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Research Article

Study on the Queue-Length Distribution in $Geo/G(MWV)/1/N$ Queue with Working Vacations

Chuanyi Luo,[1] Jiangyi Li,[2] and Yinghui Tang[3]

[1] *School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China*
[2] *Research Institute of Economics and Management, Southwestern University of Finance and Economics, Chengdu 611130, China*
[3] *School of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, China*

Correspondence should be addressed to Chuanyi Luo; lcy@swufe.edu.cn

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This paper analyzes a finite buffer size discrete-time $Geo/G/1/N$ queue with multiple working vacations and different input rate. Using supplementary variable technique and embedded Markov chain method, the queue-length distribution solution in the form of formula at arbitrary epoch is obtained. Some performance measures associated with operating cost are also discussed based on the obtained queue-length distribution. Then, several numerical experiments follow to demonstrate the effectiveness of the obtained formulae. Finally, a state-dependent operating cost function is constructed to model an express logistics service center. Regarding the service rate during working vacation as a control variable, the optimization analysis on the cost function is carried out by using parabolic method.

1. Introduction

Discrete-time queues with classical vacation policies have been explored more in depth during the last few decades due to their widespread application in telecommunication system, electronic information network, production system, and so on (see Takagi [1], Tian et al. [2], Alfa [3], and the references therein). In the queueing systems with classical vacation policies, the server is assumed completely inactive (does not provide any service) during his vacation time. Motivated by the study of reconfigurable wavelength-division multiplexing optical access network, Servi and Finn [4] developed a different vacation policy against the classical one, working vacation (WV), in which the server did not stop service for customers; instead, he remained semiactive. From then on, a large number of researchers were attracted to study queueing systems with working vacation policy. By using matrix-analytic method, Li et al. [5], Li and Tian [6], and Li [7] studied the discrete-time $GI/Geo/1$ queues with working vacations. Yi [8] considered disasters in $Geo/G/1$ queue with working vacations. Li et al. [9] and Gao and Liu [10] added the bach-arrival schedule to a discrete-time queue with working vacations. On the basis of working-vacation $Geo/G/1$ queue, Goswami and Selvaraju [11] extended the working-vacation policy to queueing model with Markovian arrival process and general phase-type distributed service time.

These mentioned research works all concentrated on working-vacation queueing models with infinite buffer size; however, the finite buffer size counterparts received little attention. In real situations, queues with finite buffer size are more suitable than queues with infinite buffer space as it is used to store arrived customers if server is busy. Among the existing references, very few papers considered the working-vacation queue with finite buffer size; see Goswami and Samanta [12], Yu et al. [13], Yu et al. [14], Zhang and Hou [15], Gao et al. [16], and Banerjee et al. [17]. Nevertheless, the existing research topics with finite buffer size and working vacations mentioned above all concentrate on $GI/Geo/1/N$ queueing system and its different varieties. We note that the finite buffer $Geo/G/1/N$ queue with working vacations has not been studied up to now. It motivates us to fill this gap.

In addition, as far as the queueing systems with working vacations are concerned, the assumption assumes in general that the customers arrive in system at a fixed rate. However, the customer's choice of entering into system or not usually depends on the system's status what they see at the arrival epoch. For example, in a make-to-order production system where the system information such as server's status and queue length is fully observable to an arriving customer, the arriving customer with rate of λ may choose to leave the system when he finds that the server is not active, or the service rate is lower than the normal rate, or too many customers accumulate in front of the server, and so on. Thus, we assume in this paper that the customers arrive in system at different rates. Consequently, the changeable arrival rate of order brings influence on some critical performance measures associated with operating cost such as queue length, waiting times, delay probability for an order, and blocking probability (see Bhaskar and Lallement [18]).

On account of the introduction mentioned above, we study the finite buffer $Geo/G/1$ queue with working vacations and different arrival rates, denoted by $Geo^{\lambda_1,\lambda_2}/G(MWV)/1/N$. Using supplementary variable method and embedded Markov chain techniques, the queue-length distribution in the form of formula at arbitrary epoch is obtained. Some performance measures associated with operating cost are also discussed under the achievement of the queue-length distribution solution. To demonstrate the effectiveness of the achieved formulae, a numerical experiment is carried out with respect to a state-dependent operating cost function.

The rest of this paper is organized as follows. Section 2 describes the queueing model. Section 3 analyzes the queue-length distribution at arbitrary epoch. In Section 4, various performance measures are obtained. Section 5 focuses on the numerical performance characteristics. To demonstrate the application of the model studied in this paper, Section 6 constructs a state-dependent cost function from an express logistics service center and discusses a cost minimization problem concerning the function. Finally, some conclusions and topics for future research are mentioned in Section 7.

2. Model Description

We consider a discrete-time single server queue with vacations, in which a potential customer arrives in time interval (n^-, n) with delayed access and departs in (n, n^+) (we call it LAS-DA discipline). Upon the server working at a normal service rate, customers arrive in system according to a Bernoulli process with parameter of $\lambda_1 \in (0,1)$. We denote the arrival interval in this case by random variable $\tau^{(1)}$; that is, $P\{\tau^{(1)} = j\} = \lambda_1 \overline{\lambda}_1^{j-1}$, where $\overline{\lambda}_1 = 1 - \lambda_1$; in another case, when the server is on vacation, customers arrive in system according to another Poison process with parameter of $\lambda_2 \in (0,1)$. The arrival intervals in this case are denoted by $\tau^{(2)}$; that is, $P\{\tau^{(2)} = j\} = \lambda_2 \overline{\lambda}_2^{j-1}$.

Service and Vacation Rules. The server serves the waiting customers (if there is any) according to FCFS (first come, first served) discipline. The service begins with a normal rate when the first customer arrives in system and ends when the system becomes empty. We call this time interval the "normal busy period." The duration of the service time for a customer, denoted by $\chi^{(b)}$, is a random variable with arbitrary

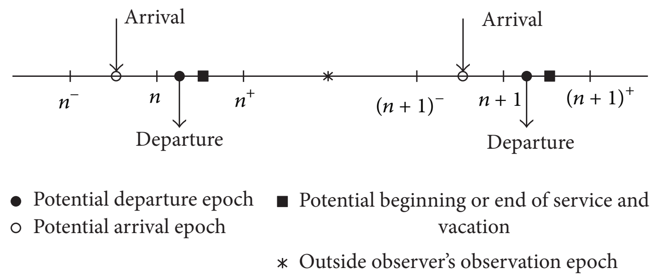

Figure 1: Various time epochs in a late arrival system with delayed access (LAS-DA).

probability mass function (PMF) $g_j^{(b)} = P\{\chi^{(b)} = j\}$, $j \ge 1$, probability generating function (PGF) $G^{(b)}(z) = \sum_{j=1}^{\infty} g_j^{(b)} z^j$, and finite mean service time $E[\chi^{(b)}] = \alpha_1$. When the system becomes empty, the server takes multiple working vacations [2]. The length of a working vacation, denoted by V, is geometrically distributed with PMF $P\{V = k\} = v\overline{v}^{k-1}$. The length of the service time for a customer during working vacation period, denoted by $\chi^{(v)}$, is also a random variable following another arbitrary distribution with PMF $g_j^{(v)} = P\{\chi_i^{(v)} = j\}$, $j \ge 1$ and PGF $G^{(v)}(z) = \sum_{j=1}^{\infty} g_j^{(v)} z^j$ and finite mean service time $E[\chi^{(v)}] = \alpha_2 > \alpha_1$. If the server finds that the system is nonempty upon comes back from a working vacation, he returns the service rate to the normal level and restarts the service interrupted at the end of vacation from the beginning; otherwise, he takes the next working vacation. To avoid confusion, the different time epochs at which events occur are shown in Figure 1. Finally, we assume the service, arrival, and the vacations are mutually independent.

To describe the system state, the following random variables are introduced:

$N(n^+)$: queue length (including the customer in service) at epoch n^+;

$X(n^+)$: remaining service time of the customer being served at epoch n^+;

$\eta(n^+) = \{0$, server is in working vacation at epoch n^+; 1, server is in normal busy period at epoch $n^+\}$.

Let $M(n^+) = \{N(n^+), \eta(n^+), X(n^+)\}$; then $M(n^+)$ is a Markov process with state space $S = \{(0,0)\} \cup \{(i,j,k), 1 \le i \le N;\ j = 0,1;\ k \ge 1\}$. We define the joint probabilities by

$$\begin{aligned} P_{0,0}(n^+) &= P\{N(n^+) = 0, \eta(n^+) = 0\}, \\ P_{i,j}(n^+,k) &= P\{N(n^+) = i, \eta(n^+) = j, X(n^+) = k\}, \\ &\qquad 1 \le i \le N;\ j = 0,1;\ k \ge 1. \end{aligned} \tag{1}$$

As $n \to +\infty$, the mentioned probabilities above are denoted by $P_{0,0}$ and $P_{i,j}(k)$, $i = 1, \ldots, N$; $j = 0, 1$; $k \ge 1$, respectively.

3. The Queue-Length Solutions in the Form of Formula

In this section, by combining embedded Markov chain and supplementary variable methods, the queue-length distribution at arbitrary epoch is obtained.

3.1. Steady State Queue Length at an Arbitrary Epoch n^+. Firstly, we develop the usual Chapman-Kolmogorov (C-K) difference equations by regarding the remaining service time as the supplementary variable. Generally, using one-step transition probability, the system can get to the state of $(i,0,k)$ (assume at epoch n) from four types of the up-step state (assume at epoch $n-1$): one is from state of $(i-1,0,k+1)$ by probability $\lambda_2\bar{v}$; that is, the system is in vacation and one customer arrives during time interval $((n-1)^+, n^+]$; the second is from state of $(i+1,0,1)$ by probability $\bar{\lambda}_2\bar{v}g_k^{(v)}$; that is, the system is in vacation and no customer arrives during time interval $((n-1)^+, n^+]$, and the customer being served at epoch $(n-1)^+$ completes his service at n^+; the third is from state of $(i,0,1)$ by probability $\lambda_2\bar{v}g_k^{(v)}$; that is, the system is in vacation and one customer arrives during time interval $((n-1)^+, n^+]$, and the customer being served at epoch $(n-1)^+$ completes his service at n^+; the last one is from state of $(i,0,k+1)$ by probability $\bar{\lambda}_2\bar{v}$, that is, the system is in vacation and no customer arrives during time interval $((n-1)^+, n^+]$. So, under steady state ($n \to +\infty$), we get

$$P_{i,0}(k) = \lambda_2\bar{v}P_{i-1,0}(k+1) + \bar{\lambda}_2\bar{v}g_k^{(v)}P_{i+1,0}(1) + \lambda_2\bar{v}g_k^{(v)}P_{i,0}(1) + \bar{\lambda}_2\bar{v}P_{i,0}(k+1), \quad 2 \le i \le N-2. \tag{2}$$

