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Review Article

Existence of Positive Solutions for a Functional Fractional Boundary Value Problem

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We study the existence of positive solutions for a boundary value problem of fractional-order functional differential equations. Several new existence results are obtained.

1. Introduction

Fractional differential equations can describe many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, control, and engineering. Due to their considerable importance and application, significant progress has been made in there are a great number of excellent works about ordinary and partial differential equations involving fractional derivatives; see, for instance, [1–15].

As pointed out in [16], boundary value problems associated with functional differential equations have arisen from problems of physics and variational problems of control theory appeared early in the literature; see [17, 18]. Since then many authors (see, e.g., [19–23]) investigated the existence of solutions for boundary value problems concerning functional differential equations. Recently an increasing interest in studying the existence of solutions for boundary value problems of fractional-order functional differential equations is observed; see for example, [24–26].

For $\tau > 0$, we denote by C_τ the Banach space of all continuous functions $\psi : [-\tau, 0] \to R$ endowed with the sup-norm

$$\|\psi\|_{[-\tau,0]} := \sup\{|\psi(s)| : s \in [-\tau, 0]\}. \tag{1.1}$$

If $u : [-\tau, 1] \to R$, then for any $t \in [0,1]$, we denote by u_t the element of C_τ defined by

$$u_t(\theta) = u(t+\theta), \quad \text{for } \theta \in [-\tau, 0]. \tag{1.2}$$

In this paper we investigate a fractional-order functional differential equation of the form

$$D^\rho u(t) = f(t, u_t), \quad t \in [0,1], \tag{1.3}$$

where $\rho \in (m-1, m]$ ($m \geq 3$ is a natural number), $f(t, u_t) : [0,1] \times C_\tau \to R$ is a continuous function, associated with the boundary condition

$$u'(0) = \cdots = u^{(m-2)}(0) = 0, \quad u^{(m-2)}(1) = 0, \tag{1.4}$$

and the initial condition

$$u_0 = \phi, \tag{1.5}$$

where ϕ is an element of the space

$$C_\tau^+(0) := \{\psi \in C_\tau : \psi(s) \geq 0, \ s \in [-\tau, 0], \ \psi(0) = 0\}. \tag{1.6}$$

To the best of the authors knowledge, no one has studied the existence of positive solutions for problem (1.3)–(1.5). The aim of this paper is to fill the gap in the relevant literatures. The key tool in finding our main results is the following well-known fixed point theorem due to Krasnoselski [27].

Theorem 1.1. *Let B be a Banach space and let K be a cone in B. Assume that Ω_1 and Ω_2 are open subsets of B, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

$$\|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1, \qquad \|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2, \tag{1.7}$$

or

$$\|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1, \qquad \|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2, \tag{1.8}$$

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

Firstly, we recall some definitions of fractional calculus, which can be found in [11–14].

Definition 2.1. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.1}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional integral of order α of a function $f : (0, \infty) \to R$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} ds, \tag{2.2}$$

provided that the integral exists.

The following lemma is crucial in finding an integral representation of the boundary value problem (1.3)–(1.5).

Lemma 2.3 (see [4]). *Suppose that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.3}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.

From Lemma 2.3, we now give an integral representation of the solution of the linearized problem.

Lemma 2.4. *If $h \in C[0,1]$, then the boundary value problem*

$$D^{\rho} u(t) + h(t) = 0, \quad 0 < t < 1, \; m - 1 < \rho \le m, \tag{2.4}$$

$$u(0) = u'(0) = \cdots = u^{(m-2)}(0) = 0, \quad u^{(m-2)}(1) = 0 \tag{2.5}$$

has a unique solution

$$u(t) = \int_{0}^{1} G(t,s) h(s) ds, \tag{2.6}$$

where

$$G(t,s) = \frac{1}{\Gamma(\rho)} \begin{cases} t^{\rho-1}(1-s)^{\rho-m+1} - (t-s)^{\rho-1}, & s < t, \\ t^{\rho-1}(1-s)^{\rho-m+1}, & t \leq s. \end{cases} \tag{2.7}$$

