

## *Research Article*

# **Dynamically Switching among Bundled and Single Tickets with Time-Dependent Demand Rates**

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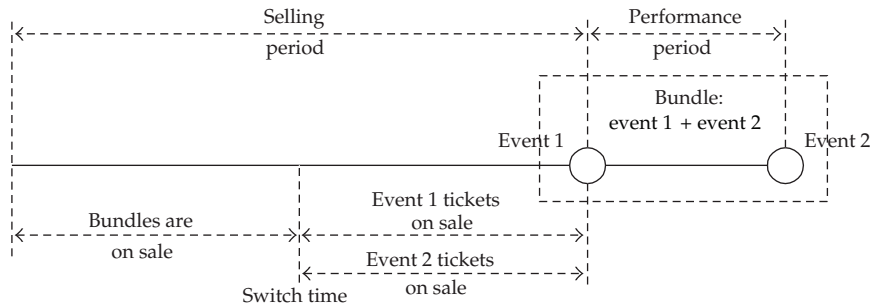
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The most important market segmentation in sports and entertainment industry is the competition between customers that buy bundled and single tickets. A common selling practice is starting the selling season with bundled ticket sales and switching to selling single tickets later on. The aim of this practice is to increase the number of customers that buy bundles, which in return increases the load factor of the events with low demand. In this paper, we investigate the effect of time dependent demand on dynamic switching times from bundled to single ticket sales and the potential revenue gain over the case where the demand rate of events is assumed to be constant with time.

## **1. Introduction**

There are several fundamental decision problems that every seller faces regarding how to price, how many to allocate to a specific customer group, and when to make a specific service or a specific product available, and there are several uncertainties involved in such decisions. Revenue management (RM) is the application of analytical tools that mathematically assist to make those decisions and reduce uncertainties involved in order to maximize profitability.

Although RM has made its outstanding reputation upon the successful application in airline industry, it can be applied to other industries as well. Sports & Entertainment (S&E) is one of the industries that revenue management can be applied appreciably considering the three basic properties of the services or products of the S&E industry. First of all, the services or products of S&E industry are perishable since event tickets have no value after the events take place. Moreover, there is always a limited capacity for the service or product since events



**Figure 1:** Event time line.

are held in venues like a stadium or a theater hall. In addition, customers of S&E products can be successfully segmented into different types of customers.

Other than the student or regular public differentiation in pricing as observed in many industries, segmentation of the market into bundled and single-event ticket buyers is more effective, S&E industry specific and scientifically more interested to focus on. Due to team loyalty or genuine personal interest, some customers prefer to purchase bundled tickets to *all* events during the season, while others prefer to purchase single-tickets to each individual event. Most of the times, bundled tickets are offered starting from the beginning of the selling period, and single-ticket sales are allowed later on in the selling period before the performance period begins. Selling the bundled tickets first permits the firm to sell the premium quality seats to season ticket buyers who are most of the time, lucrative fans valuable to the firm.

We consider that there are two events in the performance period. The selling period starts with sales of bundled tickets which are composed of two tickets, one to each event. We assume that nonhomogenous Poisson arrivals for bundled ticket customers since the interest in purchasing bundles may change with time due to various reasons. Later in the selling period, bundled tickets sale is ceased at the switch time and selling of single-tickets is started and continues until the end of the selling period which is marked by the performance time of the first event. After the switch, customer arrivals split into two independent nonhomogenous Poisson process with time-dependent demand rates for each event. The event time line of the problem is illustrated in Figure 1.

To maximize profitability from ticket sales with such a selling policy, the capacity of the venue should be managed optimally between bundled and single-ticket customers with the help of an optimal switch time. In this work, we study the specific question of timing the switch from bundled tickets to single-tickets when time-dependent customer arrivals are observed, and we compare the results to the case where customer arrivals are homogeneous through time.

## 2. Literature Review

The literature on bundling was started by Stigler [1] and followed by researchers in economics (e.g., Adams and Yellen [2]). The initial studies assume that strict additivity is valid which is the case when products are valued independently. However, when the products are complements or substitutes, the strict additivity assumption often is not valid. The bundling

strategies examined in the literature consider the correlation between reservation prices, heterogeneity of valuations, substitutability and complementarity relations as major factors (Gürler et al. [3]).

Stremersch and Tellis [4] define price bundling as “the sale of two or more separate products in a package at a discount without any integration of products” and product bundling as “the integration and sale of two or more separate products or services at any price.” Although these definitions are more suitable for retail environments, the practice of ticket bundling in S&E industry can be positioned closer to price bundling since there is no integration of products, and a discount is applied most of the time.

From a pure mathematical perspective, our work is closely related to Feng and Gallego [5], which determine the time to switch from one predetermined price to a second higher or lower one dynamically as demand realizes so as to maximize revenue from a given amount of retailing items. Demand rate is assumed to be higher for the lower price, and the optimal timing policy is shown to be a time threshold depending on the remaining stock amount. But in S&E industry, prices are kept constant after the initial announcement throughout the selling period which is known as the price stickiness (Courty [6]). Therefore, timing of the availability of different product groups is of interest within the RM problems of S&E industry than timing of a price change.

Some organizations in S&E industries announce switching time from bundled ticket sales to single-ticket sale in advance before observing any demand realizations. Drake et al. [7] study the optimal timing of such a static switch. However, demand realizations in real world may not be as smooth as considered in the static switching problem. Therefore, most of the organizations *dynamically* select their switching times between different products or announce promotions based on the realized demand throughout the selling period. Accordingly in our work, we study the *dynamic* switching time, where the switch time is determined while observing demand realizations.