Let $i = 0, 1, N-1, N$, respectively; we get the degenerate equations

$$P_{0,0} = \bar{\lambda}_2 P_{0,0} + \bar{\lambda}_2 P_{1,0}(1) + \bar{\lambda}_1 P_{1,1}(1), \tag{3}$$

$$P_{1,0}(k) = \bar{\lambda}_2\bar{v}P_{1,0}(k+1) + \lambda_2\bar{v}g_k^{(v)}P_{0,0} + \lambda_2\bar{v}g_k^{(v)}P_{1,0}(1) + \bar{\lambda}_2\bar{v}g_k^{(v)}P_{2,0}(1), \tag{4}$$

$$P_{N-1,0}(k) = \lambda_2\bar{v}P_{N-2,0}(k+1) + \bar{v}g_k^{(v)}P_{N,0}(1) + \lambda_2\bar{v}g_k^{(v)}P_{N-1,0}(1) + \bar{\lambda}_2\bar{v}P_{N-1,0}(k+1), \tag{5}$$

$$P_{N,0}(k) = \lambda_2\bar{v}P_{N-1,0}(k+1) + \bar{v}P_{N,0}(k+1). \tag{6}$$

Similarly, we also obtain the C-K equations when system is in normal busy period:

$$\begin{aligned}
P_{1,1}(k) &= \lambda_2 v g_k^{(b)} P_{0,0} + \lambda_1 g_k^{(b)} P_{1,1}(1) + \bar{\lambda}_1 P_{1,1}(k+1) + \bar{\lambda}_1 g_k^{(b)} P_{2,1}(1) \\
&\quad + \lambda_2 v g_k^{(b)} P_{1,0}(1) + \bar{\lambda}_2 v g_k^{(b)} P_{2,0}(1) + \bar{\lambda}_2 v g_k^{(b)} \sum_{t=2}^{\infty} P_{1,0}(t), \\
P_{i,1}(k) &= \lambda_1 g_k^{(b)} P_{i,1}(1) + \bar{\lambda}_1 g_k^{(b)} P_{i+1,1}(1) + \bar{\lambda}_1 P_{i,1}(k+1) + \lambda_1 P_{i-1,1}(k+1) \\
&\quad + \lambda_2 v g_k^{(b)} P_{i,0}(1) + \bar{\lambda}_2 v g_k^{(b)} P_{i+1,0}(1) + \bar{\lambda}_2 v g_k^{(b)} \sum_{t=2}^{\infty} P_{i,0}(t) \\
&\quad + \lambda_2 v g_k^{(b)} \sum_{t=2}^{\infty} P_{i-1,0}(t), \quad 2 \le i \le N-2, \\
P_{N-1,1}(k) &= \lambda_1 g_k^{(b)} P_{N-1,1}(1) + g_k^{(b)} P_{N,1}(1) + \bar{\lambda}_1 P_{N-1,1}(k+1) + \lambda_1 P_{N-2,1}(k+1) \\
&\quad + \lambda_2 v g_k^{(b)} P_{N-1,0}(1) + v g_k^{(b)} P_{N,0}(1) + \bar{\lambda}_2 v g_k^{(b)} \sum_{t=2}^{\infty} P_{N-1,0}(t) \\
&\quad + \lambda_2 v g_k^{(b)} \sum_{t=2}^{\infty} P_{N-2,0}(t), \\
P_{N,1}(k) &= P_{N,1}(k+1) + \lambda_1 P_{N-1,1}(k+1) + v g_k^{(b)} \sum_{t=2}^{\infty} P_{N,0}(t) \\
&\quad + \lambda_2 v g_k^{(b)} \sum_{t=2}^{\infty} P_{N-1,0}(t).
\end{aligned} \tag{7}$$

Define the z-transforms of $P_{i,0}(k)$ and $P_{i,1}(k)$, respectively:

$$\begin{aligned}
P_{i,0}^*(u) &= \sum_{k=1}^{\infty} P_{i,0}(k)u^k, \quad i = 1,2,\dots,N, \\
P_{i,1}^*(u) &= \sum_{k=1}^{\infty} P_{i,1}(k)u^k, \quad i = 1,2,\dots,N.
\end{aligned} \tag{8}$$

We propose the following notations:

$$\begin{aligned}
P_{i,0} &= \lim_{u\to 1^-} P_{i,0}^*(u) = P_{i,0}^*(1), \quad i = 1,2,\dots,N; \\
P_{i,1} &= \lim_{u\to 1^-} P_{i,1}^*(u) = P_{i,1}^*(1), \quad i = 1,2,\dots,N.
\end{aligned} \tag{9}$$

Multiplying (2) and (4)–(7) by u^k and summing over k from 1 to ∞, we obtain after some simplification

$$\left(1-\frac{\overline{\lambda}_2\overline{v}}{u}\right)P^*_{1,0}(u)=-\overline{\lambda}_2\overline{v}P_{1,0}(1)+\overline{v}G^{(v)}(u)\left[\lambda_2P_{0,0}+\lambda_2P_{1,0}(1)+\overline{\lambda}_2P_{2,0}(1)\right],$$

$$\left(1-\frac{\overline{\lambda}_2\overline{v}}{u}\right)P^*_{i,0}(u)=\frac{\lambda_2\overline{v}}{u}P^*_{i-1,0}(u)-\lambda_2\overline{v}P_{i-1,0}(1)-\overline{\lambda}_2\overline{v}P_{i,0}(1)+\overline{v}G^{(v)}(u)\left[\overline{\lambda}_2P_{i+1,0}(1)+\lambda_2P_{i,0}(1)\right],\quad i=2,3,\ldots,N-2,$$

$$\left(1-\frac{\overline{\lambda}_2\overline{v}}{u}\right)P^*_{N-1,0}(u)=\frac{\lambda_2\overline{v}}{u}P^*_{N-2,0}(u)-\lambda_2\overline{v}P_{N-2,0}(1)-\overline{\lambda}_2\overline{v}P_{N-1,0}(1)+\overline{v}G^{(v)}(u)\cdot\left[P_{N,0}(1)+\lambda_2P_{N-1,0}(1)\right],$$

$$\left(1-\frac{\overline{v}}{u}\right)P^*_{N,0}(u)=\frac{\lambda_2\overline{v}}{u}P^*_{N-1,0}(u)-\lambda_2\overline{v}P_{N-1,0}(1)-\overline{v}P_{N,0}(1),$$

$$\left(1-\frac{\overline{\lambda}_1}{u}\right)P^*_{1,1}(u)=-\lambda_1P_{1,1}(1)+G^{(b)}(u)\left[\lambda_2vP_{0,0}+\lambda_1P_{1,1}(1)+\overline{\lambda}_1P_{2,1}(1)+\lambda_2vP_{1,0}(1)+\overline{\lambda}_2vP_{2,0}(1)+\overline{\lambda}_2v\sum_{t=2}^{\infty}P_{1,0}(t)\right],$$

$$\left(1-\frac{\overline{\lambda}_1}{u}\right)P^*_{i,1}(u)=\frac{\lambda_1}{u}P^*_{i-1,1}(u)-\lambda_1P_{i-1,1}(1)-\overline{\lambda}_1P_{i,1}(1)+G^{(b)}(u)\left[\lambda_1P_{i,1}(1)+\overline{\lambda}_1P_{i+1,1}(1)+\lambda_2vP_{i,0}(1)+\overline{\lambda}_2vP_{i+1,0}(1)+\overline{\lambda}_2v\sum_{t=2}^{\infty}P_{i,0}(t)+\lambda_2v\sum_{t=2}^{\infty}P_{i-1,0}(t)\right],\quad i=2,3,\ldots,N-2,$$

$$\left(1-\frac{\overline{\lambda}_1}{u}\right)P^*_{N-1,1}(u)=\frac{\lambda_1}{u}P^*_{N-2,1}(u)-\lambda_1P_{N-2,1}(1)-\overline{\lambda}_1P_{N-1,1}(1)+G^{(b)}(u)\left[\lambda_1P_{N-1,1}(1)+P_{N,1}(1)+\lambda_2vP_{N-1,0}(1)+vP_{N,0}(1)+\overline{\lambda}_2v\sum_{t=2}^{\infty}P_{N-1,0}(t)+\lambda_2v\sum_{t=2}^{\infty}P_{N-2,0}(t)\right],$$

$$\left(1-\frac{1}{u}\right)P^*_{N,1}(u)=\frac{\lambda_1}{u}P^*_{N-1,1}(u)-\lambda_1P_{N-1,1}(1)-P_{N,1}(1)+vG^{(b)}(u)\left[\sum_{t=2}^{\infty}P_{N,0}(t)+\lambda_2\sum_{t=2}^{\infty}P_{N-1,0}(t)\right].\tag{10}$$

Adding (10) over all possible values of i ($i=1,2,\ldots,N$) and using (3), we get

$$\left(1-\frac{\overline{v}}{u}\right)\sum_{i=1}^{N}P^*_{i,0}(u)=\overline{v}G^{(v)}(u)\left[\sum_{i=1}^{N}P_{i,0}(1)+\overline{\lambda}_1P_{1,1}(1)\right]-\overline{v}\sum_{i=1}^{N}P_{i,0}(1),\tag{11}$$

$$\sum_{i=1}^{N}P^*_{i,1}(u)=\frac{u}{u-1}\left\{G^{(b)}(u)\cdot\left[\sum_{i=1}^{N}P_{i,1}(1)+v\sum_{i=1}^{N}P_{i,0}-\overline{v}\overline{\lambda}_1P_{1,1}(1)\right]-\sum_{i=1}^{N}P_{i,1}(1)\right\}.\tag{12}$$

Setting $u=1$ in (11), we have

$$\overline{\lambda}_1P_{1,1}(1)=\frac{v}{\overline{v}}\sum_{i=1}^{N}P_{i,0}.\tag{13}$$

Substituting (13) into (11) and letting $u=\overline{v}$ lead to

$$\sum_{i=1}^{N}P_{i,0}=\frac{\overline{v}\left[1-G^{(v)}(\overline{v})\right]}{vG^{(v)}(\overline{v})}\sum_{i=1}^{N}P_{i,0}(1).\tag{14}$$

Calculating limit on both sides of (12) as $u\rightarrow1^-$ after substituting (13) into (12) yields

$$\sum_{i=1}^{N}P_{i,1}=\alpha_1\sum_{i=1}^{N}P_{i,1}(1),\tag{15}$$

where $\alpha_1=E[\chi^{(b)}]$, $i=1,2,\ldots$.

Substituting (14) and (15) into the normalization condition, $P_{0,0}+\sum_{i=1}^{N}P_{i,0}+\sum_{i=1}^{N}P_{i,1}=1$, we get

$$P_{0,0}+\frac{\overline{v}\left[1-G^{(v)}(\overline{v})\right]}{vG^{(v)}(\overline{v})}\sum_{i=1}^{N}P_{i,0}(1)+\alpha_1\sum_{i=1}^{N}P_{i,1}(1)=1.\tag{16}$$

From (3), (13), (14), and (16), the terms of $\sum_{i=1}^{N}P_{i,0}(1)$ and $\sum_{i=1}^{N}P_{i,1}(1)$ can be expressed by $P_{0,0}$ and $P_{1,0}(1)$ as follows:

$$\sum_{i=1}^{N}P_{i,0}(1)=\frac{\left[\lambda_2P_{0,0}-\overline{\lambda}_2P_{1,0}(1)\right]G^{(v)}(\overline{v})}{1-G^{(v)}(\overline{v})},$$
$$\sum_{i=1}^{N}P_{i,1}(1)=\frac{1}{\alpha_1v}\left[v-\left(v+\lambda_2\overline{v}\right)P_{0,0}+\overline{\lambda}_2\overline{v}P_{1,0}(1)\right].\tag{17}$$

Let r^+, $r = 1, 2, \ldots$ be the departure epoch and let L_r denote the queue length immediately after the epoch r^+. $P_{i,j}^{d_r} = P\{L(r^+) = i;\ \eta(r^+) = j\}$ denotes the joint probability of $L(r^+)$ and $\eta(r^+)$ with steady state $P_{i,j}^{d} = \lim_{r\to+\infty} P_{i,j}^{d_r}$, $0 \le i \le N-1$; $j = 0, 1$.

