UDC 517.93

CORRESPONDENCE

A Reliable Method for the Numerical Integration of a Large Class of Ordinary and Partial Differential Equations

R. S. LINDZEN and H.-L. KUO

Department of the Geophysical Sciences, University of Chicago, Illinois

The purpose of this note is to describe the simple extension of a popular method of solving second-order ordinary differential equations with two end-point boundary conditions to nth order ordinary differential equations and to partial differential equations that are second order in one direction. The orginal method is simply a version of Gaussian elimination; and the extension (to be described) has, we have discovered, been published, slightly differently, before. We feel, however, that the present note will be of value, since the extension has proven very useful to both of us, and is seldom used among numerical analysts and meteorologists.

We begin by reviewing second-order ordinary differential equations. Consider

$$\frac{d^2f}{dx^2}+g(x)\frac{df}{dx}+h(x)f=r(x) \tag{1}$$

where

$$\frac{df}{dx}+a_1f=b_1 \text{ at } x=0$$

and

$$\frac{df}{dx}+a_2f=b_2 \text{ at } x=1.$$

In finite differences this becomes

$$A_nf_{n-1}+B_nf_n+C_nf_{n+1}=D_n, \tag{2}$$

$$n=1, 2, 3, \ldots, N-1,$$

where

$$A_n=\frac{1}{(\delta x)^2}-\frac{g(x_n)}{2\delta x},$$

$$B_n=-\frac{2}{(\delta x)^2}+h(x_n),$$

$$C_n=\frac{1}{(\delta x)^2}+\frac{g(x_n)}{2\delta x},$$

and

$$D_n=r(x_n).$$

δx is the grid interval used in finite-difference approximation to equation (1),

$$A_bf_0+B_bf_1=D_b,$$

and

$$A_tf_{N-1}+B_tf_N=D_t$$

where N is the level number corresponding to $x=1$. The solution of equation (2) (following Richtmyer, 1957) goes as follows:

$$f_n=\alpha_nf_{n+1}+\beta_n \tag{3}$$

where α_n and β_n are newly introduced variables. Then

$$f_{n-1}=\alpha_{n-1}f_n+\beta_{n-1}. \tag{4}$$

Substituting equation (4) into (2) we obtain

$$\alpha_n=\frac{-C_n}{(A_n\alpha_{n-1}+B_n)} \tag{5}$$

and

$$\beta_n=\frac{(D_n-A_n\beta_{n-1})}{(A_n\alpha_{n-1}+B_n)}. \tag{6}$$

Thus, knowing α_0, β_0 we may readily obtain all α_n's and β_n's. From the lower boundary condition

$$\alpha_0=-\frac{B_n}{A_b} \tag{7}$$

and

$$\beta_0=\frac{D_b}{A_b}. \tag{8}$$

Equation (3) may be used to obtain f at all n's, provided we know f_N. With the upper boundary condition we have

$$A_tf_{N-1}+B_tf_N=D_t. \tag{9}$$

We also have

$$f_{N-1}=\alpha_{N-1}f_N+\beta_{N-1}. \tag{10}$$

These may be solved to obtain

$$f_N=\frac{D_t-A_t\beta_{N-1}}{(B_t+\alpha_{N-1}A_t)}. \tag{11}$$

Thus our solution is formally complete. The procedure is valid provided that

$$A_n\alpha_{n-1}+B_n\neq 0$$

for all n. A sufficient condition for this to be so is that

$$0<H_*\leq -h\leq H^*$$

and

$$\delta x\leq\frac{2}{G}$$

where $G=\max|g(x)|$; H_* and H^* are positive constants.

These are, however, *by no means necessary conditions.* The authors have yet to find an inhomogeneous, well-posed problem for which the method fails. In particular, many wave-type problems where $h>0$ have been solved. It should be added that when $h=\text{constant}$ and $g=0$, beyond a certain point in the domain, the requirement of two end points is readily extended to include a radiation condition. Let $h=\lambda^2$ for $x>x_1$. If we wish our solution to behave as $e^{i\lambda x}$ beyond x_1, then we simply impose

$$df/dx=i\lambda f$$

at some $x>x_1$ as a boundary condition. Such an application may be found in Lindzen (1968). Also, when $h=-\lambda^2$ the method has no difficulty in separating growing from decaying solutions (Carrier and Pearson, 1968).