Our study is an extension of Duran et al. [8] where dynamic switching is considered first in the RM literature of S&E industry. Their ultimate procedure finds a set of threshold values which depend on the elapsed time and corresponding remaining inventory. However, they consider homogenous Poisson processes to model customer demand implying constant demand rates over time. In our work, we consider nonhomogenous Poisson processes with time-dependent demand rates for both bundled and single-tickets to capture the real life situation more closely.

### 3. Model and Assumptions

#### 3.1. Assumptions

We assume that the selling period begins with the selling of bundled tickets at  $t = 0$  and ends with the performance time of the first event at  $t = T$ , where  $T \in \mathbb{R}^+$ . Also, both of the events take place at a venue with  $M \in \mathbb{Z}^+$  number of available seats. We assume that prices for both bundled ticket and single-tickets are announced before the selling period begins and remain constant during the selling period; bundled tickets are offered first at price  $p_B$ , and single-tickets are offered later at price  $p_i$  for  $i = 1, 2$ . We assumed that bundled tickets are offered at a discount ( $p_B \leq p_1 + p_2$ ) to motivate more customers to buy bundled tickets.

We assume a perfect market segmentation between bundles and singles and time-dependent arrival processes for both bundled and single-tickets. Therefore, we assume that

there is a corresponding Non-homogenous Poisson process with time-dependent demand rate for each ticket group:  $N_B(s)$ ,  $0 \leq s \leq t$ , with known time-dependent demand rate  $\lambda_B(s)$  for the bundled events;  $N_1(s)$ ,  $0 \leq s \leq t$ , with known time-dependent demand rate  $\lambda_1(s)$ , and  $N_2(s)$ ,  $0 \leq s \leq t$ , with known time-dependent demand rate  $\lambda_2(s)$  for the two single events, respectively. In addition, we assume that demand rates  $\lambda_i(s)$  for  $i = B, 1, 2$  are linearly changing or constant over time. The major objective of our study is to find an optimal method to allocate  $M$  seats among bundled and single-ticket buyers within the selling period when time-dependent customer arrivals with rates  $\lambda_i(t)$  ( $i = B, 1, 2$ ) are observed. Revenue rates for bundled and single-tickets are defined as  $r_B(t) = \lambda_B(t)p_B$  and  $r_i(t) = \lambda_i(t)p_i$ ,  $i = 1, 2$ , respectively. We assume that at the beginning of the selling period, the revenue rate for the bundle is higher than the sum of the revenue rates of the single-tickets, that is,  $r_B(0) > r_1(0) + r_2(0)$ , and  $r_B(t) - r_1(t) - r_2(t)$  is a nondecreasing function. Otherwise, switching immediately would be optimal for all states.

Since we assumed a two-event selling horizon, both of the events considered must be appropriate to be bundled. Therefore, demand rate for the bundled tickets is assumed to be larger than each of the individual events' demand rates ( $\lambda_B(t) > \lambda_i(t)$ ,  $i = 1, 2$ ). Moreover, we assume that if the event demand rates increase (decrease) over time, that increase (decrease) rate is higher for singles (bundles), that is,  $\lambda_B(t) - \lambda_i(t)$ ,  $i = 1, 2$  is nonincreasing in  $t$ .

### 3.2. The Dynamic Timing Problem

We will follow the same procedure used in Duran et al. [8]; start with the calculation of the total expected revenue from ticket sales for a specific switching time, then study the effects of delaying the switch by comparing two different switching options. In order to compare those options, we will define a generator function and use this quantification to discover the characteristic of the optimal time to switch. Subsequently, we will develop a function that allows us to compute the optimal switching times for each (elapsed time and left seat) pair. Finally, we will use this function to show the structure of the optimal switching times.

For a switching time  $\tau$  such that  $\tau \in \mathcal{T}$ , where  $\mathcal{T}$  is the set of switching times satisfying  $t \leq \tau \leq T$ , only bundled tickets are sold up to the switching time  $\tau$ , and single-tickets will be sold thereafter in the period from switching time  $\tau$  to the end of the selling period. Therefore, expected total revenue from ticket sales over the entire time horizon  $[t, T]$  with  $n$  seats available for sale and when the switch is exercised at time  $\tau$  is composed of two parts. The first part is the revenue from bundled ticket sales  $B(t, n) = E[p_B((N_B(\tau) - N_B(t)) \wedge n)]$ , where  $(x \wedge y)$  is the function whose value is the minimum of the  $x$  and  $y$ . The second part is the revenue from single-ticket sales from switch time  $\tau$  to the end of the selling period  $T$ ;

$$S(\tau, n(\tau)) = p_1 E[(N_1(T) - N_1(\tau)) \wedge n(\tau)] + p_2 E[(N_2(T) - N_2(\tau)) \wedge n(\tau)], \quad (3.1)$$

where  $n(\tau) = [n - N_B(\tau) + N_B(t)]^+$ , where  $x^+ = \max\{0, x\}$ .

It is possible to investigate the effects of delaying the switch to a later time by comparing two different switching options. If the switch is made immediately at time  $t$ , the expected revenue will be  $S(t, n)$ . However, instead of switching immediately, switch may be delayed to a later time  $\tau$  ( $t \leq \tau \leq T$ ) and the expected revenue in this case will be  $E[p_B((N_B(\tau) - N_B(t)) \wedge n)] + S(\tau, n(\tau))$ . To make the value comparison of those two options, we

utilize (see Pakyardım [9] for details of the derivation) the following infinitesimal generator  $\mathcal{G}$  with respect to the Non-homogenous Poisson process (NPP) for bundles

$$\mathcal{G}g(t, n) = \frac{\partial g(t, n)}{\partial t} + \lambda_B(t)[g(t, n-1) - g(t, n)]. \quad (3.2)$$

When we apply this generator function  $\mathcal{G}$  to the expected total revenue function calculated at the switch time  $\tau$ , we will obtain the net marginal gain (or loss) for delaying the switch from  $t$  to a later time at state  $(t, n)$  by

$$\mathcal{G}S(t, n) + \lambda_B(t)p_B = \frac{\partial S(t, n)}{\partial t} + \lambda_B(t)[S(t, n-1) - S(t, n)] + \lambda_B(t)p_B. \quad (3.3)$$

The net marginal gain (or loss) expression will be one of the main terms in our analysis. Following lemma will focus on this term more closely.

