Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 965105, 23 pages
http://dx.doi.org/10.1155/2014/965105

Research Article

Number of Spanning Trees of Different Products of Complete and Complete Bipartite Graphs

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Received 28 December 2013; Revised 31 May 2014; Accepted 31 May 2014; Published 2 July 2014

Academic Editor: Efstratios Tzirtzilakis

Spanning trees have been found to be structures of paramount importance in both theoretical and practical problems. In this paper we derive new formulas for the complexity, number of spanning trees, of some products of complete and complete bipartite graphs such as Cartesian product, normal product, composition product, tensor product, symmetric product, and strong sum, using linear algebra and matrix theory techniques.

1. Introduction

The number of spanning trees of a graph is an important, well-studied quantity in graph theory and appears in a number of applications. The most notable application fields are network reliability [1–4], enumerating certain chemical isomers [5], and counting of the Eulerian circuits in a graph [6]. Every connected graph has a spanning tree. A spanning tree of a graph G is a tree that (i) is a subgraph of G (i.e., that includes only edges from G) and (ii) includes every vertex of G. The most classical interest concerning a spanning tree is the number of spanning trees, also called the complexity of the graph G and denoted by $\tau(G)$. Kirchhoff [7] gave a formula for determining it, which is known as the matrix tree theorem. The spanning trees of a graph G are the value of any cofactor of the matrix $D(G)-A(G)$, where $D(G)$ is the degree matrix (the ith diagonal entry is equal to the degree of the ith vertex and the other entry is equal to zero) and $A(G)$ is the adjacency matrix of G (the entry (i, j) is equal to the number of edges between ith vertex and jth vertex), respectively. This topic is still much studied, in particular, in explicit formulas of the number of spanning trees of some special classes. That for complete graphs is most famous among such classes; the number of spanning trees of K_n is n^{n-2}, called Cayley's formula [8]. Several proofs of Cayley's formula are known, and the most famous one is due to Prüfer [9]. The explicit formulas of the number of spanning trees are known for other classes than complete graphs: complete bipartite graphs [10–13], regular graphs [14], circulant graphs [15–19], pyramid graphs [20], and so on.

Now we introduce the following Lemma which describes a way to calculate the number of spanning trees by an extension of Kirchhoff formula.

Lemma 1 (see [21]). *Let G be a graph with n vertices. Then*

$$\tau(G) = \frac{1}{n^2}\det\left(nI-\overline{D}+\overline{A}\right), \quad (1)$$

where $\overline{A}$, $\overline{D}$ are the adjacency and degree matrices of $\overline{G}$, the complement of G, respectively, and I is the $n\times n$ unit matrix.

Lemma 2. *Let $A_n(x)$ be $n \times n$ matrix, $x \geq 2$ such that*

$$A_n(x) = \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix}. \quad (2)$$

Then

$$\det(A_n(x)) = (x+n-1)(x-1)^{n-1}. \quad (3)$$

Proof. From the definition of the circulant determinants, we have

$$\begin{aligned} \det(A_n(x)) &= \det \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix} \\ &= \prod_{j=1}^{n} \left(x + \omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1} \right) \\ &= (x+1+1+\cdots+1) \\ &\quad \times \prod_{j=1, \omega_j \neq 1}^{n} \left(x + \underbrace{\omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}}_{=-1} \right) \\ &= (x+n-1) \times (x-1)^{n-1}. \end{aligned} \quad (4)$$

We can generalize the above lemma as follows. □

Lemma 3. *Let $A, B \in F^{n\times n}$, and $\mathbb{F} \in F^{kn\times kn}$ such that*

$$\mathbb{F} = \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix}. \quad (5)$$

Then

$$\det \mathbb{F} = [\det(A-B)]^{k-1} \det[A + (k-1)B]. \quad (6)$$

Lemma 4 (see [22]). *Let $A \in F^{n\times n}$, $B \in F^{n\times m}$, $C \in F^{m\times n}$, and $D \in F^{m\times m}$. Assume that A and D are nonsingular matrices. Then:*

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= (-1)^{nm} \det\left(A - BD^{-1}C\right) \det D \\ &= (-1)^{nm} \det A \det\left(D - CA^{-1}B\right). \end{aligned} \quad (7)$$

Formulas in Lemmas 2, 3, and 4 give some sort of symmetry in some matrices which facilitates our calculation of determinants.

