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Research Article

New Eighth-Order Derivative-Free Methods for Solving Nonlinear Equations

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A new family of eighth-order derivative-free methods for solving nonlinear equations is presented. It is proved that these methods have the convergence order of eight. These new methods are derivative-free and only use four evaluations of the function per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations could achieve optimal convergence order of 2^{n-1}. Thus, we present new derivative-free methods which agree with Kung and Traub conjecture for $n = 4$. Numerical comparisons are made to demonstrate the performance of the methods presented.

1. Introduction

In this paper, we present a new family of the eighth-order methods to find a simple root α of the nonlinear equation:

$$f(x) = 0, \tag{1.1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D and it is sufficiently smooth in a neighbourhood of α. It is well known that the techniques to solve nonlinear equations have many applications in science and engineering. We will compare our new methods with well-known methods, namely, the classical Steffensen method for its simplicity [1, 2] and recently introduced eighth-order methods [3–5].

The eighth-order methods presented in this paper are derivative-free and only use four evaluations of the function per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. Thus, we present new derivative-free methods which agree with the Kung and Traub conjecture for $n = 4$. In addition, these new eighth-order derivative-free methods have an equivalent efficiency index to the established eighth-order derivative based methods presented in [3–5]. Furthermore, the new eighth-order derivative-free methods have a better efficiency index than the sixth-order derivative-free methods presented recently in [6, 7] and in view of this fact, the new methods are significantly better when compared with the established methods. Consequently, we have found that the new eighth-order derivative-free methods are consistent, stable, and convergent.

This paper is organised as follows. In Section 2, we describe the eighth-order methods that are free from derivatives and prove the important fact that the methods obtained preserve their convergence order. In Section 3, we will briefly state the established methods in order to compare the effectiveness of the new methods. Finally, in Section 4 we demonstrate the performance of each of the methods described.

2. Development of the Eighth-Order Derivative-Free Methods and Analysis of Convergence

In this section, we will define a new family of eighth-order derivative-free methods. In order to establish the order of convergence of these new methods, we state three essential definitions.

Definition 2.1. Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence m is given by

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^m}=\zeta\neq 0, \tag{2.1}$$

where ζ is the asymptotic error constant and $m \in \mathbb{R}^+$.

Definition 2.2. Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1.1). Then, the computational order of convergence [8] may be approximated by

$$\text{COC} \approx \frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}, \tag{2.2}$$

where $n \in \mathbb{N}$.

Definition 2.3. Let β be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [9, 10] and defined as

$$\mu^{1/\beta}, \tag{2.3}$$

where μ is the order of the method.

2.1. The Eighth-Order Derivative-Free Methods

In this subsection, we will define the new eighth-order derivative-free iterative method. In fact, we define different types of eighth-order method by varying the parameters $\beta_i, \phi_j, \omega_k$, and ξ_l. Therefore, the general formula of the new eighth-order method for determining the simple root of (1.1) is given as:

$$w_n = x_n + \beta_i^{-1} f(x_n), \tag{2.4}$$

$$y_n = x_n - \left(\frac{f(x_n)^2}{f(w_n) - f(x_n)}\right), \tag{2.5}$$

$$z_n = y_n - \phi_j \left(\frac{x_n - y_n}{f(x_n) - f(y_n)}\right) f(y_n), \tag{2.6}$$

$$x_{n+1} = z_n - \omega_k \xi_l \left(\frac{f(z_n) - f(y_n)}{z_n - y_n} - \frac{f(y_n) - f(x_n)}{y_n - x_n} + \frac{f(z_n) - f(x_n)}{z_n - x_n}\right)^{-1} f(z_n), \tag{2.7}$$

where $n \in \mathbb{N}$, $\beta \in \mathbb{R}^+$, provided that the denominators (2.5)–(2.7) are not equal to zero.