**Lemma 3.1.** *The net marginal gain from delaying the switch at time  $t$  for  $0 \leq t \leq T$  can be written as*

$$\begin{aligned} \mathcal{G}S(t, n) + \lambda_B(t)p_B &= (r_B(t) - r_1(t) - r_2(t)) + p_1(\lambda_1(t) - \lambda_B(t))P[N_1(T) - N_1(t) \geq n] \\ &+ p_2(\lambda_2(t) - \lambda_B(t))P[N_2(T) - N_2(t) \geq n], \end{aligned} \quad (3.4)$$

and is nondecreasing in both  $n$  and  $t$ , when  $\lambda_B(t) > \lambda_i(t)$  for all  $t$  and  $\lambda_B(t) - \lambda_i(t)$  is nonincreasing in  $t$  ( $i = 1, 2$ ) for all  $t$ .

*Proof.* We will begin the proof by stating the simplification of some general terms. Let  $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ . From the properties of NPP, we know that

$$\begin{aligned} \frac{\partial P[N_i(T) - N_i(t) \geq k]}{\partial t} &= \frac{\partial}{\partial t} \left( 1 - \frac{e^{-(\Lambda_i(T) - \Lambda_i(t))} (\Lambda_i(T) - \Lambda_i(t))^{k-1}}{(k-1)!} \right. \\ &\quad \vdots \\ &\quad \left. - \frac{e^{-(\Lambda_i(T) - \Lambda_i(t))} (\Lambda_i(T) - \Lambda_i(t))^0}{(0)!} \right). \end{aligned} \quad (3.5)$$

After eliminations we have

$$\begin{aligned} \frac{\partial P[N_i(T) - N_i(t) \geq k]}{\partial t} &= -\lambda_i(t)P[N_i(T) - N_i(t) = k-1], \\ \frac{\partial \sum_{k=1}^n P[N_i(T) - N_i(t) \geq k]}{\partial t} &= -\lambda_i(t)P[N_i(T) - N_i(t) \leq n-1]. \end{aligned} \quad (3.6)$$

Also note that  $E[(N_i(T) - N_i(t)) \wedge n] = \sum_{k=1}^n P[N_i(T) - N_i(t) \geq k]$ . Therefore, we have

$$\begin{aligned}
\mathcal{G}S(t, n) &= \frac{\partial S(t, n)}{\partial t} + \lambda_B(t)[S(t, n-1) - S(t, n)] \\
&= -\lambda_1(t)p_1P[N_1(T) - N_1(t) \leq n-1] - \lambda_2(t)p_2P[N_2(T) - N_2(t) \leq n-1] \\
&\quad - \lambda_B(t)p_1P[N_1(T) - N_1(t) \geq n] - \lambda_B(t)p_2P[N_2(T) - N_2(t) \geq n] \\
&= -\lambda_1(t)p_1 - \lambda_2(t)p_2 + p_1(\lambda_1(t) - \lambda_B(t))P[N_1(T) - N_1(t) \geq n] \\
&\quad + p_2(\lambda_2(t) - \lambda_B(t))P[N_2(T) - N_2(t) \geq n].
\end{aligned} \tag{3.7}$$

Finally we have

$$\begin{aligned}
\mathcal{G}S(t, n) + \lambda_B(t)p_B &= r_B(t) - r_1(t) - r_2(t) + p_1(\lambda_1(t) - \lambda_B(t))P[N_1(T) - N_1(t) \geq n] \\
&\quad + p_2(\lambda_2(t) - \lambda_B(t))P[N_2(T) - N_2(t) \geq n].
\end{aligned} \tag{3.8}$$

It is easy to see that  $\mathcal{G}S(t, n) + \lambda_B(t)p_B$  is nondecreasing in  $n$ . Defining  $R(t) = r_B(t) - r_1(t) - r_2(t)$  and  $k_i(t, n) = P[N_i(T) - N_i(t) \geq n]$ , we may write the net marginal gain (or loss) expression simply as  $\mathcal{G}S(t, n) + \lambda_B(t)p_B = R(t) - [p_1(\lambda_B(t) - \lambda_1(t))k_1(t, n) + p_2(\lambda_B(t) - \lambda_2(t))k_2(t, n)]$ .

Also defining  $f(t, n) = [p_1(\lambda_B(t) - \lambda_1(t))k_1(t, n) + p_2(\lambda_B(t) - \lambda_2(t))k_2(t, n)]$ , marginal gain (or loss) expression simply turns into  $\mathcal{G}S(t, n) + \lambda_B(t)p_B = R(t, n) - f(t, n)$ . For any fixed  $n$ , we have the following marginal gain (or loss) values at the beginning and at the end of the selling period:

$$(i) \text{ at } t = 0, \mathcal{G}S(0, n) + \lambda_B(0)p_B = R(0) - f(0, n),$$

$$(ii) \text{ at } t = T, \mathcal{G}S(T, n) + \lambda_B(T)p_B = R(T).$$

Noting the assumption that  $R(t) = r_B(t) - r_1(t) - r_2(t)$  being a nondecreasing function on  $[0, T]$ , and  $f(0, n)$  being nonnegative for any  $n$ ; we get  $R(0) - f(0, n) < R(0) \leq R(T)$ .

