > 0$,*

$$\frac{1}{n}\sum_{k=1}^{n} P\left(|X_k| > x\right) \le CP\left(|X| > x\right). \tag{9}$$

*Then for any $u > 0$, $t > 0$, and $n \ge 1$, the following three statements hold:*

$$\text{If } E|X|^u < \infty, \quad \text{then } \frac{1}{n}\sum_{k=1}^{n} E|X_k|^u \le CE|X|^u, \tag{10}$$

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n} E|X_k|^u I\left(|X_k| \le t\right) \\ &\quad \le C\left[E|X|^u I\left(|X| \le t\right) + t^u P\left(|X| > t\right)\right],\end{aligned} \tag{11}$$

$$\frac{1}{n}\sum_{k=1}^{n} E|X_k|^u I\left(|X_k| > t\right) \le CE|X|^u I\left(|X| > t\right). \tag{12}$$

**Lemma 10.** *Let $\{X_n, n \ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables. Then there exists a positive constant $C$ such that, for any $x \ge 0$ and all $n \ge 1$,*

$$\begin{aligned}&\left(\frac{1}{2} - P\left(\max_{1\le k\le n}|X_k| > x\right)\right)\sum_{k=1}^{n} P\left(|X_k| > x\right) \\ &\quad \le \left(\frac{C}{2} + 1\right) P\left(\max_{1\le k\le n}|X_k| > x\right).\end{aligned} \tag{13}$$

*Proof of Lemma 10.* Since $\{\max_{1\le k\le n}|X_k| > x\} = \bigcup_{k=1}^{n}\{|X_k| > x, \max_{1\le j\le k-1}|X_j| \le x\}$, we have

$$\begin{aligned}&\sum_{k=1}^{n} P\left(|X_k| > x\right) \\ &\quad = \sum_{k=1}^{n} P\left(|X_k| > x, \max_{1\le j\le k-1}|X_j| \le x\right) \\ &\qquad + \sum_{k=1}^{n} P\left(|X_k| > x, \max_{1\le j\le k-1}|X_j| > x\right) \\ &\quad = P\left(\max_{1\le k\le n}|X_k| > x\right) \\ &\qquad + \sum_{k=1}^{n} P\left(|X_j| > x, \max_{1\le j\le k-1}|X_j| > x\right).\end{aligned} \tag{14}$$

Note that

$$\begin{aligned}&\sum_{k=1}^{n} P\left(|X_k| > x, \max_{1\le j\le k-1}|X_j| > x\right) \\ &\quad = \sum_{k=1}^{n} E\left(I\left(|X_k| > x\right) I\left(\max_{1\le j\le k-1}|X_j| > x\right)\right) \\ &\quad \le E\left(\sum_{k=1}^{n} I\left(|X_k| > x\right) - E\left(I\left(|X_k| > x\right)\right)\right) \\ &\qquad \times I\left(\max_{1\le j\le n}|X_j| > x\right) \\ &\qquad + \sum_{k=1}^{n} P\left(|X_k| > x\right) P\left(\max_{1\le j\le n}|X_j| > x\right) \triangleq J_1 + J_2.\end{aligned} \tag{15}$$

Obviously, by Lemma 8, we get

$$E\left(\left|\sum_{k=1}^{n} X_k\right|^q\right) \leq C\left[\sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2\right)^{q/2}\right]. \quad (16)$$