The extension of the above method to nth order ordinary differential equations is straightforward. Consider

$$\frac{d^n f}{dx^n}+g_1(x)\frac{d^{n-1}f}{dx^{n-1}}+g_2(x)\frac{d^{n-2}f}{dx^{n-2}}+\ldots=r(x). \qquad (12)$$

For simplicity let n be even. Also, let there be appropriate boundary conditions at $x=0$ and 1. What is meant by appropriate will become evident. Let

$$f_1=\frac{d^{n-2}f}{dx^{n-2}}$$

$$\vdots$$

$$f_{\frac{n}{2}-1}=\frac{d^2 f}{dx^2}. \qquad (13a)$$

Equation (13a) may be rewritten

$$\frac{d^2 f_K}{dx^2}=f_{K-1};\ K=1,2,\ldots,m-1 \qquad (13b)$$

where $m=n/2$. Equation (12) becomes

$$\frac{d^2 f_1}{dx^2}+g_1\frac{df_1}{dx}+g_2 f_1+\sum_{l=2}^{m}\left(g_{2l-1}\frac{df_l}{dx}+g_{2l}f_l\right)=r(x) \qquad (14)$$

where $f_m=f$. Equations (13b) and (14) may be rewritten

$$\frac{d^2}{dx^2}\mathbf{f}+\mathcal{A}(x)\frac{d}{dx}\mathbf{f}+\mathcal{B}(x)\mathbf{f}=\mathbf{r}(x) \qquad (15)$$

where

$$\mathbf{f}=(f_1,f_2,\ldots,f_{\frac{n}{2}-1},f)$$

and

$$\mathcal{A}=\begin{pmatrix} g_1 & g_3 & g_5 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{pmatrix}$$

$$\mathcal{B}=\begin{pmatrix} g_2 & g_4 & g_6 & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix} r \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Instead of equation (2), we now write

$$\mathbf{A}_n\mathbf{f}_{n-1}+\mathbf{B}_n\mathbf{f}_n+\mathbf{C}_n\mathbf{f}_{n+1}=\mathbf{D}_n \qquad (16)$$

where equation (16) is the finite-difference approximation to (15);

$$\mathbf{A}_n=\frac{1}{(\delta x)^2}I-\frac{1}{2\delta x}\mathcal{A}(x),$$

$$\mathbf{B}_n=-\frac{2}{(\delta x)^2}I+\mathcal{B}(x),$$

$$\mathbf{C}_n=\frac{1}{(\delta x)^2}I+\frac{1}{2\delta x}\mathcal{A}(x);$$

$$\mathbf{D}_n=\mathbf{r}(x_n)$$

and instead of equation (3), we write

$$\mathbf{f}_n=\boldsymbol{\alpha}_n\mathbf{f}_{n+1}+\boldsymbol{\beta}_n \qquad (17)$$

where $\boldsymbol{\alpha}_n$ is now an $(n/2)\times(n/2)$ matrix and $\boldsymbol{\beta}_n$ is as an n-dimensional vector. It is easily shown that

$$\boldsymbol{\alpha}_n=-(\mathbf{A}_n\boldsymbol{\alpha}_{n-1}+\mathbf{B}_n)^{-1}\mathbf{C}_n \qquad (18)$$

and

$$\boldsymbol{\beta}_n=(\mathbf{A}_n\boldsymbol{\alpha}_{n-1}+\mathbf{B}_n)^{-1}(\mathbf{D}_n-\mathbf{A}_n\boldsymbol{\beta}_{n-1}). \qquad (19)$$

Thus, if we obtain $\boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}_0$ from our boundary condition at $x=0$, we may readily obtain all the other $\boldsymbol{\alpha}_n$'s and $\boldsymbol{\beta}_n$'s. At each step, however, we must invert an $(n/2)\times(n/2)$ matrix. For $n\leq 8$, this is a trivial matter. Even for $n\gtrsim 180$, share routines (involving Gaussian elimination) are remarkably effective. As before, the value of $\mathbf{f}_N$ is obtained from the upper boundary condition together with the equation

$$\mathbf{f}_{N-1}=\boldsymbol{\alpha}_{N-1}\mathbf{f}_N+\boldsymbol{\beta}_{N-1}.$$

It should be added that many high-order differential equations result from combining several lower order differentials. Thus, a set of second-order equations may present themselves in the course of analysis—prior to the derivation of the single nth order equation.

Although we are not normally interested in 180th order ordinary differential equations, the limit becomes quite meaningful when we come to partial differential equations.