Thus, it is easy to compare the net marginal gain (or loss) values at the beginning of the selling period at  $t = 0$  and at the end of the selling period at  $t = T$  as

$$\mathcal{G}S(0, n) + \lambda_B(0)p_B = R(0) - f(0, n) < R(T) = \mathcal{G}S(T, n) + \lambda_B(T)p_B. \tag{3.9}$$

Since  $f(t, n)$  is the product of two positive and nonincreasing functions;  $\lambda_B(t) - \lambda_i(t)$  and  $k_i(t, n)$ , is also positive and nonincreasing on  $[0, T)$ . Thus, marginal gain (or loss) expression  $\mathcal{G}S(t, n) + \lambda_B(t)p_B$  is nondecreasing in  $t$  starting from the value  $R(0) - f(0, n)$  at  $t = 0$  and reaching to a higher value  $R(T)$  at  $t = T$ .  $\square$

Taking the supremum of the expected total revenue function over all stopping times  $\tau \in \mathcal{T}$  will give us the optimal expected total revenue from bundled and single-ticket sales over  $[t, T]$  with  $n$  remaining seats, and it is given by  $V(t, n)$

$$V(t, n) = \sup_{\tau \in \mathcal{T}} E[p_B((N_B(\tau) - N_B(t)) \wedge n) + S(\tau, n(\tau))]. \tag{3.10}$$

Using properties of the martingales and Doob's optional stopping theory (see Rogers and Williams [10]), it is possible to show that

$$V(t, n) = S(t, n) + \sup_{t \leq \tau \leq T} E \int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t)p_B I_{\{n(u) > 0\}}] du. \quad (3.11)$$

If we define that

$$\tilde{V}(t, n) = \sup_{t \leq \tau \leq T} E \int_t^\tau [\mathcal{G}S(u, n(u)) + \lambda_B(t)p_B I_{\{n(u) > 0\}}] du, \quad (3.12)$$

we get that  $V(t, n) = S(t, n) + \tilde{V}(t, n)$ . This implies that the optimal revenue over  $[t, T]$  consists of the revenue from immediately switching and the additional revenue from delaying the switch to a later time, respectively. Since  $\tilde{V}(t, n)$  can not be negative by its definition, when  $\tilde{V}(t, n) = 0$ , delaying the switch further is not optimal, whereas  $\tilde{V}(t, n) > 0$  implies a revenue potential from delaying the switch.

At this point, we introduce  $\bar{V}(t, n)$  which is identical to  $\tilde{V}(t, n)$  and can be derived recursively when the conditions in Theorem 3.2 are satisfied.

**Theorem 3.2.** *Suppose that there exists a function  $\bar{V}(t, n)$  such that  $\bar{V}(t, n)$  is continuous and differentiable with right continuous derivatives in  $[0, T]$  for each fixed  $n$ .  $\bar{V}(t, n) = \tilde{V}(t, n)$  if  $\bar{V}(t, n)$  satisfies*

- (i)  $\bar{V}(t, n) \geq 0$ ,  $0 \leq t \leq T$  and  $0 \leq n \leq M$ ;
- (ii)  $\bar{V}(T, n) = 0$  for  $0 \leq n \leq M$  and  $\bar{V}(t, 0) = 0$  for  $0 \leq t \leq T$ ;
- (iii)  $\bar{V}(t, n) = 0 \Rightarrow \mathcal{G}(\bar{V} + S)(t, n) + \lambda_B(t)p_B \leq 0$ ,  $0 \leq t \leq T$  and  $0 \leq n \leq M$ ;
- (iv)  $\bar{V}(t, n) > 0 \Rightarrow \mathcal{G}(\bar{V} + S)(t, n) + \lambda_B(t)p_B = 0$ ,  $0 \leq t \leq T$  and  $0 \leq n \leq M$ .

*Proof.* Proof of Theorem 3.2 is along the lines of Theorem 1 in Duran et al. [8]. Therefore, we provide here just a sketch of the proof, and details can be seen at Pakyardım [9]. Proof starts with taking a function  $\bar{V}$  that satisfies the conditions (i) to (iv) of the theorem. By Dynkin's Lemma, we know that following equation is a martingale:

$$m(s) = \bar{V}(s, n(s)) - \bar{V}(t, n) - \int_t^s \mathcal{G}\bar{V}(u, n(u)) du. \quad (3.13)$$

Since  $Em(s) = 0$  and by replacing  $s$  with any stopping time  $\tau$  due to optional stopping theorem, we obtain that  $E[\bar{V}(\tau, n(\tau))] - E \int_t^\tau \mathcal{G}\bar{V}(u, n(u)) du = \bar{V}(t, n)$ . With a little arithmetic it is easy to show that  $\bar{V}(t, n) \geq \tilde{V}(t, n)$ . To prove that  $\bar{V}(t, n) \leq \tilde{V}(t, n)$ , the specific stopping time  $\sigma$  is defined as  $\sigma = \inf\{t \leq s \leq T : \bar{V}(s, n(s)) = 0\}$  and utilized.  $\square$

### 3.3. Calculation of Optimal $(x_n, n)$ Pairs

From condition (iv) of Theorem 3.2, we know that  $\mathcal{G}\bar{V}(t, n) = -\mathcal{G}S(t, n) - \lambda_B(t)p_B$  when  $\bar{V}(t, n) > 0$ . Applying the infinitesimal generator  $\mathcal{G}$  to  $\bar{V}(t, n)$ , we get the differential equation

$$\frac{\partial \bar{V}(t, n)}{\partial t} - \lambda_B(t)\bar{V}(t, n) = -[\lambda_B(t)\bar{V}(t, n-1) + \mathcal{G}S(t, n) + \lambda_B(t)p_B]. \quad (3.14)$$