Combining with the Cauchy-Schwarz inequality and (16), we obtain

$$\begin{aligned} J_1 &= E\left(\sum_{k=1}^{n} I\left(|X_k| > x\right) - E\left(I\left(|X_k| > x\right)\right)\right) \\ &\quad \times I\left(\max_{1\leq j\leq n}|X_j| > x\right) \\ &\leq \left[E\left(\sum_{k=1}^{n} I\left(|X_k| > x\right) - E\left(I\left(|X_k| > x\right)\right)\right)^2\right. \\ &\quad \left.\times P\left(\max_{1\leq j\leq n}|X_j| > x\right)\right]^{1/2} \\ &\leq \left[C\sum_{k=1}^{n} P\left(|X_k| > x\right) P\left(\max_{1\leq j\leq n}|X_j| > x\right)\right]^{1/2} \\ &\leq \frac{1}{2}\sum_{k=1}^{n} P\left(|X_k| > x\right) + \frac{C}{2} P\left(\max_{1\leq k\leq n}|X_k| > x\right). \end{aligned} \quad (17)$$

Now, we substitute (17) into (15) and then into (14), which implies that (13) holds. □

Consequently, we prove our main results.

*Proof of Theorem 3.* First, we prove that (i) ⇒ (ii).

Note that $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, where $a_{ni}^{+} = \max\{0, a_{ni}\}$ and $a_{ni}^{-} = \max\{0, -a_{ni}\}$. To prove (ii) it suffices to show that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}^{\pm} X_i\right| > \epsilon\right) < \infty, \quad \forall \epsilon > 0. \quad (18)$$

Thus, without loss of generality, we may assume that $a_{ni} > 0$ for all $1 \leq i \leq n, n \geq 1$.

For fixed $n \geq 1$, denote that

$$X_{ni} = X_i I\left(|X_i| \leq n^{\alpha}\right), \quad 1 \leq i \leq n. \quad (19)$$

Firstly, we show that

$$\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} E\left(a_{ni} X_{ni}\right)\right| \longrightarrow 0, \quad n \longrightarrow \infty. \quad (20)$$

If $1/2 < \alpha \leq 1$, by $EX_n = 0$, (i), (5), (12) of Lemma 9, and $\alpha p \geq 1$, we have

$$\begin{aligned} &\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} E\left(a_{ni} X_{ni}\right)\right| \\ &\quad \leq \sum_{i=1}^{n} E\left|a_{ni} X_{ni}\right| \ll n^{-\alpha}\sum_{i=1}^{n} E\left|X_{ni}\right| \\ &\quad = n^{-\alpha}\sum_{i=1}^{n} E\left|X_i\right| I\left(|X_i| > n^{\alpha}\right) \\ &\quad \leq n^{1-\alpha} E|X|\left(\frac{|X|}{n^{\alpha}}\right)^{p-1} I\left(|X| > n^{\alpha}\right) \\ &\quad \ll n^{1-\alpha p} E|X|^p I\left(|X| > n^{\alpha}\right) \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \quad (21)$$

If $\alpha > 1$, $p \geq 1$, by (5), (11) of Lemma 9, Markov inequality, and $E|X| < \infty$ from (i), we get

$$\begin{aligned} &\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} E\left(a_{ni} X_{ni}\right)\right| \\ &\quad \ll n^{-\alpha}\sum_{i=1}^{n} E\left|X_i\right| I\left(|X_i| \leq n^{\alpha}\right) \\ &\quad \ll n^{1-\alpha}\left[E|X| I\left(|X| \leq n^{\alpha}\right) + n^{\alpha} P\left(|X| > n^{\alpha}\right)\right] \\ &\quad \ll n^{1-\alpha} E|X| \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \quad (22)$$

If $\alpha > 1$, $0 < p < 1$, by $\alpha p \geq 1$, we can get $\lim_{n\to\infty} nP(|X| > n^{\alpha}) = 0$, and thus

$$\begin{aligned} &\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} E\left(a_{ni} X_{ni}\right)\right| \\ &\quad \ll n^{-\alpha}\sum_{i=1}^{n} E\left|X_i\right| I\left(|X_i| \leq n^{\alpha}\right) \\ &\quad \ll n^{1-\alpha}\left[E|X| I\left(|X| \leq n^{\alpha}\right) + n^{\alpha} P\left(|X| > n^{\alpha}\right)\right] \\ &\quad \ll n^{1-\alpha}\sum_{i=1}^{n} E|X| I\left((i-1)^{\alpha} < |X| \leq i^{\alpha}\right) \\ &\qquad + nP\left(|X| > n^{\alpha}\right). \end{aligned} \quad (23)$$