For $P_{i,0}^{d_r}$, $0 \le i \le N-1$, we have

$$\begin{aligned}
P_{0,0}^{d_r} &= P\{L(r^+)=0;\ \eta(r^+)=0\} = \overline{\lambda}_2 P\{N((r-1)^+) \\
&= 1;\ \eta((r-1)^+)=0 \mid X((r-1)^+)=1\} \\
&+ \overline{\lambda}_1 P\{N((r-1)^+)=1;\ \eta((r-1)^+) \\
&= 1 \mid X((r-1)^+)=1\} = \overline{\lambda}_2 \frac{P_{1,0}((r-1)^+,1)}{P\{X((r-1)^+)=1\}} \\
&+ \overline{\lambda}_1 \frac{P_{1,1}((r-1)^+,1)}{P\{X((r-1)^+)=1\}}, \\
P_{i,0}^{d_r} &= P\{L(r^+)=i;\ \eta(r^+)=0\} = \overline{v}\overline{\lambda}_2 P\{N((r-1)^+) \\
&= i+1;\ \eta((r-1)^+)=0 \mid X((r-1)^+)=1\} \\
&+ \overline{v}\lambda_2 P\{N((r-1)^+)=i;\ \eta((r-1)^+) \\
&= 0 \mid X((r-1)^+)=1\} = \overline{v}\overline{\lambda}_2 \frac{P_{i+1,0}((r-1)^+,1)}{P\{X((r-1)^+)=1\}} \\
&+ \overline{v}\lambda_2 \frac{P_{i,0}((r-1)^+,1)}{P\{X((r-1)^+)=1\}}, \quad 1 \le i \le N-2, \\
P_{N-1,0}^{d_r} &= P\{L(r^+)=N-1;\ \eta(r^+)=0\} \\
&= \overline{v} P\{N((r-1)^+)=N;\ \eta((r-1)^+) \\
&= 0 \mid X((r-1)^+)=1\} + \overline{v}\lambda_2 P\{N((r-1)^+)=N \\
&- 1;\ \eta((r-1)^+)=0 \mid X((r-1)^+)=1\} = \overline{v} \\
&\cdot \frac{P_{N,0}((r-1)^+,1)}{P\{X((r-1)^+)=1\}} + \overline{v}\lambda_2 \frac{P_{N-1,0}((r-1)^+,1)}{P\{X((r-1)^+)=1\}}.
\end{aligned} \tag{18}$$

Calculating limit on both sides of (18) as $r \to +\infty$ leads to

$$P_{0,0}^{d} = \frac{\overline{\lambda}_2 P_{1,0}(1) + \overline{\lambda}_1 P_{1,1}(1)}{\sum_{i=1}^{N} P_{i,0}(1) + \sum_{i=1}^{N} P_{i,1}(1)}, \tag{19}$$

$$P_{i,0}^{d} = \frac{\overline{v}\overline{\lambda}_2 P_{i+1,0}(1) + \overline{v}\lambda_2 P_{i,0}(1)}{\sum_{i=1}^{N} P_{i,0}(1) + \sum_{i=1}^{N} P_{i,1}(1)}, \quad 1 \le i \le N-2, \tag{20}$$

$$P_{N-1,0}^{d} = \frac{\overline{v} P_{N,0}(1) + \overline{v}\lambda_2 P_{N-1,0}(1)}{\sum_{i=1}^{N} P_{i,0}(1) + \sum_{i=1}^{N} P_{i,1}(1)}. \tag{21}$$

Similarly, for $P_{i,1}^{d}$, $1 \le i \le N-1$, we have

$$\begin{aligned}
&P_{i,1}^{d} \\
&\quad = \frac{\overline{\lambda}_1 P_{i+1,1}(1) + \lambda_1 P_{i,1}(1) + \overline{\lambda}_2 v P_{i+1,0}(1) + \lambda_2 v P_{i,0}(1)}{\sum_{i=1}^{N} P_{i,0}(1) + \sum_{i=1}^{N} P_{i,1}(1)}, \\
&\qquad 1 \le i \le N-2,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&P_{N-1,1}^{d} \\
&\quad = \frac{P_{N,1}(1) + \lambda_1 P_{N-1,1}(1) + v P_{N,0}(1) + \lambda_2 v P_{N-1,0}(1)}{\sum_{i=1}^{N} P_{i,0}(1) + \sum_{i=1}^{N} P_{i,1}(1)}.
\end{aligned} \tag{23}$$

From (3), (17), and (19), we get

$$\sum_{i=1}^{N} P_{i,0}(1) + \sum_{i=1}^{N} P_{i,1}(1) = \frac{\lambda_2 P_{0,0}}{P_{0,0}^{d}}, \tag{24}$$

$$P_{1,0}(1) = \frac{1 + P_{0,0}\left[\lambda_2 \alpha_1 \left(\Delta - 1/P_{0,0}^{d}\right) - 1\right]}{\overline{\lambda}_2 \alpha_1 \Delta}, \tag{25}$$

$$P_{1,1}(1) = \frac{P_{0,0}\left(1 + \lambda_2 \alpha_1 / P_{0,0}^{d}\right) - 1}{\overline{\lambda}_1 \alpha_1 \Delta}, \tag{26}$$

where $\Delta = G^{(v)}(\overline{v})/(1 - G^{(v)}(\overline{v})) - \overline{v}/\alpha_1 v$.

In order to find the solution of $P_{i,j}$, we would adopt matrix equations to express (20)–(23). Firstly, some vector notations are introduced as follows:

$$\begin{aligned}
\mathbb{P}_0(1) &= \left[P_{2,0}(1), P_{3,0}(1), \ldots, P_{N,0}(1)\right]^T, \\
\mathbb{P}_1(1) &= \left[P_{2,1}(1), P_{3,1}(1), \ldots, P_{N,1}(1)\right]^T, \\
\mathbb{P}_0^{d} &= \left[P_{1,0}^{d}, P_{2,0}^{d}, \ldots, P_{N-1,0}^{d}\right]^T, \\
\mathbb{P}_1^{d} &= \left[P_{1,1}^{d}, P_{2,1}^{d}, \ldots, P_{N-1,1}^{d}\right]^T, \\
\mathbb{P}_0 &= \left[P_{1,0}, P_{2,0}, \ldots, P_{N,0}\right]^T, \\
\mathbb{P}_1 &= \left[P_{0,0}, P_{1,1}, \ldots, P_{N-1,1}\right]^T.
\end{aligned} \tag{27}$$

Using (24), the group equations of (20)-(21) and (22)-(23) are expressed in matrix form, respectively:

$$\mathbb{A}_{N-1}\mathbb{P}_0(1) = \frac{\lambda_2 P_{0,0}}{\overline{v} P_{0,0}^{d}} \cdot \mathbb{P}_0^{d} - \mathbf{X}_1, \tag{28}$$

$$v \cdot \mathbb{A}_{N-1}\mathbb{P}_0(1) + \mathbb{B}_{N-1}\mathbb{P}_1(1) = \frac{\lambda_2 P_{0,0}}{P_{0,0}^{d}} \cdot \mathbb{P}_1^{d} - \mathbf{X}_2, \tag{29}$$

where

$$\begin{aligned}
\mathbf{X}_1 &= \left[\lambda_2 P_{1,0}(1), 0, \ldots, 0\right]^T, \\
\mathbf{X}_2 & \\
&= \left[\frac{\lambda_1 \lambda_2 P_{0,0}}{\overline{\lambda}_1} + \left(\lambda_2 v - \frac{\lambda_1 \overline{\lambda}_2}{\overline{\lambda}_1}\right) P_{1,0}(1), 0, \ldots, 0\right]^T;
\end{aligned} \tag{30}$$

$P_{1,0}(1)$ is determined by (28):

$$\mathbb{A}_{N-1} = \begin{pmatrix} \overline{\lambda}_2 & & & & & \\ \lambda_2 & \overline{\lambda}_2 & & & & \\ & \lambda_2 & \overline{\lambda}_2 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_2 & \overline{\lambda}_2 & \\ & & & & \lambda_2 & 1 \end{pmatrix}_{N-1},$$
$$\mathbb{B}_{N-1} = \begin{pmatrix} \overline{\lambda}_1 & & & & & \\ \lambda_1 & \overline{\lambda}_1 & & & & \\ & \lambda_1 & \overline{\lambda}_1 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_1 & \overline{\lambda}_1 & \\ & & & & \lambda_1 & 1 \end{pmatrix}_{N-1}. \quad (31)$$

Solving the simultaneous matrix equations (28) and (29) leads to

$$\mathbb{P}_0(1) = A_{N-1}^{-1}\left[\frac{\lambda_2 P_{0,0}}{\overline{v}P_{0,0}^d}\cdot\mathbb{P}_0^d - \mathbf{X}_1\right], \quad (32)$$

$$\mathbb{P}_1(1) = B_{N-1}^{-1}\left[\frac{\lambda_2 P_{0,0}}{\overline{v}P_{0,0}^d}\left(\overline{v}\mathbb{P}_1^d - v\mathbb{P}_0^d\right) + v\mathbf{X}_1 - \mathbf{X}_2\right]. \quad (33)$$

Let $u = 1$ in (10); we have

$$\left(1-\overline{\lambda}_2\overline{v}\right)P_{1,0} = \left(2\lambda_2-1\right)\overline{v}P_{1,0}(1) + \overline{\lambda}_2\overline{v}P_{2,0}(1) + \lambda_2\overline{v}P_{0,0}, \quad (34)$$

$$\begin{aligned}\left(1-\overline{\lambda}_2\overline{v}\right)P_{i,0} - \lambda_2\overline{v}P_{i-1,0} &= -\lambda_2\overline{v}P_{i-1,0}(1) + \left(2\lambda_2-1\right)\overline{v}P_{i,0}(1) \\ &\quad + \overline{v}\overline{\lambda}_2P_{i+1,0}(1), \quad i = 2,3,\dots,N-2,\end{aligned} \quad (35)$$

$$\begin{aligned}\left(1-\overline{\lambda}_2\overline{v}\right)P_{N-1,0} - \lambda_2\overline{v}P_{N-2,0} &= -\lambda_2\overline{v}P_{N-2,0}(1) + \left(2\lambda_2-1\right)\overline{v}P_{N-1,0}(1) \\ &\quad + \overline{v}P_{N,0}(1),\end{aligned} \quad (36)$$

$$vP_{N,0} - \lambda_2\overline{v}P_{N-1,0} = -\lambda_2\overline{v}P_{N-1,0}(1) - \overline{v}P_{N,0}(1), \quad (37)$$

$$\begin{aligned}\lambda_1P_{1,1} - \lambda_2vP_{0,0} &= \overline{\lambda}_2vP_{1,0} + \left(2\lambda_1-1\right)P_{1,1}(1) + \overline{\lambda}_1P_{2,1}(1) \\ &\quad + \left(2\lambda_2-1\right)vP_{1,0}(1) + \overline{\lambda}_2vP_{2,0}(1),\end{aligned} \quad (38)$$

$$\begin{aligned}\lambda_1P_{i,1} - \lambda_1P_{i-1,1} &= \overline{\lambda}_2vP_{i,0} + \lambda_2vP_{i-1,0} - \lambda_1P_{i-1,1}(1) \\ &\quad + \left(2\lambda_1-1\right)P_{i,1}(1) + \overline{\lambda}_1P_{i+1,1}(1) - \lambda_2vP_{i-1,0}(1) \\ &\quad + \left(2\lambda_2-1\right)vP_{i,0}(1) + \overline{\lambda}_2vP_{i+1,0}(1), \\ &\qquad i = 2,3,\dots,N-2,\end{aligned} \quad (39)$$

$$\begin{aligned}\lambda_1P_{N-1,1} - \lambda_1P_{N-2,1} &= \overline{\lambda}_2vP_{N-1,0} + \lambda_2vP_{N-2,0} - \lambda_1P_{N-2,1}(1) \\ &\quad + \left(2\lambda_1-1\right)P_{N-1,1}(1) + P_{N,1}(1) \\ &\quad - \lambda_2vP_{N-2,0}(1) + \left(2\lambda_2-1\right)vP_{N-1,0}(1) \\ &\quad + vP_{N,0}(1),\end{aligned} \quad (40)$$

$$\begin{aligned}-\lambda_1P_{N-1,1} &= vP_{N,0} + \lambda_2vP_{N-1,0} - \lambda_1P_{N-1,1}(1) - P_{N,1}(1) \\ &\quad - \lambda_2vP_{N-1,0}(1) - vP_{N,0}(1).\end{aligned} \quad (41)$$

