Research Article

On Growth of Meromorphic Solutions of Complex Functional Difference Equations

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The main purpose of this paper is to investigate the growth order of the meromorphic solutions of complex functional difference equation of the form $(\sum_{\lambda \in I} \alpha_\lambda(z)(\prod_{\nu=1}^{n} f(z + c_\nu)^{l_{\lambda,\nu}}))/(\sum_{\mu \in J} \beta_\mu(z)(\prod_{\nu=1}^{n} f(z + c_\nu)^{m_{\mu,\nu}})) = Q(z, f(p(z)))$, where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ are two finite index sets, c_ν ($\nu = 1, 2, \ldots, n$) are distinct complex numbers, $\alpha_\lambda(z)$ ($\lambda \in I$) and $\beta_\mu(z)$ ($\mu \in J$) are small functions relative to $f(z)$, and $Q(z, u)$ is a rational function in u with coefficients which are small functions of $f(z)$, $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_0 \in \mathbb{C}[z]$ of degree $k \geq 1$. We also give some examples to show that our results are sharp.

1. Introduction and Main Results

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$, and the first and second main theorems (see, e.g., [1–4]). We also use $\overline{N}(r, f)$ to denote the counting function of the poles of $f(z)$ whose every pole is counted only once. The notations $\rho(f)$ and $\mu(f)$ denote the order and the lower order of $f(z)$, respectively. $S(r, f)$ denotes any quantity that satisfies the condition: $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside an exceptional set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ or a small function relative to $f(z)$ if and only if $T(r, a(z)) = S(r, f)$.

Recently, some papers (see, e.g., [5–7]) focusing on complex difference and functional difference equations emerged. In 2005, Laine et al. [5] firstly considered the growth of meromorphic solutions of the complex functional difference equations by utilizing Nevanlinna theory. They obtained the following result.

Theorem A. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z + c_j) \right) = f(p(z)), \qquad (1)$$

where $\{J\}$ is a collection of all subsets of $\{1, 2, \ldots, n\}$, c_j's are distinct complex constants, and $p(z)$ is a polynomial of degree $k \geq 2$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to f and that $n \geq k$. Then

$$T(r, f) = O\left((\log r)^{\alpha + \varepsilon}\right), \qquad (2)$$

where $\alpha = \log n / \log k$.

In 2007, Rieppo [6] gave an estimation of growth of meromorphic solutions of complex functional equations as follows.

Theorem B. *Suppose that f is a transcendental meromorphic function. Let $Q(z, f)$, $R(z, f)$ be rational functions in f with small meromorphic coefficients relative to f such that $0 < q := \deg_f Q \leq d := \deg_f R$ and $p(z) = p_k z^k + p_{k-1} z^{k-1} +$*

$\cdots + p_0 \in \mathbb{C}[z]$ of degree $k > 1$. If f is a solution of the functional equation

$$R(z, f(z)) = Q(z, f(p(z))), \qquad (3)$$

then $qk \le d$, and for any $\varepsilon, 0 < \varepsilon < 1$, there exist positive real constants K_1 and K_2 such that

$$K_1(\log r)^{\alpha-\varepsilon} \le T(r, f) \le K_2(\log r)^{\alpha+\varepsilon}, \quad \alpha = \frac{\log d - \log q}{\log k}, \qquad (4)$$

when r is large enough.

Rieppo [6] also considered the growth order of meromorphic solutions of functional equation (3) when $k = 1$ and got the following.

Theorem C. *Suppose that f is a transcendental meromorphic solution of (3), where $p(z) = az + b$, $a, b \in \mathbb{C}$, $a \ne 0$ and $|a| \ne 1$. Then*

$$\mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}. \qquad (5)$$

Two years later, Zheng et al. [7] extended Theorem A to more general type and obtained a similar result of Theorem C. In fact, they got the following two results.

Theorem D. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j \in J} f(z + c_j) \right) = Q(z, f(p(z))), \qquad (6)$$

where $\{J\}$ is a collection of all nonempty subsets of $\{1, 2, \ldots, n\}$, c_j $(j = 1, \ldots, n)$ are distinct complex constants, $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_0 \in \mathbb{C}[z]$ of degree $k > 1$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (6) are small functions relative to f. Then $qk \le n$, and

$$T(r, f) = O\left((\log r)^{\alpha+\varepsilon}\right), \qquad (7)$$

where $\alpha = (\log n - \log q)/\log k$.

Theorem E. *Suppose that f is a transcendental meromorphic solution of (6), where $\{J\}$ is a collection of all nonempty subsets of $\{1, 2, \ldots, n\}$, c_j $(j = 1, \ldots, n)$ are distinct complex constants, $p(z) = az + b$, $a, b \in \mathbb{C}$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (6) are small functions relative to f.*

(i) *If $0 < |a| < 1$, then we have*

$$\mu(f) \ge \frac{\log q - \log n}{-\log |a|}. \qquad (8)$$

(ii) *If $|a| > 1$, then we have $q \le n$ and*

$$\rho(f) \le \frac{\log n - \log q}{\log |a|}. \qquad (9)$$

(iii) *If $|a| = 1, q > n$, then we have $\rho(f) = \mu(f) = \infty$.*

In this paper, we will consider a more general class of complex functional difference equations. We prove the following results, which generalize the above related results.