For convenience let  $[\lambda_B(t)\bar{V}(t, n-1) + \mathcal{G}S(t, n) + \lambda_B(t)p_B] = A(t, n)$  and  $\Lambda_B(t) = \int_0^t \lambda_B(s)ds$ . Multiply both side by  $e^{-\Lambda_B(t)}$  in order to transform (3.14) into a form that integration can be performed,

$$e^{-\Lambda_B(t)} \frac{\partial \bar{V}(t, n)}{\partial t} - e^{-\Lambda_B(t)} \lambda_B(t) \bar{V}(t, n) = -e^{-\Lambda_B(t)} A(t, n). \quad (3.15)$$

It is easy to see that left hand side of the (3.15) is the derivative of the product of  $e^{-\Lambda_B(t)}$  and  $\bar{V}(t, n)$ . Therefore, we get that

$$\bar{V}(t, n) = e^{\Lambda_B(t)} \int [-e^{-\Lambda_B(t)} A(t, n)] dt. \quad (3.16)$$

Using the boundary conditions on  $\bar{V}(t, n)$ , we can convert (3.16) into a definite form as

$$\bar{V}(t, n) = \bar{V}(T, n) - e^{\Lambda_B(t)} \int_t^T [-e^{-\Lambda_B(u)} A(u, n)] du = e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(u)} A(u, n) du. \quad (3.17)$$

$\bar{V}(t, n)$  can be calculated if  $\bar{V}(t, n-1)$  is known. Since  $\bar{V}(t, 0) = 0$ ,  $\bar{V}(t, n)$  can be recursively determined for any  $(t, n)$  pair starting from  $\bar{V}(t, 0) = 0$ . The formal procedure is given in Theorem 3.3. Theorem 3.3 also provides information on the structure of the optimal switching times.

**Theorem 3.3.** For all  $1 \leq n \leq M$  and  $\lambda_B(t) > \lambda_i(t)$ , for  $i = 1, 2$ , the switching-time thresholds  $\{x_n\}$  and  $\bar{V}(t, n)$  are recursively determined by

$$\bar{V}(t, n) = \begin{cases} e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s, n) ds & \text{if } t > x_n, \\ 0 & \text{otherwise,} \end{cases} \quad (3.18)$$



where

$$\begin{aligned}
 x_n &= \inf \left\{ 0 \leq t \leq T : e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s, n) ds > 0 \right\}, \\
 x_1 &\geq x_2 \geq \dots \geq x_n, \\
 A(t, n) &= \mathcal{C}S(t, n) + \lambda_B(t)p_B + \lambda_B \bar{V}(t, n-1), \quad 0 \leq t \leq T, \\
 \bar{V}(t, 0) &= 0, \quad 0 \leq t \leq T.
 \end{aligned} \tag{3.19}$$

*Proof.* Proof of Theorem 3.3 is along the lines of Theorem 3 in Duran et al. [8]. Therefore, we also provide only a sketch of the proof, and details are presented in Appendix. Proof is done by mathematical induction. Starting with  $n = 1$ , the function  $\bar{V}(t, n)$  calculated with the procedure given in theorem is shown to satisfy condition (iii) of Theorem 3.2 when  $t \leq x_n$  and condition (iv) of Theorem 3.2 when  $t > x_n$  utilizing the fact that  $\mathcal{C}S(t, n) + \lambda_B(t)p_B$  is nondecreasing in both  $n$  and  $t$ . Also at each  $n$ , we show that  $\bar{V}(t, n) \geq \bar{V}(t, n-1)$  holds.  $\square$

### 3.4. Details for the Approximation of the $\bar{V}$ Function

To calculate the  $\bar{V}(t, n)$  values for each  $(t, n)$  pair, we will use a discrete time approximation. For any  $1 \leq n \leq M$ , and  $x_n < t < T$  with some  $\delta > 0$  such that  $t + \delta \leq T$ , we have the following

$$\begin{aligned}
 \bar{V}(t, n) &= e^{\Lambda_B(t)} \int_{t+\delta}^T e^{-\Lambda_B(u)} A(u, n) du + e^{\Lambda_B(t)} \int_t^{t+\delta} e^{-\Lambda_B(u)} A(u, n) du \\
 &= e^{-(\Lambda_B(t+\delta)-\Lambda_B(t))} \bar{V}(t+\delta, n) + \int_t^{t+\delta} e^{-(\Lambda_B(u)-\Lambda_B(t))} A(u, n) du \\
 &= e^{-(\Lambda_B(t+\delta)-\Lambda_B(t))} \bar{V}(t+\delta, n) + \int_t^{t+\delta} e^{-(\Lambda_B(u)-\Lambda_B(t))} \left[ \frac{\partial S(t, n)}{\partial t} + \lambda_B(t)[S(t, n-1) - S(t, n)] \right. \\
 &\quad \left. + \lambda_B(t)p_B + \lambda_B(t)\bar{V}(u, n-1) \right] du.
 \end{aligned} \tag{3.20}$$

Within a small time interval from  $t$  to  $t + \delta$ , we may take  $S(t, n)$ ,  $\lambda_B(t)p_B$ , and  $\bar{V}(t, n)$  as constants, and defining the function  $\theta(t, \delta) = e^{-(\Lambda_B(t+\delta)+\Lambda_B(t))}$ , we obtain the following approximation:

$$\bar{V}(t, n) \cong \left( \bar{V} + S \right)(t + \delta, n) \theta(t, \delta) + (1 - \theta(t, \delta)) \left[ p_B + \left( \bar{V} + S \right)(t, n-1) \right] - S(t, n). \tag{3.21}$$