The equations of (34)–(37) and (38)–(41) are expressed by the following matrix forms, respectively:

$$\mathbb{H}_N\mathbb{P}_0 = \mathbb{F}_N\begin{pmatrix}P_{1,0}(1)\\ \mathbb{P}_0(1)\end{pmatrix}\cdot\overline{v} + \lambda_2\overline{v}\mathbf{X}_3, \quad (42)$$

$$\mathbb{C}_N\mathbb{P}_1 = \mathbb{A}_N\mathbb{P}_0\cdot v + \mathbb{E}_N\begin{pmatrix}P_{1,1}(1)\\ \mathbb{P}_1(1)\end{pmatrix} + \mathbb{F}_N\begin{pmatrix}P_{1,0}(1)\\ \mathbb{P}_0(1)\end{pmatrix}\cdot v, \quad (43)$$

where $P_{1,0}(1)$ and $P_{1,1}(1)$ are determined by (25) and (26), respectively:

$$\mathbf{X}_3 = \left[P_{0,0}, 0, \dots, 0\right]^T,$$

$$\mathbb{C}_N = \begin{pmatrix} -\lambda_2 v & \lambda_1 & & & \\ & -\lambda_1 & \lambda_1 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_1 & \lambda_1 \\ & & & & -\lambda_1 \end{pmatrix}_N,$$

$$\mathbb{H}_N = \begin{pmatrix} 1-\overline{\lambda}_2\overline{v} & & & & \\ -\lambda_2\overline{v} & 1-\overline{\lambda}_2\overline{v} & & & \\ & \ddots & \ddots & & \\ & & -\lambda_2\overline{v} & 1-\overline{\lambda}_2\overline{v} & \\ & & & -\lambda_2\overline{v} & v \end{pmatrix}_N,$$

$$\mathbb{F}_N = \begin{pmatrix} 2\lambda_2 - 1 & \overline{\lambda}_2 & & & & \\ -\lambda_2 & 2\lambda_2 - 1 & \overline{\lambda}_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\lambda_2 & 2\lambda_2 - 1 & \overline{\lambda}_2 & \\ & & & -\lambda_2 & 2\lambda_2 - 1 & 1 \\ & & & & -\lambda_2 & -1 \end{pmatrix}_N,$$

$$\mathbb{E}_N = \begin{pmatrix} 2\lambda_1 - 1 & \overline{\lambda}_1 & & & & \\ -\lambda_1 & 2\lambda_1 - 1 & \overline{\lambda}_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\lambda_1 & 2\lambda_1 - 1 & \overline{\lambda}_1 & \\ & & & -\lambda_1 & 2\lambda_1 - 1 & 1 \\ & & & & -\lambda_1 & -1 \end{pmatrix}_N. \quad (44)$$

Substituting (25)-(26) and (32)-(33) into (42)-(43), respectively, and solving the simultaneous matrix equations (42)-(43), we have

$$\mathbb{P}_0 = \overline{\nu}\mathbb{H}_N^{-1}\mathbb{F}_N\mathbb{D}_N^{-1} \begin{pmatrix} \dfrac{1 + P_{0,0}\lambda_2\alpha_1\left(\Delta - 1/P_{0,0}^d\right) - P_{0,0}}{\overline{\lambda}_2\alpha_1\Delta} \\ \dfrac{\lambda_2 P_{0,0}}{\overline{\nu}P_{0,0}^d}\cdot\mathbb{P}_0^d - \mathbf{X}_1 \end{pmatrix} + \lambda_2\overline{\nu}\mathbb{H}_N^{-1}\mathbf{X}_3, \quad (45)$$

$$\mathbb{P}_1 = \mathbb{C}_N^{-1}\left(\nu\overline{\nu}\mathbb{A}_N\mathbb{H}_N^{-1} + \nu\mathbb{I}_N\right) \cdot \mathbb{F}_N\mathbb{D}_N^{-1} \begin{pmatrix} \dfrac{1 + P_{0,0}\lambda_2\alpha_1\left(\Delta - 1/P_{0,0}^d\right) - P_{0,0}}{\overline{\lambda}_2\alpha_1\Delta} \\ \dfrac{\lambda_2 P_{0,0}}{\overline{\nu}P_{0,0}^d}\cdot\mathbb{P}_0^d - \mathbf{X}_1 \end{pmatrix}$$
$$+ \mathbb{C}_N^{-1}\mathbb{E}_N\mathbb{M}_N^{-1} \begin{pmatrix} \dfrac{P_{0,0}\left(1 + \lambda_2\alpha_1/P_{0,0}^d\right) - 1}{\overline{\lambda}_1\alpha_1\Delta} \\ \dfrac{\lambda_2 P_{0,0}}{\overline{\nu}P_{0,0}^d}\left(\overline{\nu}\mathbb{P}_1^d - \nu\mathbb{P}_0^d\right) + \nu\mathbf{X}_1 - \mathbf{X}_2 \end{pmatrix} + \lambda_2\nu\overline{\nu}\mathbb{C}_N^{-1}\mathbb{A}_N\mathbb{H}_N^{-1}\mathbf{X}_3, \quad (46)$$

where $\mathbb{D}_N = \begin{pmatrix} 1 & \\ & \mathbb{A}_{N-1} \end{pmatrix}_N$, $\mathbb{M}_N = \begin{pmatrix} 1 & \\ & \mathbb{B}_{N-1} \end{pmatrix}_N$, $\mathbb{I}_N$ is an identity matrix of degree N, and the remaining notations are consistent with the previous ones.

One may note that if we could obtain the queue-length distribution at departure epoch, $\mathbb{P}_0^d$, $\mathbb{P}_1^d$, and $P_{0,0}^d$, the arbitrary epoch probabilities, $P_{i,0}$ $(0 \le i \le N)$ and $P_{i,1}$ $(1 \le i \le N)$, can be derived from (45) and (46). Its main steps are introduced as follows:

(a) Gain the probabilities of $P_{1,1}(1)$ and $\mathbb{P}_1(1)$ expressed by $P_{0,0}$ after substituting $P_{0,0}^d$, $\mathbb{P}_0^d$, and $\mathbb{P}_1^d$ into (26) and (33), respectively.

(b) Obtain the arbitrary epoch probabilities of $\mathbb{P}_0$ and $\mathbb{P}_1$ expressed by $P_{0,0}$ from (45) and (46) as well as the arbitrary epoch probability of $P_{N,1}$ expressed by $P_{0,0}$ from the normalization condition $P_{0,0} + \sum_{i=1}^{N} P_{i,0} + \sum_{i=1}^{N} P_{i,1} = 1$.

(c) Get the value of $P_{0,0}$ after substituting $P_{1,1}(1)$, $\mathbb{P}_1(1)$, $\mathbb{P}_1$, and $P_{N,1}$ into (15).

(d) Achieve the values of $\mathbb{P}_0$ and $\mathbb{P}_1$ after substituting the value of $P_{0,0}$ back into (45) and (46), as well as the value of $P_{N,1}$ through normalization condition.

So, in the following subsection, using the embedded Markov chain technique, we investigate the queue-length distribution at a departure epoch.

3.2. Steady State Queue Length at a Departure Epoch n^+. Let L_r denote the queue length immediately after the departure epoch r^+ and define the variable of J_r:

$$J_r = \begin{cases} 0, & \text{the departure epoch } r^+ \text{ locates in a working vacation,} \\ 1, & \text{the departure epoch } r^+ \text{ locates in a normal busy period.} \end{cases} \quad (47)$$

Thus, $\{(L_r, J_r), r \ge 1\}$ is a two-dimensional Markov chain with state space $\Omega = \{(i,s), 0 \le i \le N-1;\ s = 0, 1\}$. Denote the one step transition probability by $p_{(i,s_1)(j,s_2)} = P\{L_{r+1} = j, J_{r+1} = s_2 \mid L_r = i, J_r = s_1\}$ and define $S_i^{(1)} = \sum_{k=1}^{i}\tau_k^{(1)}$, $S_0^{(1)} = 0$; $S_i^{(2)} = \sum_{k=1}^{i}\tau_k^{(2)}$, $S_0^{(2)} = 0$, where the random variables of $\tau_k^{(i)}$, $i = 1, 2$, denote the arrival intervals with the same distribution as $\tau^{(i)}$, $i = 1, 2$. Then the one step transition probability matrix (TPM), denoted by $\mathbb{Q} = (p_{(i,s_1)(j,s_2)})_{2N-1}$, will be obtained in the following work.

(1) The state transition $(i,0) \to (j,0)$ $(1 \le i \le N-1;\ i-1 \le j \le N-1;\ j \ne 0)$ occurs if the length of vacation (or

remaining vacation) is greater than a service time in working vacation; that is, $\chi^{(v)} < V$, and $j+1-i$ customers arrive during the service time $\chi^{(v)}$; that is, $S^{(2)}_{j+1-i} \le \chi^{(v)} < S^{(2)}_{j+2-i}$; then it gets

$$\theta_{j+1-i} \triangleq p_{(i,0)(j,0)} = P\left\{\chi^{(v)} < V;\ S^{(2)}_{j+1-i} \le \chi^{(v)} < S^{(2)}_{j+2-i}\right\} = \sum_{m=\max\{1,j+1-i\}}^{\infty} g^{(v)}_m \binom{m}{j+1-i} \lambda_2^{j+1-i} \overline{\lambda}_2^{m-(j+1-i)} \overline{v}^m, \quad i-1 \le j \le N-2,\ j \ne 0, \tag{48}$$

$$\theta_{N-i} \triangleq p_{(i,0)(N-1,0)} = \sum_{k=N-i}^{\infty} P\left\{\chi^{(v)} < V;\ S^{(2)}_{k} \le \chi^{(v)} < S^{(2)}_{k+1}\right\} = \sum_{k=N-i}^{\infty}\sum_{m=k}^{\infty} g^{(v)}_m \binom{m}{k} \lambda_2^{k} \overline{\lambda}_2^{m-k} \overline{v}^m.$$