Proof. We may apply Lemma 2.3 to reduce BVP (2.4), (2.5) to an equivalent integral equation

$$u(t) = c_1 t^{\rho-1} + c_2 t^{\rho-2} + \cdots + c_m t^{\rho-m} - \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} h(s)ds. \tag{2.8}$$

By the boundary condition (2.5), we easily obtain that

$$\begin{gathered} c_2 = c_3 = \cdots = c_m = 0, \\ c_1 = \frac{1}{\Gamma(\rho)} \int_0^1 (1-s)^{\rho-m+1} h(s)ds. \end{gathered} \tag{2.9}$$

Hence, the unique solution of BVP (2.4), (2.5) is

$$\begin{aligned} u(t) &= \int_0^1 \frac{1}{\Gamma(\rho)} t^{\rho-1}(1-s)^{\rho-m+1} h(s)ds - \int_0^t \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned} \tag{2.10}$$

The proof is complete. □

Lemma 2.5. $G(t,s)$ *has the following properties.*

(i) $0 \leq G(t,s) \leq B(s)$, $t, s \in [0,1]$, *where*

$$B(s) = \frac{(1-s)^{\rho-m+1} - (1-s)^{\rho-1}}{\Gamma(\rho)}; \tag{2.11}$$

(ii) $G(t,s) \geq (t^{\rho-1}/(m-2))B(s)$, *for* $0 \leq t, s \leq 1$.

Proof. It is easy to check that (i) holds. Next, we prove that (ii) holds. If $t > s$, then

$$
\begin{aligned}
\frac{G(t,s)}{B(s)} &= \frac{t^{\rho-1}(1-s)^{\rho-m+1}-(t-s)^{\rho-1}}{(1-s)^{\rho-m+1}-(1-s)^{\rho-1}} \\
&= \frac{t^{m-2}(t-ts)^{\rho-m+1}-(t-s)^{\rho-1}(t-ts)^{\rho-m+1}}{(1-s)^{\rho-m+1}-(1-s)^{\rho-1}} \\
&\geq \frac{t^{m-2}(t-ts)^{\rho-m+1}-(t-s)^{m-2}(t-ts)^{\rho-m+1}}{(1-s)^{\rho-m+1}-(1-s)^{\rho-1}} \\
&= \frac{t^{\rho-m+1}(1-s)^{\rho-m+1}\left[t^{m-2}-(t-s)^{m-2}\right]}{(1-s)^{\rho-m+1}\left[1-(1-s)^{m-2}\right]} \\
&= \frac{t^{\rho-m+1}\left[t^{m-3}+t^{m-4}(t-s)+\cdots+(t-s)^{m-3}\right]}{1+(1-s)+\cdots+(1-s)^{m-3}} \\
&\geq \frac{t^{\rho-m+1}t^{m-3}}{1+(1-s)+\cdots+(1-s)^{m-3}} \\
&\geq \frac{t^{\rho-m+1}t^{m-3}}{m-2} = \frac{t^{\rho-2}}{m-2} \geq \frac{t^{\rho-1}}{m-2}.
\end{aligned}
\tag{2.12}
$$

If $t \leq s$, then

$$
\begin{aligned}
\frac{G(t,s)}{B(s)} &= \frac{t^{\rho-1}(1-s)^{\rho-m+1}}{(1-s)^{\rho-m+1}-(1-s)^{\rho-1}} \geq \frac{t^{\rho-1}(1-s)^{\rho-m+1}}{(1-s)^{\rho-m+1}} \\
&= t^{\rho-1} \geq \frac{t^{\rho-1}}{m-2}.
\end{aligned}
\tag{2.13}
$$