### 3.5. Usage of Optimal Switching Thresholds

We have demonstrated in our analysis that, for any remaining inventory level  $n \in M$ , there exists a corresponding optimal switch time  $x_n$ . If the unsold inventory level  $n$  is reached at time  $t$  where  $t > x_n$ , delaying the switch is optimal since there is an additional revenue from delaying the switch to a later time ( $\bar{V}(t, n) > 0$  for  $t > x_n$ ). Intuition behind this is as follows: when there are  $n$  remaining seats, it takes at least  $T - x_n$  to sell out those  $n$  seats with singles only. Therefore, singles cannot sell out the remaining seats when  $t > x_n$  and bundle sale is carried on which results in a higher demand rate for each individual seat ( $\lambda_B(t) > \lambda_i(t)$ ). Conversely, if the unsold inventory level  $n$  is reached at time  $t$  where  $t \leq x_n$ , switching immediately is optimal. There is no additional revenue from delaying the switch to a later time ( $\bar{V}(t, n) = 0$  for  $t \leq x_n$ ), since singles can sell out the remaining seats at a higher price ( $p_B < p_1 + p_2$ ) per seat.

## 4. Numerical Study

In this section using computational analysis, we will first study the structure of the optimal switching times for different demand rates schemes and then compare the calculated optimal switching times to the case demand rate that is assumed to be constant. Finally, we will study the percentage improvement in revenue over constant demand rate assumption case for different demand rates schemes.

### 4.1. Optimal Switching Times and Their Behavior for Demand Rates

$\bar{V}$  and optimal switching times can be calculated using the following algorithm by the approximation given in Section 3.4:

*Algorithm 4.1.* Let

$$\Delta A(k\delta, n) = (\bar{V} + S)((k+1)\delta, n)\theta(t, \delta) + (1 - \theta(t, \delta))\left[p_B + (\bar{V} + S)(k\delta, n-1)\right] - S(k\delta, n), \quad (4.1)$$

where  $\theta(t, \delta) = e^{-(\lambda_B(t+\delta) + \lambda_B(t))}$ .

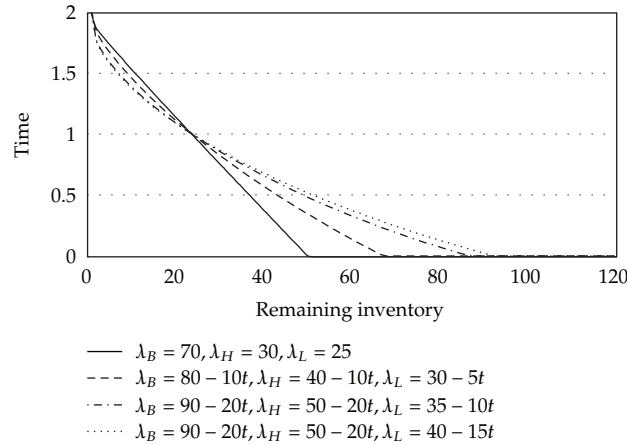
*Step 1.* Initialize  $\bar{V}(T, \cdot) = \bar{V}(K\delta, \cdot) = 0$  for all inventory levels. Set  $n = 1$  and  $k = (K - 1)$ .

*Step 2.* Calculate  $\Delta A(k\delta, n)$ .

*Step 3.* Set  $\bar{V}(k\delta, n) = (\Delta A(k\delta, n))^+$  and  $k = k - 1$ :

- (i) if  $k \neq -1$  and  $\bar{V}(k\delta, n) \geq 0$ , go to Step 2;
- (ii) otherwise set  $\bar{V}(j\delta, n) = 0$  for all  $j < k - 1$  and  $n = n + 1$ .

One high-demand and one low-demand event tickets, taking place in a 120-ticket stadium, are on sale during a 2-month selling period. The constant prices to be charged for a high-demand and low-demand event ticket will be \$200 and \$50, respectively. However, if the seats are to be sold as a bundle, the price will be discounted to \$220. We will study different demand



**Figure 2:** Optimal switching times for all decreasing demand rates.

rate schemes; linearly decreasing demand rates with different rate of changes. Regardless of their behavior, arithmetic average of demand rates over the selling period for all cases will be equal. As the constant demand rate assumption case, we will consider on average 30 customers per month for high demand, 25 customers per month for low demand, and 70 customers per month for bundled event tickets. In order to see the effect of the rate of change, the following three decreasing demand rates schemes with different slopes are investigated:

- (i)  $\lambda_B = 80 - 10t, \lambda_H = 40 - 10t, \lambda_L = 30 - 5t,$
- (ii)  $\lambda_B = 90 - 20t, \lambda_H = 50 - 20t, \lambda_L = 35 - 10t,$
- (iii)  $\lambda_B = 90 - 20t, \lambda_H = 50 - 20t, \lambda_L = 40 - 15t.$

Optimal switching thresholds for these three demand rates schemes and for constant demand rate assumption case are plotted in Figure 2.

All demand rates start from high values and decrease linearly as time passes. At early times in the selling period, demand rates for both bundles and singles are relatively higher compared to the constant demand rates assumption case. Therefore, the sales portion that is intended to be sold as bundles (this portion is also dynamically changing according to system state  $(t, n)$ ) to completely sell out the venue is realized earlier. Consequently, during early times in the selling period, we observe that areas under the threshold curves calculated for decreasing demand rate cases are greater than the area under the threshold line calculated for constant demand rate case, resulting in more frequent switches. At later times throughout the selling period, demand rates for both bundle and single-tickets are relatively lower compared to the constant demand rate case. Thus, switching occurs less frequently in later times.