Similarly, for $1 \le j \le N-1$, we have

$$\xi_j \triangleq p_{(0,0)(j,0)} = \overline{v} \cdot P\left\{\chi^{(v)} < V;\ S^{(2)}_{j} \le \chi^{(v)} < S^{(2)}_{j+1}\right\} = \sum_{m=j}^{\infty} g^{(v)}_m \binom{m}{j} \lambda_2^{j} \overline{\lambda}_2^{m-j} \overline{v}^{m+1} = \overline{v} \cdot \theta_j, \quad 1 \le j \le N-2,$$

$$\xi_{N-1} \triangleq p_{(0,0)(N-1,0)} = \overline{v} \cdot \sum_{k=N-1}^{\infty} P\left\{\chi^{(v)} < V;\ S^{(2)}_{k} \le \chi^{(v)} < S^{(2)}_{k+1}\right\} = \sum_{k=N-1}^{\infty}\sum_{m=k}^{\infty} g^{(v)}_m \binom{m}{k} \lambda_2^{k} \overline{\lambda}_2^{m-k} \overline{v}^{m+1} = \overline{v} \cdot \theta_{N-1}. \tag{49}$$

(2) The state transition $(1,1) \to (0,0)$ occurs if no customer arrives during a service time in normal busy period; that is, $\tau^{(1)} > \chi^{(b)}$. So we have

$$\omega \triangleq p_{(1,1)(0,0)} = P\left\{\tau^{(1)} > \chi^{(b)}\right\} = \sum_{m=1}^{\infty} g^{(b)}_m \overline{\lambda}_1^m. \tag{50}$$

(3) The state transition $(1,0) \to (0,0)$ occurs if the length of vacation (or remaining vacation) is not less than a service time in working vacation and no customer arrives during the service time; that is, $\chi^{(v)} \le V$; $\tau^{(2)} > \chi^{(v)}$, or the length of vacation (or remaining vacation) is less than a service time in working vacation and no customer arrives during the vacation time and the normal service time follows the vacation; that is, $\chi^{(v)} > V$; $\tau^{(2)} > V$; $\tau^{(1)} > \chi^{(b)}$. Then we have

$$\delta_1 \triangleq p_{(1,0)(0,0)} = P\left\{\chi^{(v)} \le V;\ \tau^{(2)} > \chi^{(v)}\right\} + P\left\{\chi^{(v)} > V;\ \tau^{(2)} > V;\ \tau^{(1)} > \chi^{(b)}\right\} = \sum_{m=1}^{\infty} g^{(v)}_m \overline{\lambda}_2^m \overline{v}^{m-1} + \sum_{m=1}^{\infty} v\overline{v}^{m-1} P\left\{\chi^{(v)} > m\right\} \overline{\lambda}_2^m \sum_{n=1}^{\infty} g^{(b)}_n \overline{\lambda}_1^n. \tag{51}$$

(4) The state transition $(0,0) \to (0,0)$ occurs under two cases. First, the vacation does not end immediately after the first arrival epoch behind the previous departure and the length of vacation (or remaining vacation) is not less than a service time in working vacation and no customer arrives during the service time, or the length of vacation (or remaining vacation) is less than a service time in working vacation and no customer arrives during the vacation time and the service time in normal busy period follows the vacation. Second, the vacation just ends immediately after the first arrival epoch behind the previous departure (thus, the first arrival will obtain a normal service) and no arrival occurs during the normal service time. Then we have

$$\begin{aligned}\delta_0 &\triangleq p_{(0,0)(0,0)} \\ &= \overline{v} \cdot P\left\{\chi^{(v)} \le V;\ \tau^{(2)} > \chi^{(v)}\right\} + \overline{v} \\ &\quad \cdot P\left\{\chi^{(v)} > V;\ \tau^{(2)} > V;\ \tau^{(1)} > \chi^{(b)}\right\} + v \\ &\quad \cdot P\left\{\tau^{(1)} > \chi^{(b)}\right\} \\ &= \sum_{m=1}^{\infty} g^{(v)}_m \overline{\lambda}_2^m \overline{v}^m + \sum_{m=1}^{\infty} v\overline{v}^m P\left\{\chi^{(v)} > m\right\} \overline{\lambda}_2^m \sum_{n=1}^{\infty} g^{(b)}_n \overline{\lambda}_1^n \\ &\quad + \sum_{m=1}^{\infty} g^{(b)}_m \overline{\lambda}_1^m v = \overline{v} \cdot \delta_1 + v \cdot \omega.\end{aligned} \tag{52}$$

(5) The state transition $(i,1) \to (j,1)$ $(1 \le i \le N-1;\ i-1 \le j \le N-1;\ j \ne 0)$ occurs if $j+1-i$ customers arrive during a service time in normal busy period; that is, $S^{(1)}_{j+1-i} \le \chi^{(b)} < S^{(1)}_{j+2-i}$. It yields

$$\beta_{j+1-i} \triangleq p_{(i,1)(j,1)} = P\left\{S^{(1)}_{j+1-i} \le \chi^{(b)} < S^{(1)}_{j+2-i}\right\} = \sum_{m=\max\{1,j+1-i\}}^{\infty} g^{(b)}_m \binom{m}{j+1-i} \lambda_1^{j+1-i} \overline{\lambda}_1^{m-(j+1-i)}, \quad i-1 \le j \le N-2,\ j \ne 0, \tag{53}$$

$$\beta_{N-i} \triangleq p_{(i,1)(N-1,1)} = \sum_{k=N-i}^{\infty} P\left\{S^{(1)}_{k} \le \chi^{(b)} < S^{(1)}_{k+1}\right\} = \sum_{k=N-i}^{\infty}\sum_{m=k}^{\infty} g^{(b)}_m \binom{m}{k} \lambda_1^{k} \overline{\lambda}_1^{m-k}.$$

(6) The state transition $(i,0) \rightarrow (j,1)$ $(1 \le i \le N-1;\ i-1 \le j \le N-1;\ j \ne 0)$ occurs if the length of vacation (or remaining vacation) is less than a service time in working vacation; that is, $\chi^{(v)} > V$, and $j+1-i$ customers arrive during the vacation time and the service time in normal busy period (assuming that k customers arrive during the vacation and $j+1-i-k$ customers arrive during the service time in normal busy period); that is, $S_k^{(2)} \le V < S_{k+1}^{(2)}$; $S_{j+1-i-k}^{(1)} \le \chi^{(b)} < S_{j+2-i-k}^{(1)}$, or the length of vacation (or remaining vacation) equals a service time in working vacation and $j+1-i$ customers arrive during the service time; that is, $\chi^{(v)} = V$; $S_{j+1-i}^{(2)} \le \chi^{(v)} < S_{j+2-i}^{(2)}$. Then it leads to

$$\begin{aligned}
\gamma_{j+1-i} &\triangleq p_{(i,0)(j,1)} = \sum_{k=0}^{j+1-i} P\left\{\chi^{(v)} > V;\ S_k^{(2)} \le V < S_{k+1}^{(2)};\right. \\
&\quad \left. S_{j+1-i-k}^{(1)} \le \chi^{(b)} < S_{j+2-i-k}^{(1)}\right\} + P\left\{\chi^{(v)} = V;\ S_{j+1-i}^{(2)}\right. \\
&\quad \left. \le \chi^{(v)} < S_{j+2-i}^{(2)}\right\} = \sum_{k=0}^{j+1-i} \sum_{n=\max\{1,k\}}^{\infty} v\bar{v}^{n-1} P\left\{\chi^{(v)} > n\right\} \\
&\quad \cdot \binom{n}{k} \lambda_2^k \bar{\lambda}_2^{n-k} \cdot \beta_{j+1-i-k} + \theta_{j+1-i} \cdot \frac{v}{\bar{v}}, \\
&\qquad\qquad i-1 \le j \le N-2,\ j \ne 0, \\
\gamma_{N-i} &\triangleq p_{(i,0)(N-1,1)} = \sum_{k=N-i}^{\infty} P\left\{\chi^{(v)} > V;\right. \\
&\quad k \text{ customers arrive during the length of } \chi^{(b)} \\
&\quad \left. + V\right\} + \sum_{k=N-i}^{\infty} P\left\{\chi^{(v)} = V;\right. \\
&\quad \left. k \text{ customers arrive during the length of } \chi^{(v)}\right\} \\
&= \sum_{k=0}^{N-i-1} \sum_{r=N-i-k}^{\infty} P\left\{\chi^{(v)} > V;\ S_k^{(2)} \le V < S_{k+1}^{(2)};\ S_r^{(1)}\right. \\
&\quad \left. \le \chi^{(b)} < S_{r+1}^{(1)}\right\} + \sum_{k=N-i}^{\infty} P\left\{\chi^{(v)} \ge V;\ S_k^{(2)} \le V < S_{k+1}^{(2)}\right\} \\
&= \sum_{k=0}^{N-i-1} \sum_{n=\max\{1,k\}}^{\infty} v\bar{v}^{n-1} P\left\{\chi^{(v)} > n\right\} \binom{n}{k} \lambda_2^k \bar{\lambda}_2^{n-k} \\
&\quad \cdot \beta_{N-i-k} + \sum_{k=N-i}^{\infty} \sum_{m=k}^{\infty} v\bar{v}^{m-1} P\left\{\chi^{(v)} \ge m\right\} \\
&\quad \cdot \binom{m}{k} \lambda_2^k \bar{\lambda}_2^{m-k}.
\end{aligned} \tag{54}$$

Similarly, for $1 \le j \le N-1$, we have

$$\begin{aligned}
\eta_j &\triangleq p_{(0,0)(j,1)} = \bar{v} \cdot \sum_{k=0}^{j} P\left\{\chi^{(v)} > V;\ S_k^{(2)} \le V < S_{k+1}^{(2)};\ S_{j-k}^{(1)}\right. \\
&\quad \left. \le \chi^{(b)} < S_{j-k+1}^{(1)}\right\} + \bar{v} \cdot P\left\{\chi^{(v)} = V;\ S_j^{(2)} \le V < S_{j+1}^{(2)}\right\} + v \\
&\quad \cdot P\left\{S_j^{(1)} \le \chi^{(b)} < S_{j+1}^{(1)}\right\} = \sum_{k=0}^{j} \sum_{n=\max\{1,k\}}^{\infty} v\bar{v}^n P\left\{\chi^{(v)} > n\right\} \\
&\quad \cdot \binom{n}{k} \lambda_2^k \bar{\lambda}_2^{n-k} \cdot \sum_{m=\max\{1,j-k\}}^{\infty} g_m^{(b)} \binom{m}{j-k} \lambda_1^{j-k} \bar{\lambda}_1^{m-(j-k)} \\
&\quad + \sum_{m=\max\{1,j\}}^{\infty} v\bar{v}^m g_m^{(v)} \binom{m}{j} \lambda_2^j \bar{\lambda}_2^{m-j} + v \\
&\quad \cdot \sum_{m=j}^{\infty} g_m^{(b)} \binom{m}{j} \lambda_1^j \bar{\lambda}_1^{m-j} = \bar{v} \cdot \gamma_j + v \cdot \beta_j, \quad 1 \le j \le N-2, \\
\eta_{N-1} &\triangleq p_{(0,0)(N-1,1)} = \bar{v} \cdot \sum_{k=N-1}^{\infty} P\left\{\chi^{(v)} > V;\right. \\
&\quad \left. k \text{ customers arrive during the length of } \chi^{(b)} + V\right\} + \bar{v} \\
&\quad \cdot \sum_{k=N-1}^{\infty} P\left\{\chi^{(v)} = V;\right. \\
&\quad \left. k \text{ customers arrive during the length of } \chi^{(v)}\right\} + v \\
&\quad \cdot \sum_{k=N-1}^{\infty} P\left\{k \text{ customers arrive during the length of } \chi^{(b)}\right\} \\
&= \bar{v} \cdot \sum_{k=0}^{N-2} \sum_{r=N-1-k}^{\infty} P\left\{\chi^{(v)} > V;\ S_k^{(2)} \le V < S_{k+1}^{(2)};\ S_r^{(1)} \le \chi^{(b)}\right. \\
&\quad \left. < S_{k+1}^{(1)}\right\} + \bar{v} \cdot \sum_{k=N-1}^{\infty} P\left\{\chi^{(v)} \ge V;\ S_k^{(2)} \le V < S_{k+1}^{(2)}\right\} + v \\
&\quad \cdot \sum_{k=N-1}^{\infty} P\left\{S_k^{(1)} \le \chi^{(b)} < S_{k+1}^{(1)}\right\} \\
&= \sum_{k=0}^{N-2} \sum_{r=N-1-k}^{\infty} \sum_{n=\max\{1,k\}}^{\infty} v\bar{v}^n P\left\{\chi^{(v)} > n\right\} \binom{n}{k} \lambda_2^k \bar{\lambda}_2^{n-k} \\
&\quad \cdot \sum_{m=r}^{\infty} g_m^{(b)} \binom{m}{r} \lambda_1^r \bar{\lambda}_1^{m-r} + \sum_{k=N-1}^{\infty} \sum_{m=k}^{\infty} v\bar{v}^m P\left\{\chi^{(v)} \ge m\right\} \\
&\quad \cdot \binom{m}{k} \lambda_2^k \bar{\lambda}_2^{m-k} + v \cdot \sum_{k=N-1}^{\infty} \sum_{m=k}^{\infty} g_m^{(b)} \binom{m}{k} \lambda_1^k \bar{\lambda}_1^{m-k} = \bar{v} \\
&\quad \cdot \gamma_{N-1} + v \cdot \beta_{N-1}.
\end{aligned} \tag{55}$$