Note that, if $\alpha p \geq 1$, we have

$$\begin{aligned} &\sum_{i=1}^{\infty} i^{1-\alpha} E|X| I\left((i-1)^{\alpha} < |X| \leq i^{\alpha}\right) \\ &\quad \leq \sum_{i=1}^{\infty} i^{1-\alpha p} E|X|^p I\left((i-1)^{\alpha} < |X| \leq i^{\alpha}\right) \\ &\quad \leq \sum_{i=1}^{\infty} E|X|^p I\left((i-1)^{\alpha} < |X| \leq i^{\alpha}\right) \\ &\quad = E|X|^p < \infty. \end{aligned} \quad (24)$$

Hence, by Kronecker lemma and (23), we obtain

$$\max_{1\le k\le n}\left|\sum_{i=1}^{k}E\left(a_{ni}X_{ni}\right)\right| \ll n^{1-\alpha}\sum_{i=1}^{n}E|X|I\left((i-1)^{\alpha}<|X|\le i^{\alpha}\right)\longrightarrow 0,\quad n\longrightarrow\infty. \tag{25}$$

From (21), (22), and (25) we can get (20) immediately. Hence, for all $n$ sufficiently large and any $\epsilon>0$, we have

$$\max_{1\le k\le n}\left|\sum_{i=1}^{k}E\left(a_{ni}X_{ni}\right)\right|<\frac{\epsilon}{2}. \tag{26}$$

It is easy to check that for all $n$ sufficiently large

$$\begin{aligned}&\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|>\epsilon\right\}\\&\subset\bigcup_{i=1}^{n}\left\{\left|a_{ni}X_i\right|>\epsilon\right\}\cup\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|>\epsilon\right\}\\&\subset\bigcup_{i=1}^{n}\left\{\left|a_{ni}X_i\right|>\epsilon\right\}\cup\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left(a_{ni}X_{ni}-Ea_{ni}X_{ni}\right)\right|>\frac{\epsilon}{2}\right\}\\&\triangleq A_n\cup B_n,\end{aligned} \tag{27}$$

which implies that for all $n$ sufficiently large

$$P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|>\epsilon\right)\le P\left(A_n\right)+P\left(B_n\right). \tag{28}$$

Therefore, in order to prove (ii), we only need to prove that

$$\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(A_n\right)<\infty, \tag{29}$$

$$\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(B_n\right)<\infty. \tag{30}$$

By (4), (5), and $\alpha p\ge 1$, we can get that

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(A_n\right)\\&\le\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}P\left(\left|a_{ni}X_i\right|>\epsilon\right)\\&\le\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{i=1}^{n}P\left(\left|X_i\right|>\epsilon a_{ni}^{-1}\ge C\epsilon n^{\alpha}\right)\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-1}P\left(|X|>C\epsilon n^{\alpha}\right)\\&=\sum_{n=1}^{\infty}n^{\alpha p-1}\sum_{i=n}^{\infty}P\left(C\epsilon i^{\alpha}<|X|\le C\epsilon(i+1)^{\alpha}\right)\\&=\sum_{i=1}^{\infty}\sum_{n=1}^{i}n^{\alpha p-1}P\left(C\epsilon i^{\alpha}<|X|\le C\epsilon(i+1)^{\alpha}\right)\\&\le\sum_{i=1}^{\infty}i^{\alpha p}P\left(C\epsilon i^{\alpha}<|X|\le C\epsilon(i+1)^{\alpha}\right)\\&\ll E|X|^{p}<\infty.\end{aligned} \tag{31}$$

That is, (29) holds. Thus, it remains to prove (30).