Moreover as time passes, all demand rates decrease, average inter-arrival times between events increase; that is, time is not in favor of single-ticket sales. Therefore, switching tends to become less frequent as time passes. This is the reason why the threshold curve is convex. Furthermore note that the difference between threshold curves calculated for time-dependent demand rates, and the threshold line calculated for constant demand rates is changing with time. This means that effect of constant demand rates assumption may vary according to the realization path.

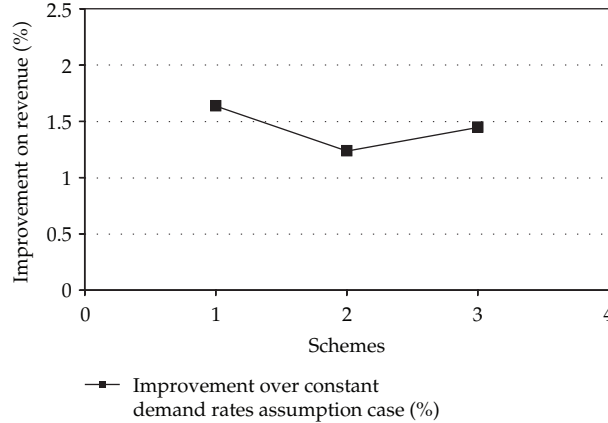


Figure 3: % Improvement over constant demand rates assumption case.

#### 4.2. Simulation Studies

To investigate the percentage improvement over constant demand rates assumption case, will generate two switching threshold sets for each scheme; one with time-dependent demand rates and one with constant demand rates. Then, we will simulate the sale horizon with different sample paths (10,000 paths for each scheme) generated by time-dependent demand rates and determine the average revenue from those demand realization paths utilizing two types of switching thresholds, calculated by constant demand rates assumption and time-dependent demand rates. Lastly, we will find the % improvement for each demand rate scheme by comparing these two average revenue values, which are summarized in Figure 3.

### 5. Conclusions

We study the problem of switching from bundled tickets to single-tickets and comparing the case where time-dependent demand rates are utilized to the constant demand rates case. We show that, for any inventory level  $n$ , there exists a time  $x_n$  after which switching from bundled ticket sales to single-ticket sales is not optimal. Moreover, we prove that thresholds are also decreasing in remaining inventory  $n$  and increasing in  $t$  as suggested in Duran et al. [8].

We numerically study the structure of optimal switching times for various demand rates schemes and compared them to the case where demand rates were assumed to be constant. We illustrate that when time-dependent demand rates are used; switching thresholds form a curve rather than a line which is suggested by constant demand rates assumption.

In simulation studies, we have expressed the value of utilizing the dynamic switching threshold policy by calculating % improvement on average revenue. We have found that revenue gain can be between 1% and 2% (varying according to demand rate schemes) over constant demand rate assumption. Along with the results of Duran [11], we conjecture that % improvement on revenue over static switch may be 2–4% when dynamic switching with time-dependent demand rates policy is utilized.

## Appendix

*Proof of Theorem 3.3.* We will prove by induction on  $n$  for the function  $\bar{V}(t, n)$  which is determined by Theorem 3.3 and satisfies condition (i)–(iv). Therefore, we will prove that  $\bar{V}(t, n)$  which is calculated by Theorem 3.3 is equivalent to  $\tilde{V}(t, n)$ . From condition (ii) we know that when  $n = 1$ ,  $\bar{V}(t, n - 1) = 0$ . Also from Lemma 3.1, we know that  $\mathcal{G}S(t, 1) + \lambda_B(t)p_B$  is a nondecreasing function in  $t$ . We require that for  $t \leq x_1$  the following inequalities must hold:

$$A(t, 1) = \mathcal{G}S(t, 1) + \lambda_B(t)p_B \leq \mathcal{G}S(x_1, 1) + \lambda_B(x_1)p_B \leq 0. \quad (\text{A.1})$$

As we already said the first inequality is from the nondecreasing property of  $\mathcal{G}S(t, 1) + \lambda_B(t)p_B$  in  $t$ . The second inequality results from the fact that if  $\mathcal{G}S(x_1, 1) + \lambda_B(x_1)p_B > 0$ , then  $e^{\lambda_B(t) \int_t^T e^{-\lambda_B(s)} A(s, n) ds} > 0$ , which contradicts the definition of  $x_1$ . Hence, for  $t \leq x_1$  (or  $\bar{V}(t, 1) = 0$  by the definition of  $\bar{V}$ )

$$\begin{aligned} \mathcal{G}(\bar{V} + S)(t, 1) + \lambda_B(t)p_B &= \frac{\partial \bar{V}(t, 1)}{\partial t} + \lambda_B(t) [\bar{V}(t, 0) - \bar{V}(t, 1)] + \mathcal{G}S(t, 1) + \lambda_B(t)p_B \\ &= \mathcal{G}S(t, 1) + \lambda_B(t)p_B = A(t, 1) \leq 0. \end{aligned} \quad (\text{A.2})$$

Thus, condition (iii) is satisfied when  $n = 1$  and  $t \leq x_1$  (or  $\bar{V}(t, 1) = 0$ ). When  $t > x_1$  (or  $\bar{V}(t, 1) > 0$ ), we have that

$$\begin{aligned} \mathcal{G}(\bar{V} + S)(t, 1) + \lambda_B(t)p_B &= \frac{\partial \bar{V}(t, 1)}{\partial t} + \lambda_B(t) [\bar{V}(t, 0) - \bar{V}(t, 1)] + \mathcal{G}S(t, 1) + \lambda_B(t)p_B \\ &= \frac{\partial \bar{V}(t, 1)}{\partial t} - \lambda_B(t)\bar{V}(t, 1) + \mathcal{G}S(t, 1) + \lambda_B(t)p_B. \end{aligned} \quad (\text{A.3})$$