The parameters used in the above eighth-order method are given as:

$$\begin{gathered}
\beta_i = i^{-1}, \quad i \in \mathbb{R}^+, \\
\phi_1 = \left(1 - \frac{f(y_n)}{f(w_n)}\right)^{-1}, \qquad \phi_2 = \left(1 + \frac{f(y_n)}{f(w_n)}\right), \\
\omega_1 = \left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1}, \qquad \omega_2 = \left(1 + \frac{f(z_n)}{f(w_n)} + \left(\frac{f(z_n)}{f(w_n)}\right)^2\right), \\
\xi_1 = \left(1 - \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right), \qquad \xi_2 = \left(1 + \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right)^{-1}.
\end{gathered} \tag{2.8}$$

In order to obtain a solution of the formula (2.7), we take one parameter from each set given above. Simply varying these parameters, we have many variants of eighth-order derivative-free methods. Furthermore, we will demonstrate the performance of the eighth-order methods with the parameters given in (2.8). To obtain the solution of (1.1) by the new derivative-free methods, we must set a particular initial approximation x_0, ideally close to the simple root. In numerical mathematics, it is very useful and essential to know the behaviour of an approximate method. Therefore, we will prove the order of convergence of the new eighth-order method.

Theorem 2.4. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ be sufficiently smooth in the interval D, the initial approximation x_0 is sufficiently close to α then the order of convergence of the new derivative-free method defined by* (2.7) *is eight.*

Proof. Let α be a simple root of $f(x)$, that is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as

$$e = x - \alpha. \tag{2.9}$$

Using the Taylor expansion, we have

$$f(x_n) = f(\alpha) + f^{'(\alpha)e_n} + 2^{-1} f^{''(\alpha)e_n^2} + 6^{-1} f^{'''(\alpha)e_n^3} + 24^{-1} f^{iv}(\alpha)e_n^4 + \cdots. \tag{2.10}$$

Taking $f(\alpha) = 0$ and simplifying, expression (2.10) becomes

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots, \tag{2.11}$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)} \quad \text{for } k = 1, 2, 3, 4, \ldots. \tag{2.12}$$

Expanding the Taylor series of $f(w_n)$ and substituting $f(x_n)$ given by (2.10), we have

$$f(w_n) = c_1(1 + c_1\beta)e_n + \left(3\beta c_1 c_2 + \beta^2 c_1^2 c_2 + c_2\right)e_n^2 + \cdots. \tag{2.13}$$

Substituting (2.11) and (2.13) in the expression (2.5) gives us

$$y_n - \alpha = x_n - \alpha - \left(\frac{x_n - w_n}{f(x_n) - f(w_n)}\right) f(x_n) = \left(\frac{c_2}{c_1}\right)(\beta c_1 + 1)e_n^2 + \cdots. \tag{2.14}$$

The expansion of $f(y_n)$ about α is given as

$$f(y_n) = \left[c_1(y_n - \alpha) + c_2(y_n - \alpha)^2 + c_3(y_n - \alpha)^3 + \cdots\right]. \tag{2.15}$$

Simplifying (2.15), we have

$$f(y_n) = c_2(c_1\beta + 1)e_n^2 + \left(\frac{\beta c_1^3 c_3 - 2c_2^2 + 3\beta c_1^2 c_3 + 2c_1 c_3 - \beta^2 c_1^2 c_2^2 - 2\beta c_1 c_2^2}{c_1}\right)e_n^3 + \cdots. \tag{2.16}$$

The expansion of the particular term used in (2.6) is given as

$$\phi_1 = \left(1 - \frac{f(y_n)}{f(w_n)}\right)^{-1} = 1 + \left(\frac{c_2}{c_1}\right)e_n + \left(\frac{\beta c_1^2 c_3 - 2\beta c_1 c_2^2 + \beta c_1 c_3 - 2c_2^2}{\lambda c_1^2}\right)e_n^2 + \cdots. \tag{2.17}$$

Substituting appropriate expressions in (2.6), we obtain

$$z_n - \alpha = y_n - \alpha - \left(1 - \frac{f(y_n)}{f(w_n)}\right)\left(\frac{x_n - y_n}{f(x_n) - f(y_n)}\right) f(y_n). \tag{2.18}$$