Theorem 1. *Suppose that $f(z)$ is a transcendental meromorphic solution of the functional difference equation*

$$\frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{l_{\lambda,\nu}} \right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{m_{\mu,\nu}} \right)} = Q(z, f(p(z))), \quad (10)$$

where c_ν $(\nu = 1, \ldots, n)$ are distinct complex constants, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ are two finite index sets, $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_0 \in \mathbb{C}[z]$ of degree $k > 1$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (10) are small functions relative to f. Denoting

$$\sigma_\nu = \max_{\lambda, \mu} \{l_{\lambda,\nu}, m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \ \sigma = \sum_{\nu=1}^n \sigma_\nu. \quad (11)$$

Then $qk \le \sigma$, and

$$T(r, f) = O\left((\log r)^{\alpha+\varepsilon}\right), \qquad (12)$$

where $\alpha = (\log \sigma - \log q)/\log k$.

Theorem 2. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{l_{\lambda,\nu}} \right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{m_{\mu,\nu}} \right)} = Q(z, f(az + b)), \qquad (13)$$

where c_ν $(\nu = 1, \ldots, n)$ are distinct complex constants, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ are two finite index sets, $a, b \in \mathbb{C}$, and $Q(z, u)$ is a rational function in u of $\deg_u Q = q(> 0)$. Also suppose that all the coefficients of (10) are small functions relative to f. Denoting

$$\sigma_\nu = \max_{\lambda, \mu} \{l_{\lambda,\nu}, m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \ \sigma = \sum_{\nu=1}^n \sigma_\nu. \quad (14)$$

(i) *If $0 < |a| < 1$, then we have*

$$\mu(f) \ge \frac{\log q - \log \sigma}{-\log |a|}. \qquad (15)$$

(ii) *If $|a| > 1$, then we have $q \le \sigma$ and*

$$\rho(f) \le \frac{\log \sigma - \log q}{\log |a|}. \qquad (16)$$

(iii) *If $|a| = 1$ and $q > \sigma$, then we have $\mu(f) = \rho(f) = \infty$.*

Next we will give some examples to show that our results are best in some extent.

Example 3. Let $c_1 = \arctan 2$, $c_2 = -\pi/4$. Then it is easy to check that $f(z) = \tan z$ solves the following equation:

$$
\frac{f(z+c_1)^2 f(z+c_2)}{f(z+c_1) + f(z+c_2)^2}
$$
$$
= \left(-4f\left(\frac{z}{2}\right)^8 + 8f\left(\frac{z}{2}\right)^7 + 28f\left(\frac{z}{2}\right)^6 - 56f\left(\frac{z}{2}\right)^5 \right.
$$
$$
\left. - 32f\left(\frac{z}{2}\right)^4 + 56f\left(\frac{z}{2}\right)^3 + 28f\left(\frac{z}{2}\right)^2 - 8f\left(\frac{z}{2}\right) - 4 \right)
$$
$$
\times \left(3f\left(\frac{z}{2}\right)^8 + 10f\left(\frac{z}{2}\right)^7 + 16f\left(\frac{z}{2}\right)^6 + 122f\left(\frac{z}{2}\right)^5 \right.
$$
$$
\left. - 6f\left(\frac{z}{2}\right)^4 - 122f\left(\frac{z}{2}\right)^3 + 16f\left(\frac{z}{2}\right)^2 - 10f\left(\frac{z}{2}\right) + 3 \right)^{-1}.
$$
(17)

Obviously, we have

$$
\mu(f) = \rho(f) = 1 = \frac{\log q - \log \sigma}{-\log |a|}, \tag{18}
$$

where $q = 8$, $\sigma = 4$ and $a = 1/2$.

Example 3 shows that the estimate in Theorem 2(i) is sharp.

Example 4. It is easy to check that $f(z) = \tan z$ satisfies the equation

$$
\frac{f(z+(\pi/3))^2 f(z+(\pi/6)) - f(z+(\pi/6))}{f(z+(\pi/3)) f(z+(\pi/6))^2 - f(z+(\pi/3))}
$$
$$
= \frac{\sqrt{3}f(2z)^2 + 4f(2z) + \sqrt{3}}{-\sqrt{3}f(2z)^2 + 4f(2z) - \sqrt{3}}.
$$
(19)

Clearly, we have

$$
\mu(f) = \rho(f) = 1 = \frac{\log \sigma - \log q}{\log |a|}, \tag{20}
$$

where $\sigma = 4$, $q = 2$ and $a = 2$.

Example 4 shows that the estimate in Theorem 2(ii) is sharp.