We previously stated that applying the infinitesimal generator  $\mathcal{G}$  to  $\bar{V}(t, n)$ , we get the following differential equation as in (3.14):

$$\begin{aligned} \frac{\partial \bar{V}(t, n)}{\partial t} - \lambda_B(t)\bar{V}(t, n) &= -\lambda_B(t)\bar{V}(t, n - 1) - \mathcal{G}S(t, n) - \lambda_B(t)p_B, \\ \frac{\partial \bar{V}(t, n)}{\partial t} &= \lambda_B(t)\bar{V}(t, n) - \lambda_B(t)\bar{V}(t, n - 1) - \mathcal{G}S(t, n) - \lambda_B(t)p_B. \end{aligned} \quad (\text{A.4})$$

Since  $\bar{V}(t, 0) = 0$ , for  $n = 1$ , we have

$$\frac{\partial \bar{V}(t, 1)}{\partial t} = \lambda_B(t)\bar{V}(t, 1) - \mathcal{G}S(t, 1) - \lambda_B(t)p_B. \quad (\text{A.5})$$

Substituting (A.5) into (A.3), we get  $\mathcal{G}(\bar{V} + S)(t, 1) + \lambda_B(t)p_B = 0$  when  $\bar{V}(t, 1) > 0$ . Therefore, condition (iv) is satisfied when  $n = 1$ . Moreover, we have  $\bar{V}(t, 1) \geq \bar{V}(t, 0) = 0$  by the definition

of  $x_1$  (there exists a time  $t$  such that  $\bar{V}(t, 1) > 0$  if  $x_1 > 0$ ). Now assume that the following statements hold for  $n \leq k < M$ ; there exist  $k$  time thresholds with  $T \geq x_1 \geq \dots \geq x_k \geq 0$  such that  $\bar{V}(t, n)$  is derived from (3.18) and satisfies conditions (i)–(iv), and the inequality  $\bar{V}(t, n) \geq \bar{V}(t, n-1)$  holds for  $n = 1, \dots, k$ . For  $n = k+1$ , we have that

$$\begin{aligned} A(t, k+1) &= \mathcal{G}S(t, k+1) + \lambda_B(t)p_B + \lambda_B(t)\bar{V}(t, k) \\ &\geq \mathcal{G}S(t, k) + \lambda_B(t)p_B + \lambda_B(t)\bar{V}(t, k-1) = A(t, k), \end{aligned} \quad (\text{A.6})$$

since  $\mathcal{G}S(t, k) + \lambda_B(t)p_B$  and  $\bar{V}(t, k)$  are increasing in  $k$  by the induction assumption. This implies that

$$e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s, k+1) ds \geq e^{\Lambda_B(t)} \int_t^T e^{-\Lambda_B(s)} A(s, k) ds. \quad (\text{A.7})$$

Together with (3.18), this implies that  $\bar{V}(t, k+1) \geq \bar{V}(t, k)$  and  $x_k \geq x_{k+1}$ . For  $t \leq x_{k+1}$  (or  $\bar{V}(t, k+1) = 0$ ),

$$\begin{aligned} \mathcal{G}(\bar{V} + S)(t, k+1) + \lambda_B(t)p_B &= \frac{\partial \bar{V}(t, k+1)}{\partial t} + \lambda_B(t) [\bar{V}(t, k) - \bar{V}(t, k+1)] \\ &\quad + \mathcal{G}S(t, k+1) + \lambda_B(t)p_B \\ &= \mathcal{G}S(t, k+1) + \lambda_B(t)p_B + \lambda_B(t)\bar{V}(t, k) = A(t, k+1) \\ &\leq A(x_{k+1}, k+1) \leq 0. \end{aligned} \quad (\text{A.8})$$

Note that  $\bar{V}(t, k) = \bar{V}(t, k+1) = 0$  since  $t \leq x_{k+1} \leq x_k$ . The first inequality follows from  $\mathcal{G}S(t, k+1) + \lambda_B(t)p_B$  being increasing in  $t$ , and the second inequality follows from the fact that if  $A(x_{k+1}, k+1) > 0$ , then this will contradict the definition of  $x_{k+1}$ . Therefore, condition (iii) is satisfied, when  $t \leq x_{k+1}$  (or  $\bar{V}(t, k+1) = 0$ ). For  $t > x_{k+1}$  (or  $\bar{V}(t, k+1) > 0$ ),

$$\begin{aligned} \mathcal{G}(\bar{V} + S)(t, k+1) + \lambda_B(t)p_B &= \frac{\partial \bar{V}(t, k+1)}{\partial t} + \lambda_B(t) [\bar{V}(t, k) - \bar{V}(t, k+1)] \\ &\quad + \mathcal{G}S(t, k+1) + \lambda_B(t)p_B \\ &= -A(t, k+1) + \lambda_B(t)\bar{V}(t, k+1) + \lambda_B(t) [\bar{V}(t, k) - \bar{V}(t, k+1)] \\ &\quad + \mathcal{G}S(t, k+1) + \lambda_B(t)p_B = 0. \end{aligned} \quad (\text{A.9})$$

Therefore condition (iv) is satisfied when  $t > x_{k+1}$  (or  $\bar{V}(t, k+1) = 0$ ). For  $n = k+1$  we showed that conditions (i)–(iv) hold. Thus, the function  $\bar{V}(t, k)$ , that is determined by the proposed procedure, is equal to  $\tilde{V}(t, k)$ . Further the switching time thresholds ( $x_n$ ) are monotonically nonincreasing in  $n$ .  $\square$

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