The Taylor series expansion of $f(z_n)$ about α is given as

$$f(z_n) = \left[c_1(z_n - \alpha) + c_2(z_n - \alpha)^2 + c_3(z_n - \alpha)^3 + \cdots\right]. \tag{2.19}$$

Simplifying (2.19), we have

$$f(z_n) = \left(\frac{2c_2^3 - c_1c_2c_3 + 4\beta c_1c_2^3 + 2\beta^2 c_1^2 c_2^3 - 2\beta c_1^2 c_2 c_3 - c_1^3 c_2 c_3}{c_1^3}\right)e_n^4 + \cdots. \tag{2.20}$$

In order to evaluate the essential terms of (2.7), we expand term by term

$$\begin{aligned}\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) &= c_1 + c_2 e_n + \left(\frac{c_1c_3 + \beta c_1 c_2^2 + c_2^2}{c_1}\right)e_n^2 + \cdots,\\ \left(\frac{f(z_n) - f(y_n)}{z_n - y_n}\right) &= c_1 + \left(\frac{\beta c_1 c_2^2 + c_2^2}{c_1}\right)e_n^2 + \cdots,\\ \left(\frac{f(z_n) - f(x_n)}{z_n - x_n}\right) &= c_1 + c_2 e_n + c_3 e_n^2 + \cdots.\end{aligned} \tag{2.21}$$

Collecting the above terms

$$\begin{aligned}\psi &= \left[\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) - \left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) + \left(\frac{f(z_n) - f(x_n)}{z_n - x_n}\right)\right]^{-1}\\ &= \frac{1}{c_1} + \left(\frac{c_2c_3 + \beta c_1 c_2 c_3}{c_1^3}\right)e_n^3 + \cdots,\\ \omega_1 &= \left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1} = 1 - \left(\frac{\beta c_1^2 c_2 c_3 - 2\beta c_1 c_2^3 + c_1 c_2 c_3 - 2c_2^3}{c_1^3}\right)e_n^3 + \cdots,\\ \xi_1 &= \left(1 - \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right) = 1 - \left(\frac{2\beta c_1 c_2^3 + 2c_2^3}{c_1^3}\right)e_n^3 + \cdots,\\ \omega_1\xi_1 &= 1 - \left(\frac{\beta c_1 c_2 c_3 + c_2 c_3}{c_1^2}\right)e_n^3 + \cdots,\end{aligned} \tag{2.22}$$

$$\begin{aligned}&\psi\omega_1\xi_1\\ &\quad= \frac{1}{c_1}\\ &\qquad+ \left(\frac{\beta^2 c_1^2 c_2^4 + 6\beta c_1^2 c_2^2 c_3 + 3\beta^2 c_1^3 c_2^2 c_3 + \beta c_1 c_2^4 - 2\beta c_1^3 c_2 c_4 + 3c_2^4 - c_1^2 c_2 c_4 + 3c_1 c_2^2 c_3 - \beta^2 c_1^4 c_2 c_4}{c_1^5}\right)\\ &\qquad\times e_n^4 + \cdots.\end{aligned} \tag{2.23}$$