Since $\{X_n, n\ge 1\}$ is a sequence of $\tilde{\rho}$-mixing random variables and $X_{ni}=X_iI(|X_i|\le n^{\alpha})$, $1\le i\le n$, thus $\{X_{ni}-EX_{ni},\ 1\le i\le n\}$ is still a sequence of $\tilde{\rho}$-mixing random variables with $E(X_{ni}-EX_{ni})=0$. By Markov inequality and Lemma 8, let $q\ge 2$. Then,

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(B_n\right)\\&=\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left(a_{ni}X_{ni}-Ea_{ni}X_{ni}\right)\right|>\frac{\epsilon}{2}\right)\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left(a_{ni}X_{ni}-Ea_{ni}X_{ni}\right)\right|^{q}\right)\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left(X_{ni}-EX_{ni}\right)\right|^{q}\right)\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}\left[\left(\sum_{i=1}^{n}EX_{ni}^{2}\right)^{q/2}+\sum_{i=1}^{n}E\left|X_{ni}\right|^{q}\right]\\&\triangleq I_1+I_2.\end{aligned} \tag{32}$$

When $p\ge 2$, taking $q>\max\{(\alpha p-1)/(\alpha-1/2),2\}$, then $\alpha p-2-\alpha q+q/2<-1$, and by Jensen inequality and (11) of Lemma 9, we have

$$\begin{aligned}I_1&=\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}\left(\sum_{i=1}^{n}EX_{ni}^{2}\right)^{q/2}\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q+q/2}\left[EX^{2}I\left(|X|\le n^{\alpha}\right)+n^{2\alpha}P\left(|X|>n^{\alpha}\right)\right]^{q/2}\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q+q/2}\left[EX^{2}\right]^{q/2}\\&\ll\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q+q/2}<\infty.\end{aligned} \tag{33}$$

Taking $q > p$, we have by (11) of Lemma 9 that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} \sum_{i=1}^{n} E\left|X_{ni}\right|^q \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+1} \left[E|X|^q I\left(|X| \le n^{\alpha}\right) + n^{\alpha q} P\left(|X| > n^{\alpha}\right)\right] \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} E|X|^q I\left(|X| \le n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{\alpha p-1} P\left(|X| > n^{\alpha}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} \sum_{i=1}^{n} E|X|^q I\left((i-1)^{\alpha} < |X| \le i^{\alpha}\right) + E|X|^p \\
&\ll \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} n^{\alpha p-1-\alpha q} E|X|^q I\left((i-1)^{\alpha} < |X| \le i^{\alpha}\right) \\
&\ll \sum_{i=1}^{\infty} i^{\alpha p-\alpha q} E|X|^q I\left((i-1)^{\alpha} < |X| \le i^{\alpha}\right) \\
&= \sum_{i=1}^{\infty} i^{\alpha p-\alpha q} E|X|^p |X|^{q-p} I\left((i-1)^{\alpha} < |X| \le i^{\alpha}\right) \\
&\le \sum_{i=1}^{\infty} E|X|^p I\left((i-1)^{\alpha} < |X| \le i^{\alpha}\right) \\
&= E|X|^p < \infty.
\end{aligned} \tag{34}$$

When $p < 2$, then, taking $q = 2$, by (32), we get

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(B_n\right) \ll \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} E\left|X_{ni}\right|^2. \tag{35}$$

Similarly to the proof of inequality (34), we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} E\left|X_{ni}\right|^2 < \infty, \tag{36}$$

which implies that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(B_n\right) < \infty. \tag{37}$$

Now, we prove the converse. To prove that (ii) implies (i), it suffices to show that

$$\sum_{n=1}^{\infty} n^{\alpha p-1} P\left(|X| > \epsilon n^{\alpha}\right) < \infty, \quad \forall \epsilon > 0. \tag{38}$$