2. Number of Spanning Trees of Cartesian Product of Graphs

The Cartesian product, $G_1 \times G_2$, of two graphs G_1 and G_2 is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \cup (V_1 \times E_2)]$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ [23].

Theorem 5. *For $m, n \geq 1$and $r \geq 2$, we have*

$$\begin{aligned} \tau(K_r \times K_{m,n}) &= r^{r-2} m^{n-1} n^{m-1} (m+r)^{(r-1)(n-1)} \\ &\quad \times (n+r)^{(r-1)(m-1)} (m+n+r)^{r-1}. \end{aligned} \quad (8)$$

Proof. Applying Lemma 1, we have

$$
\begin{aligned}
\tau\left(K_r \times K_{m,n}\right) &= \frac{1}{\left(r\left(m+n\right)\right)^2} \det\left(r\left(m+n\right) I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{r^2 (m+n)^2} \\
&\times \det \begin{pmatrix}
n+r & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & n+r & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & m+r & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+r & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & n+r & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & n+r & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & m+r & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+r
\end{pmatrix}.
\end{aligned}
\tag{9}
$$

Using Lemma 3, we get

$$
\begin{aligned}
\tau\left(K_r \times K_{m,n}\right) &= \frac{1}{r^2(m+n)^2}\det\begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix} \\
&= \frac{1}{\left(r\left(m+n\right)\right)^2}\left[\det\left(A-B\right)\right]^{r-1}\left[\det\left(A+\left(r-1\right)B\right)\right] \\
&= \frac{1}{r^2(m+n)^2} \\
&\quad \times \left(\det\begin{pmatrix} n+r & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & n+r & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & m+r & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & m+r \end{pmatrix}\right)^{r-1} \\
&\quad \times \det\begin{pmatrix} n+r & r & \cdots & r & (r-1) & \cdots & \cdots & (r-1) \\ r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & n+r & (r-1) & \cdots & \cdots & (r-1) \\ (r-1) & \cdots & \cdots & (r-1) & m+r & r & \cdots & r \\ \vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\ (r-1) & \cdots & \cdots & (r-1) & r & \cdots & r & m+r \end{pmatrix} \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}\right)^{r-1} \times \det\begin{pmatrix} D & E \\ E^T & F \end{pmatrix}.
\end{aligned}
\tag{10}
$$

Using Lemma 4, we obtain

$$\tau\left(K_r \times K_{m,n}\right) = \frac{1}{r^2(m+n)^2} \times (\det A)^{r-1}\left(\det\left(C - B^T A^{-1} B\right)\right)^{r-1} \times \det D \det\left(F - E^T D^{-1} E\right)$$

$$= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix} n+r & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+r \end{pmatrix}_{m\times m}\right)^{r-1}$$

$$\times \left(\det\begin{pmatrix} \frac{n(m+r)+(r-1)m+r^2}{n+r} & \frac{-m}{n+r} & \cdots & \frac{-m}{n+r} \\ \frac{-m}{n+r} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{n+r} \\ \frac{-m}{n+r} & \cdots & \frac{-m}{n+r} & \frac{n(m+r)+(r-1)m+r^2}{n+r} \end{pmatrix}_{n\times n}\right)^{r-1}$$

$$\times \det\begin{pmatrix} n+r & r & \cdots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & n+r \end{pmatrix}_{m\times m}$$

$$\times \det\begin{pmatrix} \frac{n(m+r)+rm^2+(2r-1)m}{n+rm} & \frac{rn+(2r-1)m}{n+rm} & \cdots & \frac{rn+(2r-1)m}{n+rm} \\ \frac{rn+(2r-1)m}{n+rm} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{rn+(2r-1)m}{n+rm} \\ \frac{rn+(2r-1)m}{n+rm} & \cdots & \frac{rn+(2r-1)m}{n+rm} & \frac{n(m+r)+rm^2+(2r-1)m}{n+rm} \end{pmatrix}_{n\times n}$$

$$= \frac{1}{r^2(m+n)^2}(n+r)^{m(r-1)}\left(\frac{-m}{n+r}\right)^{n(r-1)}$$

$$\times \left(\det\begin{pmatrix} \frac{n(m+r)+(r-1)m+r^2}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+r)+(r-1)m+r^2}{-m} \end{pmatrix}_{n\times n}\right)^{r-1}$$

$$\times r^m \det\begin{pmatrix} \frac{n+r}{r} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n+r}{r} \end{pmatrix}_{m\times m} \times \left(\frac{rn+(2r-1)m}{n+rm}\right)^n$$

$$\times \det\begin{pmatrix} \frac{n(m+r)+rm^2+(2r-1)m}{rn+(2r-1)m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+r)+rm^2+(2r-1)m}{rn+(2r-1)m} \end{pmatrix}_{n\times n}. \tag{11}$$