The proof is complete. □

3. Main Result

In the sequel we will denote by $C_0[0,1]$ the space of all continuous functions $x : [0,1] \to R$ with $x(0) = 0$. This is a Banach space when it is furnished with the usual sup-norm

$$
\|u\|_0 = \max_{t\in[0,1]}|u(t)|. \tag{3.1}
$$

For each $\phi \in C_\tau^+(0)$ and $x \in C_0[0,1]$, we define

$$x_t(s;\phi) := \begin{cases} x(t+s), & t+s \geq 0, \\ \phi(t+s), & t+s \leq 0, \end{cases} \quad s \in [-\tau, 0], \tag{3.2}$$

and observe that $x_t(;\phi) \in C_\tau$.

By a solution of the boundary value problem (1.3)–(1.5), we mean a function $u \in C_0[0,1]$ such that $D^\rho u$ exists on $[0,1]$ and u satisfies boundary condition (1.4), and for a certain ϕ, the relation

$$D^\rho u(t) = f\left(t, u_t(\cdot;\phi)\right) \tag{3.3}$$

holds for all $t \in [0,1]$.

By Lemma 2.4 we know that a function u is a solution of the boundary value problem (1.3)–(1.5) if and only if it satisfies

$$u(t) = \int_0^1 G(t,s) f\left(s, u_s(\cdot;\phi)\right) ds := \left(T_\phi u\right)(t), \quad s \in [0,1]. \tag{3.4}$$

We set

$$C_0^+[0,1] = \{u \in C_0[0,1] : u(t) \geq 0,\ t \in [0,1]\}. \tag{3.5}$$

Define the cone $P \subset C_0[0,1]$ by

$$P = \left\{ y \in C_0^+[0,1] : \min_{\tau \leq t \leq 1} y(t) \geq \frac{1}{m-2} \tau^{\rho-1} \|y\|_0 \right\}, \tag{3.6}$$

where $0 < \tau < 1$.

By Lemma 2.4, the boundary value problem (1.3)–(1.5) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u_s) ds, \quad t \in [0,1]. \tag{3.7}$$

In this paper, we assume that $0 < \tau < 1$, $\phi \in C_\tau^+(0)$, and we make use of the following assumptions.

(H_1) $f(t,\varphi) \geq 0$ for $t \in [0,1]$ and $\varphi \in C_\tau^+(0)$.

(H_2) There exist constants $M > \|\phi\|_{[-\tau,0]}$ and τ_i, $i = 1,2,\ldots,q$, with $0 < \tau_1 < \tau_2 < \cdots < \tau_q < \tau$, as well as continuous functions $a_k \in C[0,1]$ and nondecreasing continuous functions $g_k : R^+ \to R^+$ such that

$$f(t,\varphi) \leq \sum_{k=1}^{q} a_k(t) g_k\left(\|\varphi\|_{[-\tau_k,0]}\right), \quad \varphi \in C_\tau^+(0),\ \|\varphi\|_{[-\tau,0]} \leq M. \tag{3.8}$$

(H_3) There exist functions $\omega : [0,1] \to [0,\tau]$, continuous $c : [0,1] \to R^+$, and nondecreasing $\eta : R^+ \to R^+$ such that

$$f(t,\varphi) \geq c(t)\eta(\varphi(-\omega(t))), \quad \varphi \in C_\tau^+(0). \tag{3.9}$$

Similar to the proof of Lemma 3.2 in [4], we have the following

Lemma 3.1. *Let* (H_1) *holds. Then $T_\phi : P \to P$ is completely continuous.*