Substituting appropriate expressions in (2.7), we obtain

$$e_{n+1} = z_n - \alpha - \psi\omega_1\xi_1 f(z_n). \tag{2.24}$$

Simplifying (2.24), we obtain the error equation

$$\begin{aligned} e_{n+1} = c_1^{-7}\Big[&8\beta^3 c_1^5 c_2^3 c_3^2 - 10c_2^7 - 32\beta^2 c_1^2 c_2^7 - \beta c_1^3 c_2^5 c_3 + 12\beta^2 c_1^4 c_2^3 c_3^2 + 8\beta c_1^3 c_2^3 c_3^2 - \beta^4 c_1^7 c_2^2 c_3 c_4 \\ &+8\beta c_1^3 c_2^4 c_4 + 12\beta^2 c_1^4 c_2^4 c_4 - 9\beta^3 c_1^4 c_2^5 c_3 - 8\beta^2 c_1^3 c_2^5 c_3 - 3\beta^4 c_1^5 c_2^5 c_3 + 2\beta^4 c_1^6 c_2^4 c_4 \\ &- 6\beta^2 c_1^5 c_2^2 c_3 c_4 - 4\beta c_1^4 c_2^2 c_3 c_4 - 4\beta^3 c_1^6 c_2^2 c_3 c_4 + 2\beta^4 c_1^6 c_2^3 c_3^2 - c_1^3 c_2^2 c_3 c_4 - 2\beta^4 c_1^4 c_2^7 \\ &- 30\beta c_1 c_2^7 - 14\beta^3 c_1^3 c_2^7 + c_1 c_2^5 c_3 + 2c_1^2 c_2^3 c_3^2 + 2c_1^2 c_2^4 c_4 + 8\beta^3 c_1^5 c_2^4 c_4\Big] e_n^8. \end{aligned} \tag{2.25}$$

The expression (2.25) establishes the asymptotic error constant for the eighth order of convergence for the new eighth-order derivative-free method defined by (2.7). □

3. The Established Eighth-Order Methods

The eight particular eighth-order derivative-based methods considered are given in [3–5]. Since these methods are well established, we will state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new eighth-order derivative-free method.

3.1. The Bi, Wu, and Ren Methods

The first of the established eighth-order methods was presented by Bi et al. [3].

Method 1.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.1}$$

$$z_n = y_n - \left[\frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{3.2}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+2)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{3.3}$$

where $[z_n, y_n] = (f(z_n) - f(y_n))/(z_n - y_n)$, $\gamma \in \mathbb{R}$, $f(y_n)$ is given by (3.1), x_0 is the initial approximation and provided that the denominators of (3.1)–(3.3) are not equal to zero.

Method 2.

$$z_n = y_n - \left[1 + 2\frac{f(y_n)}{f(x_n)} + 5\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \left(\frac{f(y_n)}{f(x_n)}\right)^3\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{3.4}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+2)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{3.5}$$

where $\gamma, \mu \in \mathbb{R}$, $f(y_n)$ is given by (3.1) and provided that the denominators of (3.4) and (3.5) are not equal to zero.

Method 3.

$$z_n = y_n - \left[1 - 2\frac{f(y_n)}{f(x_n)} - \left(\frac{f(y_n)}{f(x_n)}\right)^2 + \left(\frac{f(y_n)}{f(x_n)}\right)^3\right]^{-1} \left(\frac{f(y_n)}{f'(x_n)}\right), \tag{3.6}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma + 2) f(z_n)}{f(x_n) + \gamma f(z_n)}\right] \left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{3.7}$$

where $\gamma, \mu \in \mathbb{R}$, $f(y_n)$ is given by (3.1), x_0 is the initial approximation and provided that the denominators of (3.6) and (3.7) are not equal to zero.

Method 4.

$$z_n = y_n - \left[\frac{f(x_n) - 3f(y_n)}{f(x_n)}\right]^{(-2/3)} \left(\frac{f(y_n)}{f'(x_n)}\right), \tag{3.8}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma + 2) f(z_n)}{f(x_n) + \gamma f(z_n)}\right] \left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{3.9}$$

where $\gamma \in \mathbb{R}$, $f(y_n)$ is given by (3.1) and provided that the denominators of (3.8) and (3.9) are not equal to zero.

3.2. *The Sharma Methods*

The three particular eighth-order methods considered are given in [4]. Since these methods are well established, we will state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new iterative eighth-order method.

Method 5.

$$z_n = y_n - \left[\frac{f(x_n)}{f(x_n) - 2f(y_n)}\right] \left(\frac{f(y_n)}{f'(x_n)}\right), \tag{3.10}$$

$$x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \gamma \left(\frac{f(z_n)}{f(x_n)}\right)^2\right] \left(\frac{f[x_n, y_n] f(z_n)}{f[y_n, z_n] f[x_n, z_n]}\right), \tag{3.11}$$

where $\gamma \in \mathbb{R}$, $f(y_n)$ is given by (3.1) and provided that the denominators of (3.10) and (3.11) are not equal to zero.