Consider an equation of the form

$$\frac{\partial^2 f}{\partial x^2}+\mathcal{L}_{y,x}[f]=r(y,x) \tag{20}$$

where $\mathcal{L}_{y,x}$ is a differential operator of arbitrary order in y—but of no greater than first order in x. The finite-difference form of equation (20) is also given by (16) where, however, $\mathbf{f}_n$ is now the set of the values of f at the nth level in x at all the grid points in y. The boundary conditions at $x=0$, 1 (or any other two points) are introduced as before; the boundary conditions at $y=0$, 1 are included in $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$. The present method appears to be genuinely insensitive to equation (20)'s type. Several problems with mixed hyperbolic-elliptic equations have been solved with no difficulty. It is our impression that whenever equation (20) together with its boundary conditions has a continuous solution the present method will determine it. In this respect, our method appears superior to iterative procedures which usually fail for operators that are not purely elliptic. Our method is similar to those described by Cornock (1954), Karlqvist (1952), and Schechter (1960) in connection with the solution of particular partial differential equations. While the application of the method to high-order ordinary differential equations is obvious, we are not familiar with earlier references in this connection. The disadvantage of our method (minor for our purposes) is that it requires the inversion of N (where N=the number of levels) $J\times J$ matrices (where J is either the number of grid points in the y-direction, or—in the case of ordinary differential equations—one-half the order of the differential equation), and the storage of N $J\times J$ matrices and N J-dimensional vectors for use in the backward sweep. In an elegant extension of the method described here, Schechter (1960) reduced the solution of the system of equations (16) to the inversion of a single $J\times J$ matrix. Schechter's method has, however, a serious disadvantage. As the number of levels increases, the condition number of the matrix to be inverted increases. If the equation to be inverted is hyperbolic over a significant part of its domain, the rise in condition number can be astronomical—the matrix becoming uninvertible for practical purposes. Thus, Schechter's method is type-sensitive.

The method described in this paper has been successfully used by the authors to investigate the propagation of planetary scale equatorial waves through shear zones with and without critical levels, the propagation of internal gravity waves with arbitrary distributions of temperature, viscosity, conductivity, anisotropic ion drag, Newtonian cooling and thermal excitation, and the nonlinear flows in the boundary layer of a vortex. The method has also been used by Matsuno (personal communication) to study the propagation of internal Rossby waves in an atmosphere with an arbitrary distribution of zonal wind with latitude and altitude. The results of all these calculations will be published separately. In each case, however, all the matrices to be inverted were of low condition number, and accurately and easily inverted using standard "share" routines.

As a final comment, we should state that many equations of the form of equation (20) are more efficiently solved by relaxation methods. Moreover, when equation (20)'s x and y dependence is separable, a common method of solution is to Fourier transform out one of the dependencies and use the present method for solving the resulting second-order ordinary differential equations. The virtue of the present method is not that it is the most efficient method, but that it appears to be generally reliable.

ACKNOWLEDGMENTS

R. S. Lindzen and H.-L. Kuo wish to acknowledge the support of NSF Grants GA 1622 and GA 1339 respectively.

REFERENCES

Carrier, G. F., and Pearson, C. E., *Ordinary Differential Equations*, Blaisdell Publishing Co., Waltham, Mass., **1968**, 229 pp.

Cornock, A. F., "The Numerical Solution of Poisson's and the Biharmonic Equations by Matrices," *Proceedings of the Cambridge Philosophical Society*, Vol. 50, Pt. 4, University Press, Cambridge, Oct. **1954**, pp. 524–535.

Karlqvist, O., "Numerical Solution of Elliptic Difference Equations by Matrix Methods," *Tellus*, Vol. 4, No. 4, Nov. **1952**, pp. 374–384.

Lindzen, R. S., "The Application of Classical Atmospheric Tidal Theory," *Proceedings of the Royal Society of London*, Ser. A, Vol. 303, No. 1474, Mar. **1968**, pp. 299–316.

Richtmyer, R. D., *Difference Methods for Initial-Value Problems*, Interscience Press, New York, **1957**, 238 pp.

Schechter, S., "Quasi-Tridiagonal Matrices and Type-Insensitive Difference Equations," *Quarterly of Applied Mathematics*, Vol. 18, No. 3, William Byrd Press, Inc., Richmond, Va., Oct. **1960**, pp. 285–295.

[*Received May 23, 1969; revised August 8, 1969*]