Lemma 3.2. *If $0 < \tau < 1$ and $u \in P$, then we have*

$$\left\| u_t(\cdot;\phi) \right\|_{[-\tau,0]} \geq \gamma \|u\|_0, \quad t \in [\tau,1], \tag{3.10}$$

where

$$\gamma = \frac{\tau^{\rho-1}}{m-2}. \tag{3.11}$$

Proof. From the definition of $u_t(s;\phi)$, for $t \geq \tau$, we have

$$u_t(s;\phi) = u(t+s), \quad s \in [-\tau,0]. \tag{3.12}$$

Thus, we get for $u \in P$ that

$$\left\| u_t(\cdot;\phi) \right\|_{[-\tau,0]} = \max_{s\in[-\tau,0]} u(\mathrm{t}+s) \geq u(t) \geq \gamma \|u\|_0, \quad t \geq \tau. \tag{3.13}$$

We are now in a position to present and prove our main result. □

Theorem 3.3. *Let* (H_1), (H_2), *and* (H_3) *hold. If M (as in* (H_2)*), satisfying*

$$\sum_{k=1}^{q} g_k(M) \int_0^1 \left[(1-s)^{\rho-m+1} - (1-s)^{\rho-1}\right] a_k(s)ds \leq \Gamma(\rho)M, \tag{3.14}$$

and there exists a constant $m > 0$ ($m \neq M$) satisfying

$$\eta(\gamma m) \frac{\gamma}{\Gamma(\rho)} \int_\tau^1 \left[(1-s)^{\rho-m+1} - (1-s)^{\rho-1}\right] c(s)ds \geq m, \tag{3.15}$$

then (1.3)–(1.5) *has a positive solution.*

Proof. If $u \in P$ with $\|u\|_0 = M$, then from (3.8), (3.14), and Lemma 2.4 (i), we get for any $t \in [0,1]$ that

$$\begin{aligned}
(T_\phi u)(t) &= \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))\,ds \le \int_0^1 B(s) f(s, u_s(\cdot;\phi))\,ds \\
&\le \sum_{k=1}^{q} \int_0^1 B(s) a_k(s) g_k\Big(\|u_s(\cdot;\phi)\|_{[-\tau_k,0]}\Big)\,ds \\
&\le \sum_{k=1}^{q} g_k(M) \int_0^1 B(s) a_k(s)\,ds \\
&= \frac{1}{\Gamma(\rho)} \sum_{k=1}^{q} g_k(M) \int_0^1 \Big[(1-s)^{\rho-m+1} - (1-s)^{\rho-1}\Big] a_k(s)\,ds \\
&\le M.
\end{aligned} \tag{3.16}$$

Now if we set

$$\Omega_1 = \{u \in C[0,1] : \|u\|_0 < M\}, \tag{3.17}$$

then (3.16) shows that $\|T_\phi u\|_0 \le \|u\|_0$ for $u \in P \cap \partial\Omega_1$.

Without loss of generality, we suppose that $m < M$. For $u \in P$ with $\|u\|_0 = m$, we have from Lemma 3.2 and (3.9) that

$$\begin{aligned}
(T_\phi u)(t) &= \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))\,ds \ge \int_\tau^1 \gamma B(s) f(s, u_s(\cdot;\phi))\,ds \\
&\ge \int_\tau^1 \gamma B(s) c(s) \eta\big(u_s(-\omega(s);\phi)\big)\,ds = \int_\tau^1 \gamma B(s) c(s) \eta(u(s-\omega(s)))\,ds \\
&\ge \eta\big(\gamma\|u\|_0\big) \int_\tau^1 \gamma B(s) c(s)\,ds \\
&= \omega(\gamma m) \frac{\gamma}{\Gamma(\rho)} \int_\tau^1 \Big[(1-s)^{\rho-m+1} - (1-s)^{\rho-1}\Big] c(s)\,ds \ge m.
\end{aligned} \tag{3.18}$$

Now if we set

$$\Omega_2 = \{u \in C[0,1] : \|u\|_0 < m\}, \tag{3.19}$$

then (3.18) shows that $\|T_\phi u\|_0 \ge \|u\|_0$ for $u \in P \cap \partial\Omega_2$.