Using Lemma 2, we have

$$\begin{aligned} &\tau(K_r \times K_{m,n}) \\ &= \frac{1}{r^2(m+n)^2} \times (n+r)^{m(r-1)} \\ &\times \left(\frac{-m}{n+r}\right)^{n(r-1)} \left[-\frac{nm+rn+(r-1)m+r^2}{m}+n-1\right]^{r-1} \\ &\times \left[-\frac{nm+rn+(r-1)m+r^2}{m}-1\right]^{(r-1)(n-1)} \\ &\times r^m\left(\frac{n+r}{r}+m-1\right)\left(\frac{n+r}{r}-1\right)^{m-1} \\ &\times \left(\frac{nr+(2r-1)m}{n+rm}\right)^n \\ &\times \left[\frac{n(m+r)+rm^2+(2r-1)m}{rn+(2r-1)m}+n-1\right] \\ &\times \left[\frac{n(m+r)+rm^2+(2r-1)m}{rn+(2r-1)m}-1\right]^{(n-1)} \\ &= \frac{1}{r^2(m+n)^2} \times r^{r-1}(n+r)^{(m-n)(r-1)} \times (m+n+r)^{r-1} \\ &\times \left(rm+rn+mn+r^2\right)^{(r-1)(n-1)} \\ &\times (n+rm) \times n^{m-1} \times \frac{1}{(n+rm)^n} \times r(m+n)^2 \\ &\times (n+rm)^{n-1} \times m^{n-1} \\ &= r^{r-2} \times (n+r)^{(m-n)(r-1)} \times m^{n-1} \times n^{m-1} \\ &\times (m+n+r)^{r-1} \times \left(rm+rn+mn+r^2\right)^{(r-1)(n-1)} \\ &= r^{r-2} \times m^{n-1} \times n^{m-1} \times (m+r)^{(r-1)(n-1)} \\ &\times (n+r)^{(r-1)(m-1)} \times (m+n+r)^{r-1}. \end{aligned} \tag{12}$$

Specially,

$$\begin{aligned} \tau(K_r \times K_{n,n}) = r^{r-2} \times n^{2n-2} \times (2n+r)^{r-1} \\ \times (n+r)^{2(r-1)(n-1)}; \quad n \geq 1. \end{aligned} \tag{13}$$

□

3. Number of Spanning Trees of Normal Product of Graphs

The normal product, or the strong product, $G_1 \circ G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2, or u_1 is adjacent to v_1 and $u_2 = v_2$, or u_1 is adjacent to v_1 and u_2 is adjacent to v_2, [24].

Theorem 6. *For $m, n \geq 1$ and $r \geq 2$, we have*

$$\begin{aligned} \tau(K_r \circ K_{m,n}) = r^{rm+rn-2} m^{n-1} n^{m-1} \\ \times (m+1)^{n(r-1)} (n+1)^{m(r-1)}. \end{aligned} \tag{14}$$

Proof. Applying Lemma 1, we have

$$
\begin{aligned}
&\tau\left(K_r \circ K_{m,n}\right) \\
&= \frac{1}{\left(r\left(m+n\right)\right)^2}\det\left(r\left(m+n\right)I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{r^2\left(m+n\right)^2} \\
&\quad \times \det\left(\begin{array}{cccccccccccc}
r(n+1) & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & r(n+1) & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & r(m+1) & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & r(m+1) & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0
\end{array}\right.
\end{aligned}
$$

$$\left.\begin{matrix}
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & r(n+1) & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & r(n+1) & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & r(m+1) & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & r(m+1)
\end{matrix}\right). \tag{15}$$