To obtain the TPM $\mathbb{Q}$, some additional notations are introduced as follows:

$$\begin{aligned} \mathbf{Y}_i &= (\xi_i, \eta_i), \quad 1 \le i \le N-1; \\ \mathbf{\Gamma} &= (\delta_1, \omega)^T; \\ \mathbb{R}_i &= \begin{pmatrix} \theta_i & \gamma_i \\ 0 & \beta_i \end{pmatrix}, \quad 0 \le i \le N-1. \end{aligned} \tag{56}$$

Using lexicographical sequence for the states, the structure of $\mathbb{Q}$ is given by

$$\mathbb{Q} = \begin{pmatrix} \delta_0 & \mathbf{Y}_1 & \cdots & \mathbf{Y}_{N-4} & \mathbf{Y}_{N-3} & \mathbf{Y}_{N-2} & \mathbf{Y}_{N-1} \\ \mathbf{\Gamma} & \mathbf{R}_1 & \cdots & \mathbf{R}_{N-4} & \mathbf{R}_{N-3} & \mathbf{R}_{N-2} & \mathbf{R}_{N-1} \\ & \mathbf{R}_0 & \cdots & \mathbf{R}_{N-5} & \mathbf{R}_{N-4} & \mathbf{R}_{N-3} & \mathbf{R}_{N-2} \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & \mathbf{R}_0 & \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \\ & & & & \mathbf{R}_0 & \mathbf{R}_1 & \mathbf{R}_2 \\ & & & & & \mathbf{R}_0 & \mathbf{R}_1 \end{pmatrix}. \tag{57}$$

Let $\mathbb{P}^d = (P^d_{0,0}, P^d_{1,0}, P^d_{1,1}, \ldots, P^d_{N-1,0}, P^d_{N-1,1})^T$ be a column vector of departure epoch probabilities; $\mathbf{e} = (1, 1, \ldots, 1)_{1\times(2N-1)}$ is a row vector of dimension of $2N-1$; then we have the system linear equations $\{\mathbb{Q}^{\mathsf{T}}\mathbb{P}^d = \mathbb{P}^d;\ \mathbf{e}\mathbb{P}^d = 1\}$, which can be directly converted into the following equations:

$$\begin{pmatrix} \mathbb{Q}^{\mathsf{T}} - \mathbb{I}_{2N-1} \\ \mathbf{e} \end{pmatrix} \cdot \mathbb{P}^d = \begin{pmatrix} \mathbf{O}_{(2N-1)\times 1} \\ 1 \end{pmatrix}, \tag{58}$$

where $\mathbb{I}_{2N-1}$ is an identity square matrix with $(2N-1)$ dimensions and $\mathbf{O}_{(2N-1)\times 1}$ is a column vector with $2N-1$ dimensions and all of its' entries are equal to zero.

Subsequently, the queue-length probabilities at departure epoch, that is, $\mathbb{P}^d$, are obtained by using software of MATLAB. Moreover, we finally work out the queue-length distribution at arbitrary epoch.

3.3. Steady State Queue Length at Other Epochs. To find the queue-length distributions at a potential arrival epoch n^-, prearrival epoch, arbitrary epoch n, and outside observer's observation epoch, we define the following additional notations.

$P^-_{j,0}$ or $P^-_{j,1}$ is the steady state probability of j customers waiting in system at an arbitrary epoch n^- during a working vacation or normal busy period.

$P^{(a)}_{j,0}(s^-)$ or $P^{(a)}_{j,1}(s^-)$ is the transient probability of j customers waiting in system at a prearrival epoch s^- during a working vacation or normal busy period:

$$P^{(a)}_{j,0} \triangleq \lim_{s\to\infty} P^{(a)}_{j,0}(s^-), \tag{59}$$

$$P^{(a)}_{j,1} \triangleq \lim_{s\to\infty} P^{(a)}_{j,1}(s^-). \tag{60}$$

$P^{(o)}_{j,0}$ or $P^{(o)}_{j,1}$ is the steady state probability of j customers waiting in system at an outside observer's observation epoch during a working vacation or normal busy period.

$P^{(e)}_{j,0}$ or $P^{(e)}_{j,1}$ is the steady state probability of j customers waiting in system at an arbitrary epoch n during a working vacation or normal busy period.

From Figure 1, it is clear that

$$\begin{aligned} P^{(o)}_{j,0} &= P^-_{j,0} = P_{j,0}, \quad 0 \le j \le N, \\ P^{(o)}_{j,1} &= P^-_{j,1} = P_{j,1}, \quad 0 \le j \le N. \end{aligned} \tag{61}$$

The prearrival epoch probabilities are determined by the following relations:

$$\begin{aligned} P^{(a)}_{j,0} &= \lim_{s\to\infty} P^{(a)}_{j,0}(s^-) = \lim_{s\to\infty} P\{N(s^-) = j;\ \eta(s^-) = 0 \mid \text{A customer arrives in time interval } (s^-, s)\} \\ &= \lim_{s\to\infty} \frac{P\{N(s^-) = j;\ \eta(s^-) = 0;\ \text{A customer arrives in time interval } (s^-, s)\}}{P\{\text{A customer arrives in time interval } (s^-, s)\}} = \frac{\lambda_2 P_{j,0}}{\lambda_2 \sum_{j=0}^N P_{j,0} + \lambda_1 \sum_{j=1}^N P_{j,1}}, \\ & \qquad 0 \le j \le N. \end{aligned} \tag{62}$$

Similarly, we have

$$P^{(a)}_{j,1} = \frac{\lambda_1 P_{j,1}}{\lambda_2 \sum_{j=0}^N P_{j,0} + \lambda_1 \sum_{j=1}^N P_{j,1}}, \quad 1 \le j \le N. \tag{63}$$

We can derive the conclusion of $P^{(a)}_{j,i} = P_{j,i}$ from (62) and (63) under the condition of $\lambda_1 = \lambda_2$; it means the GASTA (geometric arrivals see time average) property does not hold if $\lambda_1 \ne \lambda_2$.

The relations between $P^{(e)}_{j,i}$ and $P_{j,i}$ can also be conducted by considering the arbitrary epochs n and n^- in Figure 1:

$$\begin{aligned} P^{(e)}_{0,0} &= \overline{\lambda}_2 P_{0,0}, \\ P^{(e)}_{j,0} &= \overline{\lambda}_2 P_{j,0} + \lambda_2 P_{j-1,0}, \quad 1 \le j \le N-1, \end{aligned}$$

$$
\begin{aligned}
P_{N,0}^{(e)} &= P_{N,0} + \lambda_2 P_{N-1,0}, \\
P_{1,1}^{(e)} &= \overline{\lambda}_1 P_{1,1}, \\
P_{j,1}^{(e)} &= \overline{\lambda}_1 P_{j,1} + \lambda_1 P_{j-1,1}, \quad 2 \le j \le N-1, \\
P_{N,1}^{(e)} &= P_{N,1} + \lambda_1 P_{N-1,1}.
\end{aligned} \tag{64}
$$

Thus the arbitrary epoch (n) probabilities are determined by (64).

4. Performance Evaluation

Performance measures of the queueing system can easily be obtained based on the achieved queue-length distribution in Section 3.

4.1. Blocking Probability. Blocking probability of customer (denoted by P_{loss}) is pretty important in a finite buffer queueing system:

$$P_{\text{loss}} = P_{N,0}^{(a)} + P_{N,1}^{(a)}, \tag{65}$$

where $P_{N,0}^{(a)}$ and $P_{N,1}^{(a)}$ are given by (62) and (63), respectively.

4.2. Average Queue Length. From (61), three sorts of average queue length are presented, respectively, as follows:

(1) Average queue length is as follows:

$$\overline{N}^{(o)} = \sum_{j=1}^{N} jP_{j,0} + \sum_{j=1}^{N} jP_{j,1}. \tag{66}$$

(2) Average queue length during working vacation is as follows:

$$\overline{N}_1^{(o)} = \sum_{j=1}^{N} jP_{j,0}. \tag{67}$$

(3) Average queue length during normal busy period is as follows:

$$\overline{N}_2^{(o)} = \sum_{j=1}^{N} jP_{j,1}. \tag{68}$$

4.3. Average Sojourn Time for a Customer. Denoting the average sojourn time by $\overline{S}$ and using Little's rule, we have

$$\overline{S} = \frac{\overline{N}^{(o)}}{\lambda_e}, \tag{69}$$

where $\lambda_e = (1-P_{\text{loss}})(\lambda_2 \sum_{j=0}^{N} P_{j,0} + \lambda_1 \sum_{j=1}^{N} P_{j,1})$ is the effective average arrival rate.

5. Discussion of Numerical Performance Characteristics

In this section, some numerical conclusions in the form of table or graph are reported. Moreover, using the MATLAB software package, the effects of some parameters on the blocking probability and average sojourn time are also investigated.

Table 1: System queue length distribution at various epochs when service time is geometric with $\alpha_1 = 4$, $\alpha_2 = 7.1429$, $\lambda_1 = 0.22$, $\lambda_2 = 0.12$, $\nu = 0.025$, and $N = 15$.

j	$P_{j,0}^{(o)}$	$P_{j,1}^{(o)}$	$P_{j,0}^{(a)}$	$P_{j,1}^{(a)}$	$P_{j,0}^{(e)}$	$P_{j,1}^{(e)}$
0	0.1925		0.1347		0.1694	
1	0.1267	0.0384	0.0886	0.0492	0.1346	0.0300
2	0.0717	0.0543	0.0502	0.0696	0.0783	0.0508
3	0.0406	0.0583	0.0284	0.0748	0.0443	0.0574
4	0.0230	0.0564	0.0161	0.0723	0.0251	0.0568
5	0.0130	0.0517	0.0091	0.0663	0.0142	0.0527
6	0.0073	0.0461	0.0051	0.0591	0.0080	0.0473
7	0.0041	0.0404	0.0029	0.0518	0.0045	0.0417
8	0.0023	0.0350	0.0016	0.0449	0.0025	0.0362
9	0.0013	0.0301	0.0009	0.0386	0.0014	0.0312
10	0.0008	0.0258	0.0006	0.0331	0.0009	0.0267
11	0.0004	0.0221	0.0003	0.0283	0.0005	0.0230
12	0.0003	0.0189	0.0002	0.0242	0.0003	0.0200
13	0.0002	0.0162	0.0001	0.0208	0.0002	0.0168
14	0.0001	0.0138	0.0000	0.0177	0.0001	0.0143
15	0.0001	0.0081	0.0000	0.0104	0.0001	0.0111

$\overline{N}^{(o)} = 3.8425$, $\overline{N}_1^{(o)} = 0.6730$, $\overline{N}_2^{(o)} = 3.1695$, $P_{\text{loss}} = 0.0105$, $\overline{S} = 22.397$.