Hence by the first part of Theorem 1.1, T_ϕ has a fixed point $u \in P \cap (\overline{\Omega}_1 \setminus \Omega_2)$, and accordingly, u is a solution of (1.3)–(1.5). □

Having in mind the proof of Theorem 3.3, one can easily conclude the following results.

Theorem 3.4. *Let* (H_1)*,* (H_2)*,* (H_3)*, and* (3.14) *hold. If the function* η *satisfies the condition*

$$\limsup_{x\to+\infty}\frac{\eta(x)}{x}>\frac{\Gamma(\rho)}{\gamma^2\int_\tau^1\left[(1-s)^{\rho-m+1}-(1-s)^{\rho-1}\right]c(s)ds},\tag{3.20}$$

where γ *is as in* (3.11)*. Then* (1.3)–(1.5) *has a positive solution.*

Theorem 3.5. *Let* (H_1)*,* (H_2)*,* (H_3)*, and* (3.14) *hold. If the function* η *satisfies the condition*

$$\limsup_{x\to 0+}\frac{\eta(x)}{x}>\frac{\Gamma(\rho)}{\gamma^2\int_\tau^1\left[(1-s)^{\rho-m+1}-(1-s)^{\rho-1}\right]c(s)ds}.\tag{3.21}$$

Then (1.3)–(1.5) *has a positive solution.*

Theorem 3.6. *Let* (H_1)*,* (H_3) *and* (3.15) *hold. Assume that*

(H_4) $f_\infty=0$,

uniformly for $t\in[0,1]$*, where*

$$f_\infty:=\lim_{\psi\in C_\tau^+,\|\psi\|_{[-\tau,0]}\to\infty}\frac{f(t,\psi)}{\|\psi\|_{[-\tau,0]}}.\tag{3.22}$$

Then (1.3)–(1.5) *has a positive solution.*

Proof. Since (H_3) and (3.15) hold, we have from the proof of Theorem 3.3 that

$$\|T_\phi u\|_0\geq\|u\|_0\quad\text{for } u\in P\cap\partial\Omega_2.\tag{3.23}$$

From (H_4), we will consider two cases in the following.

Case 1 (f is bounded). In this case, there exists a positive constant $L>0$ such that

$$|f(t,\psi)|\leq L,\quad t\in[0,1],\ \psi\in C_\tau^+.\tag{3.24}$$

We choose a positive constant

$$M_1\geq\frac{L(m-2)}{\Gamma(\rho+1)(\rho-m+2)}.\tag{3.25}$$

For $u \in P$ with $\|u\|_0 = M_1$, we have

$$\begin{aligned}(T_\phi u)(t) &= \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))ds \le \int_0^1 B(s) f(s, u_s(\cdot;\phi))ds \\ &\le \int_0^1 LB(s)ds = \frac{L}{\Gamma(\rho)} \int_0^1 \left[(1-s)^{\rho-m+1} - (1-s)^{\rho-1}\right] ds \\ &= \frac{L}{\Gamma(\rho)} \left(\frac{1}{\rho-m+2} - \frac{1}{\rho}\right) \le M_1.\end{aligned} \tag{3.26}$$

Case 2 (f is unbounded). In this case, there exists a positive constant $M_2 \ge \|\phi\|_{[-\tau,0]}$ such that

$$f(t,\varphi) \le \epsilon \|\varphi\|_{[-\tau,0]}, \quad t \in [0,1], \ \varphi \in C_\tau^+ \text{ with } \|\varphi\|_{[-\tau,0]} \ge M_2, \tag{3.27}$$

where $\epsilon > 0$ is a constant satisfying

$$\epsilon \le \frac{\Gamma(\rho+1)(\rho-m+2)}{m-2}. \tag{3.28}$$

By the definition of $u_s(\cdot;\phi)$, we easily obtain that

$$\|u_s(\cdot;\phi)\|_{[-\tau,0]} \le \max\left\{\|u\|_0, \|\phi\|_{[-\tau,0]}\right\}. \tag{3.29}$$