Method 6.

$$z_n = y_n - \left[\frac{f(x_n)}{f(x_n) - 2f(y_n)}\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \quad (3.12)$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+1)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f[x_n, y_n]f(z_n)}{f[y_n, z_n]f[x_n, z_n]}\right), \quad (3.13)$$

where $\gamma, \beta \in \mathbb{R}$, $f(y_n)$ is given by (3.1) and provided that the denominators of (3.12) and (3.13) are not equal to zero.

Method 7.

$$z_n = y_n - \left[\frac{f(x_n)}{f(x_n) - 2f(y_n)}\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \quad (3.14)$$

$$x_{n+1} = z_n - \left[1 + \gamma\frac{f(z_n)}{f(x_n)}\right]^{(1/\gamma)}\left(\frac{f[x_n, y_n]f(z_n)}{f[y_n, z_n]f[x_n, z_n]}\right), \quad (3.15)$$

where $\gamma \in \mathbb{R}$, $f(y_n)$ is given by (3.1), x_0 is the initial approximation and provided that the denominator of (3.14) and (3.15) are not equal to zero.

3.3. The Thukral Eighth-Order Method

The following eighth-order method is actually presented in [5] and since it is well established, we will state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new iterative eighth-order method. The Newton-type eighth-order iterative method is expressed as

$$z_n = x_n - \frac{f(x_n)^2 + f(y_n)^2}{f'(x_n)(f(x_n) - f(y_n))}, \quad (3.16)$$

$$x_{n+1} = z_n - \left[\left(\frac{1+\mu_n^2}{1-\mu_n}\right)^2 - 2(\mu_n)^2 - 6(\mu_n)^3 + \frac{f(z_n)}{f(y_n)} + 4\frac{f(z_n)}{f(x_n)}\right]\left(\frac{f(z_n)}{f'(x_n)}\right), \quad (3.17)$$

where $\mu_n = (f(y_n)/f(x_n))$ $n \in \mathbb{N}$, $f(y_n)$ is given by (3.1) and provided that the denominators of (3.16) and (3.17) are not equal to zero.

4. Application of the New Derivative-Free Iterative Methods

To demonstrate the performance of the new eighth-order methods, we take six particular nonlinear equations. We will determine the consistency and stability of results by examining the convergence of the new derivative-free iterative methods. The findings are generalised by illustrating the effectiveness of the eighth-order methods for determining the simple root of a nonlinear equation. Consequently, we will give estimates of the approximate solution

Table 1: Errors occurring in the estimates of the root of (4.1) by the methods described.

Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	COC
(2.7)	$0.690e-3$	$0.933e-34$	$0.255e-373$	11.0000
(3.3)	$0.228e-1$	$0.281e-13$	$0.213e-108$	7.9863
(3.5)	—	—	—	—
(3.7)	$0.795e-1$	$0.229e-9$	$0.295e-77$	8.0425
(3.9)	$0.281e-2$	$0.104e-20$	$0.386e-168$	8.0266
(3.11)	$0.414e-2$	$0.164e-18$	$0.997e-150$	7.9992
(3.13)	$0.654e-2$	$0.625e-17$	$0.452e-137$	7.9986
(3.15)	$0.523e-2$	$0.105e-17$	$0.295e-143$	7.9990
(3.17)	—	—	—	—

produced by the eighth-order methods and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In addition, we need to set a particular value of the parameters used in all the eighth-order formula given in this paper. Therefore, we take $i = \gamma = \beta = 1$ as an arbitrary value. In fact, the errors displayed are of absolute value.

Remark 4.1. The family of three-point methods requires four function evaluations and has the order of convergence eight. Therefore, this family is of optimal order and supports the Kung-Traub conjecture [11]. To determine the efficiency index of these new derivative-free methods, we will use the definition (2.2). Hence, the efficiency index of the eighth-order derivative-free methods given is $\sqrt[4]{8} \approx 1.68$.