Using Lemma 3, we get

$$\tau\left(K_r \circ K_{m,n}\right) = \frac{1}{r^2(m+n)^2} \times \det \begin{pmatrix}
A & B & \cdots & \cdots & \cdots & B \\
B & A & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & A & B \\
B & \cdots & \cdots & \cdots & B & A
\end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{(r(m+n))^2}[\det(A-B)]^{r-1}[\det(A+(r-1)B)] \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix}
r(n+1) & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & r(n+1) & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & r(m+1) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & r(m+1)
\end{pmatrix}\right)^{r-1} \\
&\quad \times \det\begin{pmatrix}
r(n+1) & r & \cdots & r & 0 & \cdots & \cdots & 0 \\
r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\
r & \cdots & r & r(n+1) & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & r(m+1) & r & \cdots & r \\
\vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\
0 & \cdots & \cdots & 0 & r & \cdots & r & r(m+1)
\end{pmatrix} \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix}
r(n+1) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & r(n+1)
\end{pmatrix}_{m\times m}\right)^{r-1} \\
&\quad \times \left(\det\begin{pmatrix}
r(m+1) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & r(m+1)
\end{pmatrix}_{n\times n}\right)^{r-1} \\
&\quad \times \det\begin{pmatrix}
r(n+1) & r & \cdots & r \\
r & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & r \\
r & \cdots & r & r(n+1)
\end{pmatrix}_{m\times m} \times \det\begin{pmatrix}
r(m+1) & r & \cdots & r \\
r & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & r \\
r & \cdots & r & r(m+1)
\end{pmatrix}_{n\times n}.
\end{aligned}
\tag{16}
$$

Using Lemma 2, we obtain

$$
\begin{aligned}
&\tau(K_r \circ K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2}(r(n+1))^{m(r-1)}(r(m+1))^{n(r-1)} \\
&\quad \times \left(r^m \times (n+m) \times n^{m-1}\right)\left(r^n \times (n+m) \times m^{n-1}\right) \\
&= r^{r(m+n)-2} m^{n-1} n^{m-1} (m+1)^{n(r-1)} (n+1)^{m(r-1)}.
\end{aligned}
\tag{17}
$$

Specially,

$$
\tau(K_r \circ K_{n,n}) = r^{2(rn-1)} \times n^{2(n-1)} \times (n+1)^{2n(r-1)}; \quad n \ge 1. \tag{18}
$$

□

4. Number of Spanning Trees of Composition Product of Graphs

The composition, or lexicographic product, $G_1[G_2]$, of two graphs G_1 and G_2 is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 [24].

Theorem 7. *For $m,n \geq 1$ and $r \geq 2$, we get*

$$\tau\left(K_r\left[K_{m,n}\right]\right) = r^{2(r-1)}(m+n)^{2(r-1)}(rm+rn-m)^{r(m-1)} \times (rm+rn-n)^{r(n-1)}. \tag{19}$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}
&\tau\left(K_r\left[K_{m,n}\right]\right) \\
&= \frac{1}{r^2(m+n)^2} \\
&\times \left(\det\begin{pmatrix}
rn+(r-1)m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & rn+(r-1)m+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & rm+(r-1)n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & rm+(r-1)n+1
\end{pmatrix}\right)^r \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix}
rn+(r-1)m+1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & rn+(r-1)m+1
\end{pmatrix}_{m\times m}\right)^r \\
&\times \left(\det\begin{pmatrix}
rm+(r-1)n+1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & rm+(r-1)n+1
\end{pmatrix}_{n\times n}\right)^r.
\end{aligned} \tag{20}$$

Using Lemma 2, we obtain

$$\tau\left(K_r\left[K_{m,n}\right]\right) = r^{2(r-1)}(m+n)^{2(r-1)} \times (rm+rn-m)^{r(m-1)}(rm+rn-n)^{r(n-1)}. \tag{21}$$

Specially,

$$\tau\left(K_r\left[K_{n,n}\right]\right) = (2r)^{2(r-1)}n^{2(rn-1)}(2r-1)^{2r(n-1)}; \quad n \geq 1. \tag{22}$$

□

5. Complexity of Tensor Product of Graphs

The tensor product, or Kronecker product, $G_1 \otimes G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [24].

Theorem 8. *For $m,n \geq 1$ and $r \geq 2$, we have*

$$\tau\left(K_r \otimes K_{m,n}\right) = r^{r-2}(r-2)^{r-1}(r-1)^{r(m+n-2)+1}m^{rn-1}n^{rm-1}. \tag{23}$$

Proof. Applying Lemma 1, we get

$$
\begin{aligned}
&\tau\left(K_r \otimes K_{m,n}\right) \\
&= \frac{1}{\left(r\left(m+n\right)\right)^2} \det\left(r\left(m+n\right) I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{r^2\left(m+n\right)^2} \\
&\times \det \left(\begin{array}{cccccccccccc}
(r-1)n+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & (r-1)n+1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & (r-1)m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & (r-1)m+1 & 0 & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0
\end{array}\right.
\end{aligned}
$$

$$\left.\begin{matrix}
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0 & (r-1)n+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & (r-1)n+1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & (r-1)m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & (r-1)m+1
\end{matrix}\right). \tag{24}$$