5.1. Numerical Illustration of the System Performance Measures. To illustrate the effectiveness of the queue-length solutions obtained in Section 3, the numerical results of queue-length distributions under some certain cases are given in Tables 1 and 2. And, the corresponding blocking probability, average queue length, and average sojourn time are given at the bottom of the tables. At the same time, the queue-length distributions in the form of graphs are also demonstrated in Figures 2(a) and 2(b).

5.2. Sensitivity Analysis. Figures 3(a) and 3(b) give the numerical illustrations of the effects of N on the blocking probability and average sojourn time under three kinds of service time distributions; that is, (1) service time is deterministic: $\{g_4^{(b)} = 1\}$, $\{g_6^{(v)} = 1\}$ with other parameters: $\lambda_1 = 0.23$, $\lambda_2 = 0.15$, and $\nu = 0.008$; (2) service time is geometric with average service times: $\alpha_1 = 3.3333$, $\alpha_2 = 5$, and other parameters: $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\nu = 0.008$; (3) service time is arbitrary: $\{g_3^{(b)} = 0.01, g_4^{(b)} = 0.05, g_5^{(b)} = 0.1, g_6^{(b)} = 0.2, g_7^{(b)} = 0.4, g_8^{(b)} = 0.24\}$; $\{g_4^{(v)} = 0.15, g_6^{(v)} = 0.25, g_8^{(v)} = 0.3, g_9^{(v)} = 0.25, g_{10}^{(v)} = 0.05\}$ with other parameters: $\lambda_1 = 0.136$, $\lambda_2 = 0.13$, and $\nu = 0.008$. It can be seen from Figures 3(a) and 3(b) that, for each case, the blocking probability and average sojourn time both decrease as system capacity N increases and both of them become stable with increasing N.

Figures 4(a) and 4(b) give the numerical illustrations of the effects of λ_2 on the blocking probability and average

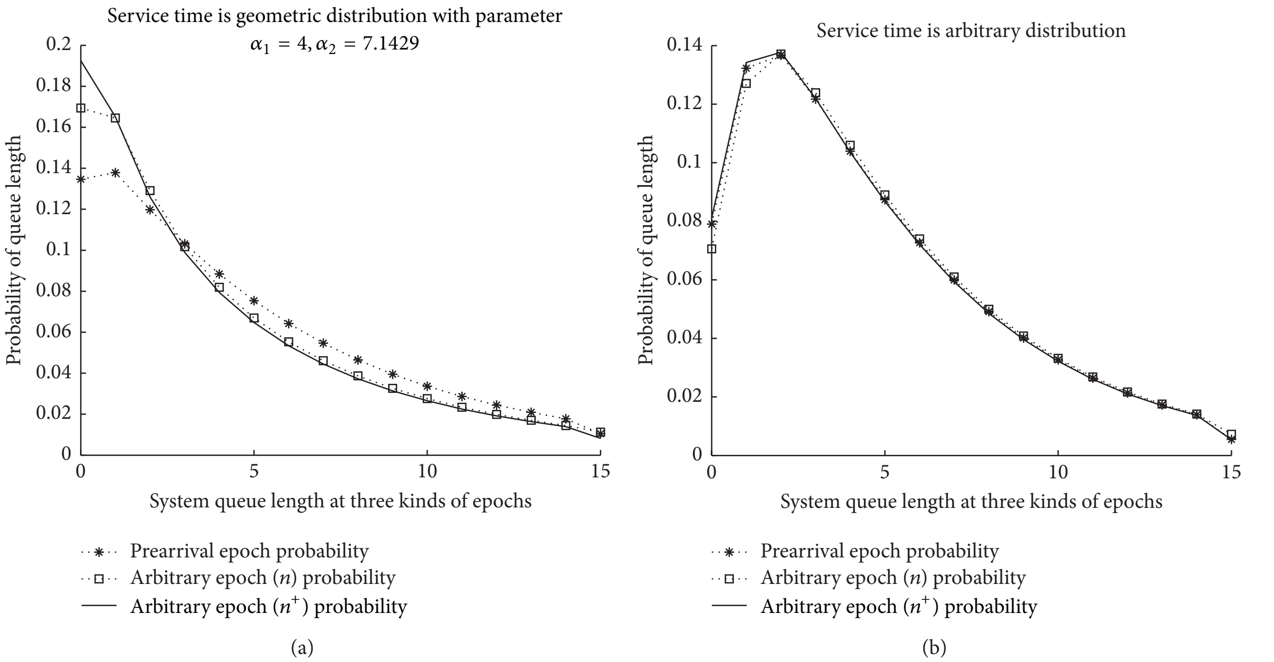

Figure 2: (a) Steady queue length distributions at different epochs. (b) Steady queue length distributions at different epochs.

Table 2: System queue length distribution at various epochs when service time is arbitrary with $\{g_3^{(b)} = 0.01, g_4^{(b)} = 0.05, g_5^{(b)} = 0.1, g_6^{(b)} = 0.2, g_7^{(b)} = 0.4, g_8^{(b)} = 0.24\}$, $\{g_4^{(v)} = 0.15, g_6^{(v)} = 0.25, g_8^{(v)} = 0.3, g_9^{(v)} = 0.25, g_{10}^{(v)} = 0.05\}$, and $\lambda_1 = 0.136$, $\lambda_2 = 0.13$, $\nu = 0.008$, and $N = 15$.

j	$P_{j,0}^{(o)}$	$P_{j,1}^{(o)}$	$P_{j,0}^{(a)}$	$P_{j,1}^{(a)}$	$P_{j,0}^{(e)}$	$P_{j,1}^{(e)}$
0	0.0811		0.0790		0.0706	
1	0.1008	0.0334	0.0982	0.0340	0.0982	0.0289
2	0.0819	0.0558	0.0798	0.0569	0.0844	0.0528
3	0.0567	0.0652	0.0553	0.0665	0.0600	0.0639
4	0.0377	0.0659	0.0367	0.0672	0.0402	0.0658
5	0.0250	0.0618	0.0244	0.0630	0.0267	0.0624
6	0.0166	0.0554	0.0162	0.0565	0.0177	0.0563
7	0.0110	0.0483	0.0107	0.0492	0.0117	0.0493
8	0.0073	0.0412	0.0071	0.0420	0.0078	0.0422
9	0.0049	0.0347	0.0048	0.0354	0.0052	0.0356
10	0.0033	0.0289	0.0032	0.0295	0.0035	0.0297
11	0.0022	0.0238	0.0021	0.0243	0.0023	0.0245
12	0.0015	0.0195	0.0015	0.0199	0.0016	0.0201
13	0.0011	0.0159	0.0011	0.0162	0.0012	0.0164
14	0.0008	0.0129	0.0008	0.0132	0.0008	0.0133
15	0.0003	0.0051	0.0003	0.0052	0.0004	0.0069

$\overline{N}^{(o)} = 4.569$, $\overline{N}_1^{(o)} = 1.0948$, $\overline{N}_2^{(o)} = 3.4742$, $P_{\text{loss}} = 0.0055$, $\overline{S} = 34.249$.

sojourn time under three kinds of service time distributions; that is, (1) service time is deterministic: $\{g_4^{(b)} = 1\}$, $\{g_6^{(v)} = 1\}$ with other parameters: $\lambda_1 = 0.24$, $N = 15$, and $\nu = 0.008$; (2) service time is geometric with average service times: $\alpha_1 = 2$, $\alpha_2 = 3.3333$ and other parameters: $\lambda_1 = 0.46$, $N = 15$, $\nu = 0.008$; (3) service time is arbitrary: $\{g_3^{(b)} = 0.01$, $g_4^{(b)} = 0.05$, $g_5^{(b)} = 0.1$, $g_6^{(b)} = 0.2$, $g_7^{(b)} = 0.4$, $g_8^{(b)} = 0.24\}$; $\{g_4^{(v)} = 0.15$, $g_6^{(v)} = 0.25$, $g_8^{(v)} = 0.3$, $g_9^{(v)} = 0.25$, $g_{10}^{(v)} = 0.05\}$ with other parameters: $\lambda_1 = 0.136$ and $\nu = 0.008$. The three curves in Figure 4(a) or Figure 4(b) exhibit an increasing shape. We observe from Figure 4(a) that, for each case, the blocking probability becomes sensitive after a certain arrival rate during working vacation (λ_2) and seems to tend to be unstable as $\lambda_2 \to \lambda_1$. For three cases in Figure 4(b), we can see that the average sojourn time gets sensitive after a certain value of λ_2 and likely tends to be stable as $\lambda_2 \to \lambda_1$.

6. A Numerical Example

To display the application of the above models, a real problem concerning express logistics services is studied. In an express logistics service center with the maximum capacity of N (in other words, if the number of express parcels accumulates to N the new arriving express parcels will be delivered to other service centers), the express parcels arrive in distribution center according to a Poisson process. The arrival rate of the express parcels depends on the state of the server in the distribution center. When the server is on duty, the arrival rate is λ_1. On the other hand, the rate is $\lambda_2 < \lambda_1$ if the server is on vacation. In this service center, the semivacation policy is adopted: after dealing with all the waiting express parcels, the server takes vacation. To improve the service efficiency, the manager requires the server to keep on providing service during his vacation with a low service rate. When returning from a vacation and finding any express parcels waiting for service, the server returns the low service rate to a normal

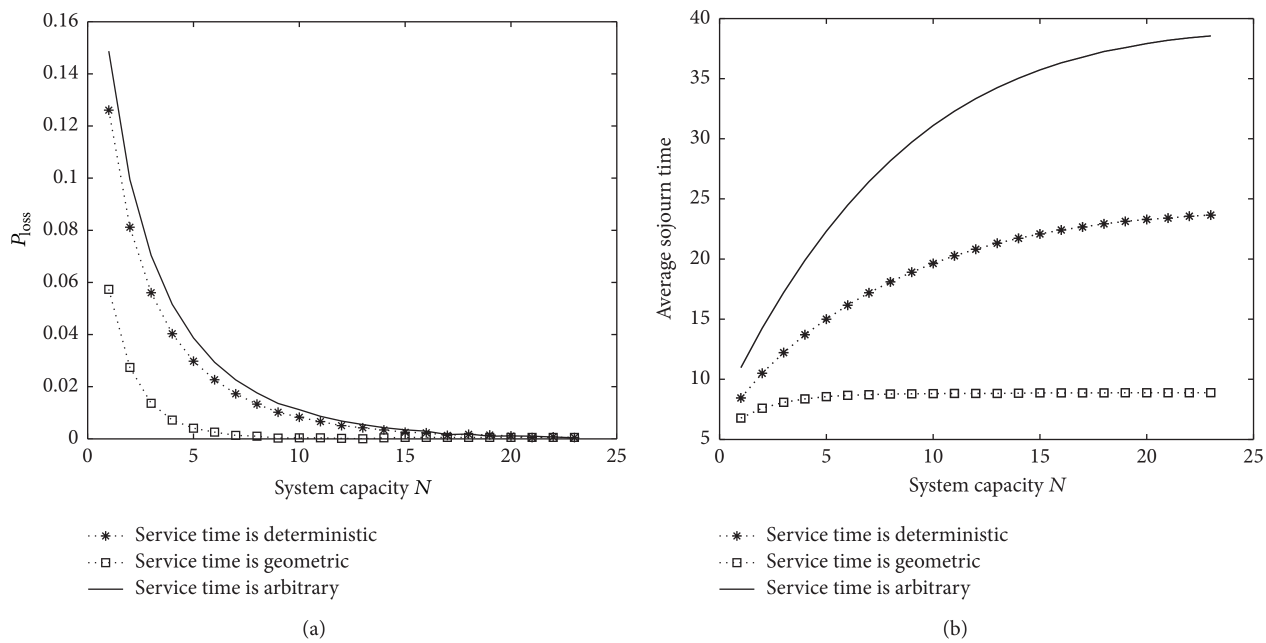

Figure 3: (a) Effect of N on blocking probability under different service time distributions. (b) Effect of N on average sojourn time under different service time distributions.