If $u \in P$ with $\|u\|_0 = M_2$, then from (3.8), (3.14), and Lemma 2.4 (i), we get for any $t \in [0,1]$ that

$$\begin{aligned}(T_\phi u)(t) &= \int_0^1 G(t,s) f(s, u_s(\cdot;\phi))ds \le \int_0^1 B(s) f(s, u_s(\cdot;\phi))ds \\ &= \epsilon M_2 \int_0^1 B(s)ds = \frac{\epsilon M_2}{\Gamma(\rho)} \int_0^1 \left[(1-s)^{\rho-m+1} - (1-s)^{\rho-1}\right] ds \\ &= \frac{\epsilon M_2 (m-2)}{\Gamma(\rho+1)(\rho-m+2)} \le M_2.\end{aligned} \tag{3.30}$$

Set $M_3 = \max\{M_1, M_2, 2m\}$. Then in either case we may put

$$\Omega_3 = \{u \in C[0,1] : \|u\|_0 < M_3\}, \tag{3.31}$$

and for $u \in P \cap \partial\Omega_3$, $\|T_\phi u\|_0 \ge \|u\|_0$. By the first part of Theorem 1.1, T_ϕ has a fixed point $u \in P \cap (\overline{\Omega}_3 \setminus \Omega_2)$, and accordingly, u is a solution of (1.3)–(1.5). □

4. An Example

To illustrate our results, we present the following example.

Example 4.1. Consider the boundary value problem of fractional-order functional differential equations

$$D^{2.6}u(t) = \sqrt{1-t}\left|u\left(t-\frac{1}{3}\right)\right|^2 + 2\exp\left(\left|u\left(t-\frac{1}{2}\right)\right|^{1/2}\right), \quad t \in [0,1],$$
$$u'(0) = u'(1) = 0, \tag{4.1}$$
$$u(t) = \phi(t), \quad t \in \left[-\frac{1}{2}, 0\right],$$

where $\phi \in C^+_{1/2}(0)$ with $\|\phi\|_{[-1/2,0]} \le 4$.

Observe that

$$\begin{aligned} f(t,\psi) &= \sqrt{1-t}|\psi(-\alpha)|^2 + 2\exp\left(|\psi(-\beta)|^{1/2}\right) \\ &\le a_1(t)g_1\left(\|\psi\|_{[-\tau_1,0]}\right) + a_2(t)g_2\left(\|\psi\|_{[-\tau_2,0]}\right), \end{aligned} \tag{4.2}$$

where we have set

$$a_1(t) := \sqrt{1-t}, \quad a_2(t) := 2, \quad \tau_1 = \alpha, \quad \tau_2 = \beta, \quad g_1(x) := x^2, \quad g_2(x) := \exp(\sqrt{x}). \tag{4.3}$$

Also we obtain

$$f(t,\psi) \ge c(t)\eta(\psi(-\omega(t))), \tag{4.4}$$

where

$$c(t) := 2, \quad \omega(t) = \beta, \quad \eta(x) = \exp(\sqrt{x}), \quad x \ge 0. \tag{4.5}$$

Thus, (H_1) and (H_2) hold. Note that here $\rho = 2.6$, $m = 3$, we have

$$\sum_{k=1}^{2} g_k(4)\int_0^1 \left[(1-s)^{0.6} - (1-s)^{1.6}\right]a_k(s)ds = 5.2615 < 5.7184 = 4\Gamma(2.6), \tag{4.6}$$

and $\|\phi\|_{[-1/2,0]} \le 4$, which implies that condition (3.14) holds with $M = 4$. Finally, we observe that $\lim_{x\to+\infty}(\eta(x)/x) = +\infty$, and therefore condition (3.20) is satisfied. Then all assumptions of Theorem 3.4 hold. Thus, with Theorem 3.4, problem (1.3)–(1.5) has at least one positive solution.

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