Using Lemma 3, we obtain

$$\tau\left(K_r \otimes K_{m,n}\right) = \frac{1}{r^2(m+n)^2} \det \begin{pmatrix}
A & B & \cdots & \cdots & \cdots & B \\
B & A & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & A & B \\
B & \cdots & \cdots & \cdots & B & A
\end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{(r(m+n))^2}[\det(A-B)]^{r-1}[\det(A+(r-1)B)] \\
&= \frac{1}{r^2(m+n)^2} \\
&\times \left(\det\begin{pmatrix}
(r-1)n & 0 & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & (r-1)n & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & (r-1)m & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & 0 & (r-1)m
\end{pmatrix}\right)^{r-1} \\
&\times \det\begin{pmatrix}
(r-1)n+r & r & \cdots & r & 1 & \cdots & \cdots & 1 \\
r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\
r & \cdots & r & (r-1)n+r & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & (r-1)m+r & r & \cdots & r \\
\vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\
1 & \cdots & \cdots & 1 & r & \cdots & r & (r-1)m+r
\end{pmatrix} \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix} A & B \\ B^T & C\end{pmatrix}\right)^{r-1} \times \det\begin{pmatrix} D & E \\ E^T & F\end{pmatrix}.
\end{aligned}
\tag{25}
$$

Using Lemma 4, we obtain

$$
\begin{aligned}
&\tau(K_r \otimes K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} \times (\det A)^{r-1}\left(\det\left(C - B^T A^{-1} B\right)\right)^{r-1} \times \det D \det\left(F - E^T D^{-1} E\right) \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix}
(r-1)n & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & (r-1)n
\end{pmatrix}_{m\times m}\right)^{r-1}
\end{aligned}
$$

$$\times\det\begin{pmatrix} \frac{\left[(r-1)^2m+\left(r^2-r\right)\right]n+\left(r^2-r\right)m^2+\left(r^2-1\right)m}{rm+(r-1)n} & \frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n} & \cdots & \frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n} \\ \frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n} \\ \frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n} & \cdots & \frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n} & \frac{\left[(r-1)^2m+\left(r^2-r\right)\right]n+\left(r^2-r\right)m^2+\left(r^2-1\right)m}{rm+(r-1)n} \end{pmatrix}_{n\times n}$$

$$\times\left(\det\begin{pmatrix} \frac{m\left[(r-1)^2n-1\right]}{(r-1)n} & \frac{-m}{(r-1)n} & \cdots & \frac{-m}{(r-1)n} \\ \frac{-m}{(r-1)n} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{(r-1)n} \\ \frac{-m}{(r-1)n} & \cdots & \frac{-m}{(r-1)n} & \frac{m\left[(r-1)^2n-1\right]}{(r-1)n} \end{pmatrix}_{n\times n}\right)^{r-1}$$

$$\times\det\begin{pmatrix} (r-1)n+r & r & \cdots & r \\ r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r \\ r & \cdots & r & (r-1)n+r \end{pmatrix}_{m\times m}$$

$$=\frac{1}{r^2(m+n)^2}((r-1)n)^{m(r-1)}$$

$$\times\left(\frac{-m}{(r-1)n}\right)^{n(r-1)}\left(\det\begin{pmatrix} -\left[(r-1)^2n-1\right] & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & -\left[(r-1)^2n-1\right] \end{pmatrix}_{n\times n}\right)^{r-1}$$

$$\times r^m \det\begin{pmatrix} \frac{(r-1)n+r}{r} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{(r-1)n+r}{r} \end{pmatrix}_{m\times m} \times \left(\frac{\left(r^2-1\right)m+\left(r^2-r\right)n}{rm+(r-1)n}\right)^n$$

$$\times \det\begin{pmatrix} \frac{\left[(r-1)^2m+\left(r^2-r\right)\right]n+\left(r^2-r\right)m^2+\left(r^2-1\right)m}{(r^2-1)m+\left(r^2-r\right)n} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{[(r-1)^2m+(r^2-r)]n+\left(r^2-r\right)m^2+\left(r^2-1\right)m}{(r^2-1)m+\left(r^2-r\right)n} \end{pmatrix}_{n\times n}. \tag{26}$$