Remark 4.2. In Tables 1–6, it is observed that the new eighth-order derivative-free methods are competitive with the existing eighth-order derivative-based methods. Furthermore, in these tables we have omitted the insignificant approximations by the various methods and the absolute errors $|x_n - \alpha| \le 10^{-1500}$ in the first three iterations are given in Tables 1–6.

4.1. Numerical Example 1

In our first example, we will demonstrate the convergence of the new eighth-order derivative-free methods for the following nonlinear equation:

$$f(x) = e^{-x} - \cos(x), \tag{4.1}$$

and the exact value of the simple root of (4.1) is $\alpha = -0.666273126\ldots$. In Table 1 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = 3^{-1}$.

4.2. Numerical Example 2

In our second example, we will demonstrate the convergence of new eighth-order derivative-free methods for a different type of nonlinear equation:

$$f(x) = \ln\left(x^2 + x + 2\right) - x + 1, \tag{4.2}$$

Table 2: Errors occurring in the estimates of the root of (4.2) by the methods described.

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
(2.7)	$0.193e-14$	$0.982e-127$	$0.443e-1025$	8.0000
(3.3)	$0.244e-12$	$0.248e-108$	$0.279e-876$	8.0000
(3.5)	$0.104e-11$	$0.169e-102$	$0.806e-829$	8.0675
(3.7)	$0.289e-12$	$0.119e-107$	$0.102e-870$	8.0693
(3.9)	$0.321e-12$	$0.320e-107$	$0.308e-867$	8.0689
(3.11)	$0.109e-11$	$0.220e-102$	$0.591e-828$	7.9999
(3.13)	$0.109e-11$	$0.220e-102$	$0.599e-828$	8.0003
(3.15)	$0.109e-11$	$0.220e-102$	$0.595e-828$	7.9999
(3.17)	$0.743e-11$	$0.760e-95$	$0.905e-767$	8.0000

Table 3: Errors occurring in the estimates of the root of (4.3) by the methods described.

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
(2.7)	$0.241e-4$	$0.872e-37$	$0.258e-296$	7.9859
(3.3)	$0.813e-2$	$0.152e-17$	$0.264e-143$	7.9961
(3.5)	0.416	0.370	$0.661e-4$	0.8817
(3.7)	$0.750e-2$	$0.599e-18$	$0.110e-146$	8.0254
(3.9)	$0.128e-2$	$0.355e-24$	$0.123e-196$	8.0317
(3.11)	$0.153e-2$	$0.101e-22$	$0.367e-184$	7.9999
(3.13)	$0.205e-2$	$0.104e-21$	$0.447e-176$	7.9998
(3.15)	$0.178e-2$	$0.332e-22$	$0.502e-180$	7.9998
(3.17)	—	—	—	—

and the exact value of the simple root of (4.2) is $\alpha = 4.15259074\ldots$. In Table 2 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = 4.4$.

4.3. Numerical Example 3

In this subsection, we take another nonlinear equation. We will demonstrate the convergence of the new eighth-order derivative-free methods for the following nonlinear equation:

$$f(x) = \sin(x)^2 - x^2 + 1, \tag{4.3}$$

and the exact value of the simple root of (4.3) is $\alpha = 1.40449165\ldots$. In Table 3 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = 1$.

4.4. Numerical Example 4

In the next examples, we take another different type of nonlinear equation. We will demonstrate the convergence of new eighth-order derivative-free methods for the following nonlinear equation:

$$f(x) = \exp\left(-x^2 + x + 2\right) - \cos(x+1) + x^3 + 1, \tag{4.4}$$

Table 4: Errors occurring in the estimates of the root of (4.4) by the methods described.