Noting that

$$\max_{1\le i\le n} \left|a_{ni} X_i\right| \le \max_{1\le i\le n} \left|\sum_{j=1}^{i} a_{nj} X_j\right| + \max_{1\le i\le n} \left|\sum_{j=1}^{i-1} a_{nj} X_j\right|, \tag{39}$$

then from (ii) and (5), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} \left|X_i\right| > \epsilon n^{\alpha}\right) \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} \left|a_{ni} X_i\right| > \epsilon\right) \\
&\quad \ll \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le i\le n} \left|\sum_{j=1}^{i} a_{nj} X_j\right| > \epsilon\right) < \infty.
\end{aligned} \tag{40}$$

Combining with the condition of $\alpha p \ge 1$,

$$P\left(\max_{1\le i\le n} \left|X_i\right| > \epsilon n^{\alpha}\right) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{41}$$

Thus, for sufficiently large $n$,

$$P\left(\max_{1\le i\le n} \left|X_i\right| > \epsilon n^{\alpha}\right) < \frac{1}{2}. \tag{42}$$

Therefore, by applying Lemma 10, it is easy to see that

$$\sum_{i=1}^{n} P\left(\left|X_i\right| > \epsilon n^{\alpha}\right) \ll P\left(\max_{1\le i\le n} \left|X_i\right| > \epsilon n^{\alpha}\right), \tag{43}$$

which, together with the conditions of (4) and (40), gives

$$\sum_{n=1}^{\infty} n^{\alpha p-1} P\left(|X| > \epsilon n^{\alpha}\right) \ll \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} P\left(\left|X_i\right| > \epsilon n^{\alpha}\right) < \infty, \tag{44}$$

which implies that (i) holds. This completes the proof of Theorem 3. □

*Proof of Corollary 7.* Taking $\alpha = 1/p$ and $a_{ni} = n^{-\alpha}$, for all $1 \le i \le n$, $n \ge 1$, in Theorem 3, we can get (7) immediately; thus

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} X_i\right| > \epsilon n^{1/p}\right) \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} X_i\right| > \epsilon n^{1/p}\right) \\
&\ge \frac{1}{2} \sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^i} \left|\sum_{i=1}^{j} X_i\right| > \epsilon 2^{(i+1)/p}\right).
\end{aligned} \tag{45}$$

It follows from Borel-Cantelli lemma that

$$P\left(\max_{1\le j\le 2^i} \left|\sum_{i=1}^{j} X_i\right| > \epsilon 2^{(i+1)/p}, \text{ i.o.}\right) = 0. \tag{46}$$

Hence,

$$\lim_{i\to\infty} \frac{1}{2^{(i+1)/p}} \max_{1\le j\le 2^i} \left|\sum_{i=1}^{j} X_i\right| = 0 \quad \text{a.s.} \tag{47}$$

For all positive integers $n$, there exists a nonnegative integer $i_0$ such that $2^{i_0-1} \leq n < 2^{i_0}$. We have by (47) that

$$\begin{aligned}
\max_{2^{i_0-1}\leq n\leq 2^{i_0}} \frac{1}{n^{1/p}}\left|\sum_{i=1}^{n} X_i\right| &\leq \frac{1}{2^{(i_0-1)/p}} \max_{1\leq j\leq 2^{i_0}}\left|\sum_{i=1}^{j} X_i\right| \\
&\leq \frac{2^{2/p}}{2^{(i_0+1)/p}} \max_{1\leq j\leq 2^{i_0}}\left|\sum_{i=1}^{j} X_i\right| \\
&\longrightarrow 0 \quad \text{a.s.}, \ i_0 \longrightarrow \infty,
\end{aligned} \tag{48}$$

which implies that

$$\frac{1}{n^{1/p}}\left|\sum_{i=1}^{n} X_i\right| \longrightarrow 0 \quad \text{a.s.}, \ n \longrightarrow \infty. \tag{49}$$

The proof of Corollary 7 is completed. □

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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