Using Lemma 2 yields

$$\begin{aligned}\tau\left(K_r \otimes K_{m,n}\right) &= \frac{1}{r^2(m+n)^2} \times (r-1)^{(r-1)(m-n)} \\ &\quad \times n^{(r-1)(m-n)} \times m^{n(r-1)} \times [rn(r-2)]^{r-1} \\ &\quad \times \left[n(r-1)^2\right]^{(r-1)(n-1)} \times [rm+(r-1)n] \\ &\quad \times [(r-1)n]^{m-1} \times \frac{1}{(rm+(r-1)n)^n} \\ &\quad \times \left[r(r-1)(m+n)^2\right] \\ &\quad \times [m(r-1)(rm+(r-1)n)]^{n-1} \\ &= r^{r-2} \times (r-1)^{r(m+n)-2r+1} \times (r-2)^{r-1} \\ &\quad \times m^{nr-1} \times n^{mr-1}. \end{aligned} \tag{27}$$

Specially,

$$\begin{aligned}\tau\left(K_r \otimes K_{n,n}\right) &= r^{r-2} \times (r-2)^{r-1} \times (r-1)^{2r(n-1)+1} \times n^{2(nr-1)}; \quad n \ge 1. \end{aligned} \tag{28}$$

□

6. Number of Spanning Trees of Symmetric Product of Graphs

The symmetric product, $G_1 \oplus G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 \oplus G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \oplus G_2$ if and only if either u_1 is adjacent to v_1 in G_1 and u_2 is not adjacent to v_2 in G_2 or u_1 is not adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [24].

Theorem 9. *For $m, n \ge 1$ and $r \ge 2$, we have*

$$\begin{aligned}\tau\left(K_r \oplus K_{m,n}\right) &= r^{r-2} \times ((r-1)m+n)^{r(m-1)} \\ &\quad \times ((r-1)n+m)^{r(n-1)} \times \left(m^2+n^2+rmn\right)^{r-1}. \end{aligned} \tag{29}$$

Proof. Applying Lemma 1, we have

$$
\begin{aligned}
&\tau\left(K_r \oplus K_{m,n}\right) \\
&= \frac{1}{\left(r\left(m+n\right)\right)^2} \det\left(r\left(m+n\right)I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{r^2\left(m+n\right)^2} \\
&\times \det \left(\begin{array}{cccccccccccc}
(r-1)m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & (r-1)m+n+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & (r-1)n+m+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & (r-1)n+m+1 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1
\end{array}\right.
\end{aligned}
$$

$$\left.\begin{matrix}
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \cdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \vdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & (r-1)\,m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & (r-1)\,m+n+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & (r-1)\,n+m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & (r-1)\,n+m+1
\end{matrix}\right). \tag{30}$$

Using Lemma 3, we obtain

$$\tau\left(K_r \oplus K_{m,n}\right) = \frac{1}{r^2(m+n)^2}\det\begin{pmatrix}
A & B & \cdots & \cdots & \cdots & B \\
B & A & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & A & B \\
B & \cdots & \cdots & \cdots & B & A
\end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{(r(m+n))^2}[\det(A-B)]^{r-1}[\det(A+(r-1)B)] \\
&= \frac{1}{r^2(m+n)^2} \\
&\times \left(\det \begin{pmatrix}
(r-1)m+n+1 & 1 & \cdots & 1 & -1 & \cdots & \cdots & -1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & (r-1)m+n+1 & -1 & \cdots & \cdots & -1 \\
-1 & \cdots & \cdots & -1 & (r-1)n+m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
-1 & \cdots & \cdots & -1 & 1 & \cdots & 1 & (r-1)n+m+1
\end{pmatrix} \right)^{r-1} \\
&\times \det \begin{pmatrix}
(r-1)m+n+1 & 1 & \cdots & 1 & (r-1) & \cdots & \cdots & (r-1) \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & (r-1)m+n+1 & (r-1) & \cdots & \cdots & (r-1) \\
(r-1) & \cdots & \cdots & (r-1) & (r-1)n+m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
(r-1) & \cdots & \cdots & (r-1) & 1 & \cdots & 1 & (r-1)n+m+1
\end{pmatrix} \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}\right)^{r-1} \times \det\begin{pmatrix} D & E \\ E^T & F \end{pmatrix}.
\end{aligned} \tag{31}
$$