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
(2.7)	$0.228e-5$	$0.453e-44$	$0.110e-353$	7.9901
(3.3)	$0.216e-4$	$0.143e-39$	$0.526e-321$	7.9940
(3.5)	$0.180e-5$	$0.500e-48$	$0.176e-388$	7.9940
(3.7)	$0.217e-4$	$0.149e-39$	$0.736e-321$	7.9940
(3.9)	$0.218e-4$	$0.153e-39$	$0.891e-321$	8.0000
(3.11)	$0.295e-5$	$0.235e-46$	$0.376e-375$	8.0428
(3.13)	$0.295e-5$	$0.235e-46$	$0.372e-375$	8.0551
(3.15)	$0.481e-3$	$0.217e-24$	$0.391e-195$	8.0553
(3.17)	$0.127e-4$	$0.959e-42$	$0.101e-338$	7.9892

Table 5: Errors occurring in the estimates of the root of (4.5) by the methods described.

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	COC
(2.7)	$0.933e-2$	$0.595e-9$	$0.211e-66$	7.7073
(3.3)	$0.225e-1$	$0.363e-8$	$0.179e-62$	7.9964
(3.5)	$0.924e-2$	$0.358e-10$	$0.102e-77$	7.4660
(3.7)	$0.247e-2$	$0.132e-15$	$0.930e-122$	7.6851
(3.9)	0.408	$0.310e-1$	$0.256e-7$	2.7679
(3.11)	$0.474e-3$	$0.192e-24$	$0.141e-195$	7.9668
(3.13)	$0.489e-3$	$0.246e-24$	$0.103e-194$	7.9666
(3.15)	$0.481e-3$	$0.217e-24$	$0.391e-195$	7.9666
(3.17)	$0.609e-2$	$0.428e-12$	$0.259e-93$	7.9992

and the exact value of the simple root of (4.4) is $\alpha = -1$. In Table 4 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = -2^{-1}$.

4.5. Numerical Example 5

In this subsection, we take another nonlinear equation. We will demonstrate the convergence of the new eighth-order derivative-free methods for the following nonlinear equation:

$$f(x) = x^{11} + x + 1, \tag{4.5}$$

and the exact value of the simple root of (4.5) is $\alpha = -0.8443975\ldots$. In Table 5 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = -1$.

4.6. Numerical Example 6

In the last but not least of the examples, we take another different type of nonlinear equation. We will demonstrate the convergence of new eighth-order derivative-free methods for the following nonlinear equation:

$$f(x) = (x-2)\left(x^{10} + x + 1\right)e^{-x-1}, \tag{4.6}$$

Table 6: Errors occurring in the estimates of the root of (4.6) by the methods described.

Methods	$\lvert x_1 - \alpha \rvert$	$\lvert x_2 - \alpha \rvert$	$\lvert x_3 - \alpha \rvert$	COC
(2.7)	$0.320e-2$	$0.116e-10$	$0.114e-75$	7.5078
(3.3)	$0.778e-2$	$0.270e-12$	$0.725e-96$	7.9904
(3.5)	1.43	—	—	—
(3.7)	0.282	—	—	—
(3.9)	$0.119e-2$	$0.568e-19$	$0.153e-149$	7.7503
(3.11)	$0.790e-3$	$0.193e-20$	$0.250e-161$	7.9997
(3.13)	$0.115e-2$	$0.390e-19$	$0.693e-151$	7.9996
(3.15)	$0.957e-3$	$0.896e-20$	$0.535e-156$	7.9997
(3.17)	—	—	—	—

and the exact value of the simple root of (4.6) is $\alpha = 2$. In Table 6 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = 1.9$.

5. Remarks and Conclusion

We have demonstrated the performance of a new family of eighth-order derivative-free methods. Convergence analysis proves that the new methods preserve their order of convergence. There are two major advantages of the eighth-order derivative-free methods. Firstly, we do not have to evaluate the derivative of the functions; therefore, they are especially efficient where the computational cost of the derivative is expensive, and secondly we have established a higher order of convergence method than the existing derivative-free methods [6, 7]. We have examined the effectiveness of the new derivative-free methods by showing the accuracy of the simple root of a nonlinear equation. The main purpose of demonstrating the new eighth-order methods for six types of nonlinear equations was purely to illustrate the accuracy of the approximate solution and the computational order of convergence.

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