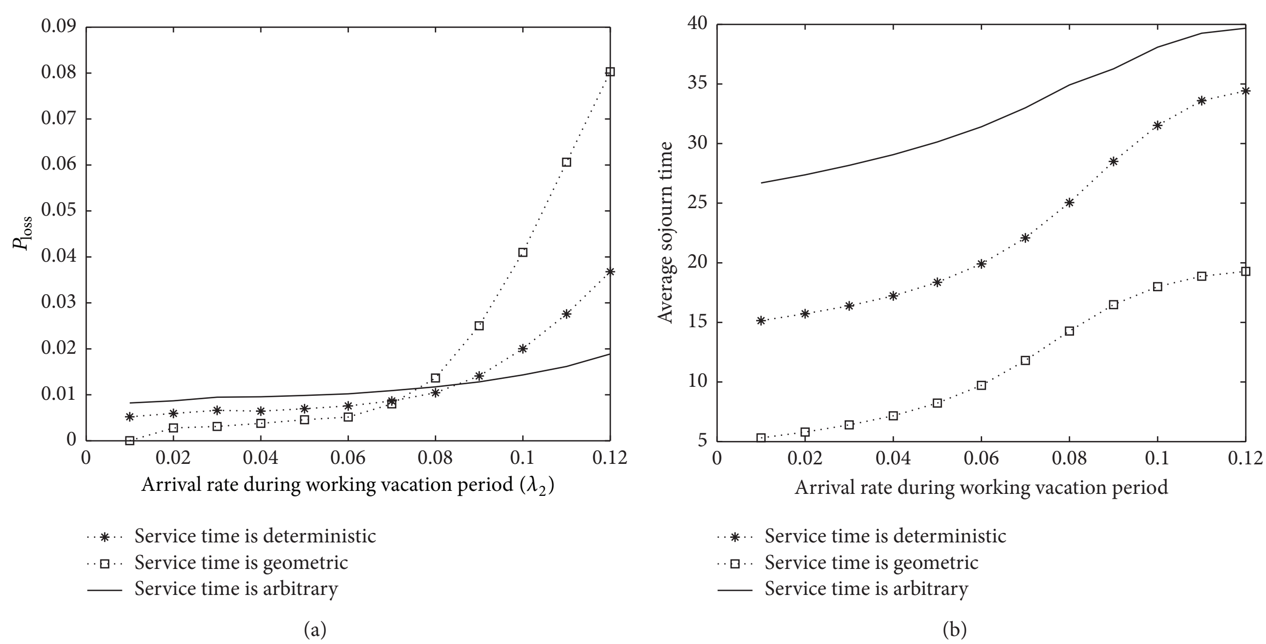

Figure 4: (a) Effect of λ_2 on blocking probability under different service time distributions. (b) Effect of λ_2 on average sojourn time under different service time distributions.

level. Otherwise, the server takes the next vacation. The length of a working vacation V is geometrically distributed with parameter ν. The service time for each parcel during normal busy period is an arbitrarily distributed variable with mean value of α_1. The service time for each parcel during working vacation is also an arbitrarily distributed variable with mean value of α_2.

Thus, this express logistics service center can be modeled by the finite-capacity $Geo/G/1/N$ queueing system with working vacation and different input rate studied in this paper. To realize precise control, the manager considers the following cost elements:

$d_1 \equiv$ fixed possessing fee per unit time for a express parcel in center;

$d_2(j) \equiv$ unit time fee for possessing j $(1 \le j \le N)$ express parcels during working vacation;

$d_3(j) \equiv$ unit time fee for possessing j $(1 \le j \le N)$ express parcels during normal busy period;

$d_4 \equiv$ fixed service fee per unit time during working vacation;

$d_5 \equiv$ fixed service fee per unit time during normal busy period.

Based on the cost elements defined above, the optimal cost problem can be modeled mathematically by

$$\min_{\alpha_2}: h(\alpha_2) = d_1 \cdot \bar{S} + \sum_{j=1}^{N} d_2(j) \cdot P_{j,0} + \sum_{j=1}^{N} d_3(j) \cdot P_{j,1} + \frac{d_4}{\alpha_2} + \frac{d_5}{\alpha_1}, \tag{70}$$

where $P_{j,0}$, $P_{j,1}$, and $\bar{S}$ are determined by (45), (46), and (69), respectively.

Regarding the service during working vacations as a piece of additional work for the server, the proper service rate during that time is a critical decision variable for the manager to arrange the optimal operation. The key task of the manager is to determine the appropriate value of service rate during working vacation (denoted by α_2^*) in (70) to minimize the operating cost.

One may note that it would be pretty difficult to solve the optimum problem (70) by using analytic method because of the highly nonlinear and complex of the given cost function. Instead, we apply the parabolic method to deal with it. Based on the introduction of parabolic method (see [19]), we design the following steps.

Step 1 (initialization). Choose a starting 3-point pattern $\{y^{(l)}, y^{(m)}, y^{(r)}\}$ along with a stopping tolerance $\varepsilon = 10^{-5}$ and initialize the iteration counter $i = 0$. Substitute these initial values into the following approximate formula

$$y^{(q)} = \frac{1}{2} \frac{\left[\left(y^{(m)}\right)^2 - \left(y^{(r)}\right)^2\right] h\left(y^{(l)}\right) + \left[\left(y^{(r)}\right)^2 - \left(y^{(l)}\right)^2\right] h\left(y^{(m)}\right) + \left[\left(y^{(l)}\right)^2 - \left(y^{(m)}\right)^2\right] h\left(y^{(r)}\right)}{\left[y^{(m)} - y^{(r)}\right] h\left(y^{(l)}\right) + \left[y^{(r)} - y^{(l)}\right] h\left(y^{(m)}\right) + \left[y^{(l)} - y^{(m)}\right] h\left(y^{(r)}\right)}. \tag{71}$$

Step 2 (stopping). If $|y^{(q)} - y^{(m)}| \le \varepsilon$, stop and report approximate optimum solution $y^{(m)}$.

Step 3 (quadratic fit). Compute a quadratic fit optimum $y^{(q)}$ according to the formulae (70) and (71). Then if $y^{(q)} \le y^{(m)}$, go to Step 4, and if $y^{(q)} > y^{(m)}$ go to Step 5.

Step 4 (left). If $h(y^{(m)})$ is superior to $h(y^{(q)})$ (less for a minimize, greater for a maximize), then update $y^{(q)} \to y^{(l)}$. Otherwise, replace $y^{(m)} \to y^{(r)}$, $y^{(q)} \to y^{(m)}$. Either way, advance $i = i + 1$ and return to Step 2.

Step 5 (right). If $h(y^{(m)})$ is superior to $h(y^{(q)})$ (less for a minimize, greater for a maximize), then update $y^{(q)} \to y^{(r)}$. Otherwise, replace $y^{(m)} \to y^{(l)}$, $y^{(q)} \to y^{(m)}$. Either way, advance $i = i + 1$ and return to Step 2.

It is assumed that the systems parameters are as follows.

(1) The input rates of express parcels during working vacation and normal busy period are $\lambda_2 = 0.3$ and $\lambda_1 = 0.38$, respectively.

(2) The working vacation time (V) follows a geometric distribution with parameter $\nu = 0.08$ and the capacity of system is $N = 15$.

(3) The service time during normal busy period is arbitrarily distributed as $\{g_1 = 0.2, g_2 = 0.4, g_3 = 0.2, g_4 = 0.1, g_5 = 0.1\}$.

The other parameters in (70) are given as follows: $d_1 = 90$, $d_2(j) = 12 \cdot j + \sqrt[5]{j^3}$, $d_3(j) = 15 \cdot j + \sqrt[3]{j^5}$, $d_4 = 1800$, $d_5 = 2200$. So, the effect of service rate during working vacation period $(1/\alpha_2)$ on the operating fee $(h(\alpha_2))$ is demonstrated in Figure 5.

From Figure 5, choosing an initial 3-point pattern $(1/\alpha)^{(l)} = 0.05$, $(1/\alpha)^{(m)} = 0.1$, $(1/\alpha)^{(r)} = 0.15$, and the stopping tolerance $\varepsilon = 10^{-5}$, it obtains the optimal value of α_2 after three iterations: $\alpha_2^* = 10.5820$ which leads to the minimum expected operating fee: $h(\alpha_2^*) = 2849.9146$.

7. Conclusions

For the first time, we carry out an analysis of finite buffer $Geo/G(MWV)/1/N$ queue with working vacations and different input rate. Using two kinds of classical analysis techniques, we derive the queue-length solutions in the form of formula at different epoches (given by Section 3). Through this solution, we also obtain some important performance characteristics (given by Section 4). Based on the queue-length distribution, a state-dependent cost function is developed from a express logistics service center and optimally analyzed in Section 6. This paper is a beginning study for finite buffer $Geo/G/1/N$ queue with working vacation. In the future, research topics such as $Geo^X/G/1/N$ with batch arrivals, $Geo/G/1/N$ with Bernoulli-schedule working vacation, $Geo/G/1/N$ with N-policy, and so on can be studied with the same analysis techniques.

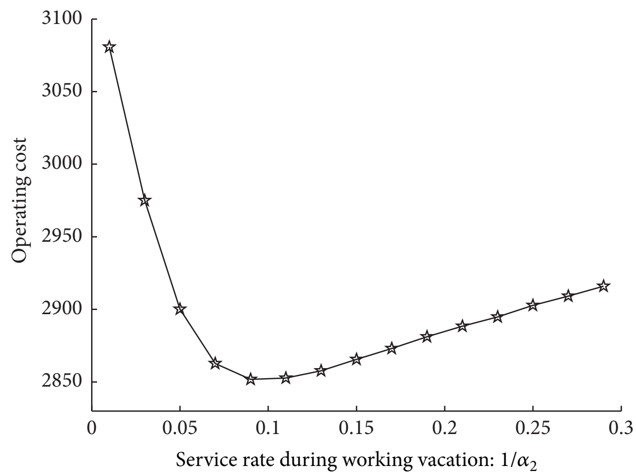

Figure 5: Effect of $1/\alpha_2$ on the operating cost per unit time.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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