Using Lemma 4, we get

$$
\begin{aligned}
&\tau(K_r \oplus K_{m,n}) \\
&= \frac{1}{r^2(m+n)^2} \times (\det A)^{r-1}\left(\det\left(C - B^T A^{-1} B\right)\right)^{r-1} \times \det D \det\left(F - E^T D^{-1} E\right) \\
&= \frac{(rm+n)^{r-1}((r-1)m+n)^{(r-1)(m-1)}}{r^2(n+m)^2}
\end{aligned}
$$

$$\times \left(\det \begin{pmatrix} \frac{(r-1)\,n^2 + \left[\left(r^2 - r + 1\right) m + 1\right] n + rm^2 + (r-1)\,m}{n+rm} & \frac{n+(r-1)\,m}{n+rm} & \cdots & \frac{n+(r-1)\,m}{n+rm} \\ \frac{n+(r-1)\,m}{n+rm} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{n+(r-1)\,m}{n+rm} \\ \frac{n+(r-1)\,m}{n+rm} & \cdots & \frac{n+(r-1)\,m}{n+rm} & \frac{(r-1)\,n^2 + \left[\left(r^2 - r + 1\right) m + 1\right] n + rm^2 + (r-1)\,m}{n+rm} \end{pmatrix}_{n\times n} \right)^{r-1}$$

$$\times (rm+n)\,((r-1)\,m+n)^{m-1}$$

$$\times \det \begin{pmatrix} \frac{(r-1)\,n^2 + \left[\left(r^2 - r + 1\right) m + 1\right] n + rm^2 + \left(-r^2 + 3r - 1\right) m}{n+rm} & \frac{n+\left(-r^2+3r-1\right) m}{n+rm} & \cdots & \frac{n+(-r^2+3r-1)m}{n+rm} \\ \frac{n+\left(-r^2+3r-1\right) m}{n+rm} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{n+\left(-r^2+3r-1\right) m}{n+rm} \\ \frac{n+\left(-r^2+3r-1\right) m}{n+rm} & \cdots & \frac{n+(-r^2+3r-1)m}{n+rm} & \frac{(r-1)n^2 + \left[\left(r^2 - r + 1\right) m + 1\right] n + rm^2 + \left(-r^2 + 3r - 1\right) m}{n+rm} \end{pmatrix}_{n\times n}$$

$$= \frac{(rm+n)^{r-1}((r-1)m+n)^{(r-1)(m-1)}}{r^2(n+m)^2} \times \left(\frac{n+(r-1)m}{n+rm}\right)^{n(r-1)}$$

$$\times\left(\det\begin{pmatrix}\frac{(r-1)n^2+\left[\left(r^2-r+1\right)m+1\right]n+rm^2+(r-1)m}{n+(r-1)m} & 1 & \cdots & 1\\ 1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 1 & \cdots & 1 & \frac{(r-1)n^2+\left[\left(r^2-r+1\right)m+1\right]n+rm^2+(r-1)m}{n+(r-1)m}\end{pmatrix}_{n\times n}\right)^{r-1}$$

$$\times(rm+n)((r-1)m+n)^{m-1}\times\left(\frac{n+\left(-r^2+3r-1\right)m}{n+rm}\right)^{n}$$

$$\times\det\begin{pmatrix}\frac{(r-1)n^2+\left[\left(r^2-r+1\right)m+1\right]n+rm^2+\left(-r^2+3r-1\right)m}{n+(-r^2+3r-1)m} & 1 & \cdots & 1\\ 1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 1 & \cdots & 1 & \frac{(r-1)n^2+[(r^2-r+1)m+1]n+rm^2+\left(-r^2+3r-1\right)m}{n+\left(-r^2+3r-1\right)m}\end{pmatrix}_{n\times n}. \tag{32}$$

Using Lemma 2, we have

$$\begin{aligned}\tau\left(K_r\oplus K_{m,n}\right) &= \frac{1}{r^2(m+n)^2}\times(rm+n)^{r-1}\times((r-1)m+n)^{(r-1)(m-1)}\\ &\times\left(\frac{n+(r-1)m}{n+rm}\right)^{(r-1)n}\times\frac{1}{(n+(r-1)m)^{(r-1)n}}\\ &\times\left(rn^2+rm^2+r^2nm\right)^{r-1}\\ &\times\left(rm^2+(r^2-r+1)nm+(r-1)n^2\right)^{(r-1)(n-1)}\\ &\times(rm+n)\times((r-1)m+n)^{m-1}\\ &\times\left(\frac{n+\left(-r^2+3r-1\right)m}{n+rm}\right)^{n}\\ &\times\frac{1}{(n+\left(-r^2+3r-1\right)m)^{n}}\times\left(rn^2+rm^2+2rmn\right)\\ &\times\left(rm^2+\left(r^2-r+1\right)nm+(r-1)n^2\right)^{n-1}\\ &= r^{r-2}\times((r-1)m+n)^{r(m-1)}\\ &\times((r-1)n+m)^{r(n-1)}\times\left(m^2+n^2+rmn\right)^{r-1}.\end{aligned}\tag{33}$$

Specially,

$$\tau\left(K_r\oplus K_{n,n}\right)=r^{2rn-r-2}\times n^{2rn-2}\times(r+2)^{r-1};\quad n\geq 1. \tag{34}$$

□

7. Number of Spanning Trees of Strong Sum of Graphs

The strong sum, $G_1 * G_2$, of two graphs G_1 and G_2 is the simple graph with $V(G_1 * G_2)=V_1\times V_2$ where (u_1,u_2) and (v_1,v_2) are adjacent in $G_1 * G_2$ if and only if u_2 is adjacent to v_2 in G_2 and either u_1 is adjacent to v_1 in G_1 or $u_1=v_1$ [24].

Theorem 10. *For $m,n\geq 1$ and $r\geq 2$, we have*

$$\tau\left(K_r * K_{m,n}\right)=r^{(m+n)r-2}\times m^{rn-1}\times n^{rm-1}. \tag{35}$$

Proof. Applying Lemma 1, we have

$$
\begin{aligned}
&\tau\left(K_r * K_{m,n}\right) \\
&= \frac{1}{\left(r\left(m+n\right)\right)^2} \det\left(r\left(m+n\right) I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{r^2\left(m+n\right)^2} \\
&\times \det \begin{pmatrix}
rn+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & rn+1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & rm+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & rm+1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & rn+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & rn+1 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & rm+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & rm+1
\end{pmatrix}.
\end{aligned}
\tag{36}
$$

Using Lemma 3, we obtain

$$
\begin{aligned}
\tau\left(K_r * K_{m,n}\right) &= \frac{1}{r^2(m+n)^2}\det\begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & A & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix} \\
&= \frac{1}{(r(m+n))^2}\left[\det(A-B)\right]^{r-1}\left[\det\left(A+(r-1)B\right)\right] \\
&= \frac{1}{r^2(m+n)^2}\left(\det\begin{pmatrix} rn & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & rn & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & rm & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & rm \end{pmatrix}\right)^{r-1} \\
&\quad \times \det\begin{pmatrix} r(n+1) & r & \cdots & r & 0 & \cdots & \cdots & 0 \\ r & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r & \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & r(n+1) & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & r(m+1) & r & \cdots & r \\ \vdots & \ddots & \ddots & \vdots & r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & r \\ 0 & \cdots & \cdots & 0 & r & \cdots & r & r(m+1) \end{pmatrix} \\
&= \frac{1}{r^2(m+n)^2}\left[(rn)^m(rm)^n\right]^{r-1} \times r^m \det\begin{pmatrix} n+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & n+1 \end{pmatrix}_{m\times m} \\
&\quad \times r^n \det\begin{pmatrix} m+1 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & m+1 \end{pmatrix}_{n\times n}.
\end{aligned}
\tag{37}
$$

Using Lemma 2, we get

$$\begin{aligned}\tau\left(K_r * K_{m,n}\right) = & \frac{1}{r^2(m+n)^2} \times r^{(m+n)(r-1)} \\ & \times n^{m(r-1)} \times m^{n(r-1)} \times r^{(m+n)} \\ & \times (n+1+m-1)(n+1-1)^{m-1} \\ & \times (m+1+n-1)(m+1-1)^{n-1} \\ = & r^{(m+n)r-2} \times m^{nr-1} \times n^{mr-1}.\end{aligned} \tag{38}$$

Specially,

$$\tau\left(K_r * K_{n,n}\right) = (nr)^{2(rn-1)}; \quad n \geq 1. \tag{39}$$

□

8. Conclusion

Driving formulas for different types of graphs can prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequence related to network reliability. Some computationally hard problems, such as the Steiner tree problem and the traveling salesperson problem, can be solved approximately by using spanning trees [25]. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and lemmas and their proofs.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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