Example 5. $f(z) = \tan z$ satisfies the equation of the form

$$
\frac{f(z+(\pi/4))^2}{f(z+(\pi/4)) + f(z-(\pi/4))^2}
$$
$$
= \frac{-\left(f(z/2)^2 - 2f(z/2) - 1\right)^3}{8f(z/2)\left(f(z/2)^2 - 1\right)\left(f(z/2)^2 + 2f(z/2) - 1\right)},
$$
(21)

where $\sigma = 4$, $q = 6$, and $a = 1/2$. $\rho(f) = \mu(f) = 1 > \log(3/2)/\log 2 = (\log q - \log \sigma)/ - \log |a|$.

Example 5 shows that the strict inequality in Theorem 2 may occur. Therefore, we do not have the same estimation as in Theorem C for the growth order of meromorphic solutions of (13).

The following Example shows that the restriction $q > \sigma$ in case (iii) in Theorem 2 is necessary.

Example 6. Meromorphic function $f(z) = \tan z$ solves the following equation:

$$
\frac{f(z+(\pi/4))^2}{f(z+(\pi/4)) + f(z-(\pi/4))^2} = \frac{(f(z)+1)^3}{4f(z)(1-f(z))}, \tag{22}
$$

where $a = 1$ and $4 = \sigma > q = 3$, but $\rho(f) = \mu(f) = 1$.

Next, we give an example to show that case (iii) in Theorem 2 may hold.

Example 7. Function $f(z) = ze^{e^z}$ satisfies the following equation:

$$
\frac{(z+\log 6)(z+\log 2)^5 \left[f(z+\log 4)^4 + f(z+\log 4) \right]}{(z+\log 4) f(z+\log 6)}
$$
$$
= \frac{(z+\log 4)^3 f(z+\log 2)^6 + (z+\log 2)^6}{f(z+\log 2)},
$$
(23)

where $a = 1$ and $q = 6 > 5 = \sigma$. Obviously, $\rho(f) = \mu(f) = \infty$.

2. Main Lemmas

In order to prove our results, we need the following lemmas.

Lemma 1 (see [4, 8]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$
R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j}, \tag{24}
$$

such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy

$$
T(r, a_i) = S(r, f), \quad i = 0, 1, \ldots, p,
$$
$$
T(r, b_j) = S(r, f), \quad j = 0, 1, \ldots, q;
$$
(25)

then one has

$$
T(r, R(z, f)) = \max\{p, q\} \cdot T(r, f) + S(r, f). \tag{26}
$$

From the proof of Theorem 1 in [9], we have the following estimate for the Nevanlinna characteristic.

Lemma 2. *Let $f_1, f_2, \ldots, f_n$ be distinct meromorphic functions and*

$$
F(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{\lambda \in I} \alpha_\lambda(z) f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \ldots f_n^{l_{\lambda,n}}}{\sum_{\mu \in J} \beta_\mu(z) f_1^{m_{\mu,1}} f_2^{m_{\mu,2}} \ldots f_n^{m_{\mu,n}}}. \tag{27}
$$

Then

$$T\left(r, F\left(z\right)\right) \leq \sum_{\nu=1}^{n} \sigma_\nu T\left(r, f_\nu\right) + S\left(r, f\right), \tag{28}$$

where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ *and* $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N} \bigcup \{0\}, \nu = 1, 2, \ldots, n\}$ *are two finite index sets,* $\sigma_\nu = \max_{\lambda,\mu} \{l_{\lambda,\nu}, m_{\mu,\nu}\}$ $(\nu = 1, 2, \ldots, n)$. $\alpha_\lambda(z) = o(T(r, f_\nu)(\lambda \in I))$ *and* $\beta_\mu(z) = o(T(r, f_\nu)(\mu \in J))$ *hold for all* $\nu \in \{1, 2, \ldots, n\}$ *and satisfy* $T(r, \alpha_\lambda) = S(r, f)$ $(\lambda \in I)$ *and* $T(r, \beta_\mu) = S(r, f)$ $(\mu \in J)$.

Lemma 3 (see [7]). *Let* c *be a complex constant. Given* $\varepsilon > 0$ *and a meromorphic function* f, *one has*

$$T\left(r, f\left(z \pm c\right)\right) \leq (1 + \varepsilon) T\left(r + |c|, f\right), \tag{29}$$

for all $r > r_0$, *where* r_0 *is some positive constant.*

Lemma 4 (see [4]). *Let* $g : (0, +\infty) \to \mathbb{R}, h : (0, +\infty) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set* E *of finite linear measure. Then, for any* $\alpha > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \leq h(\alpha r)$ *for all* $r > r_0$.

Lemma 5 (see [10]). *Let* f *be a transcendental meromorphic function, and* $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$, $a_k \neq 0$, *be a nonconstant polynomial of degree* k. *Given* $0 < \delta < |a_k|$, *denote* $\lambda = |a_k| + \delta$ *and* $\mu = |a_k| - \delta$. *Then given* $\varepsilon > 0$ *and* $a \in \mathbb{C} \bigcup \{\infty\}$, *one has*

$$kn\left(\mu r^k, a, f\right) \leq n\left(r, a, f\left(p\left(z\right)\right)\right) \leq kn\left(\lambda r^k, a, f\right)$$

$$N\left(\mu r^k, a, f\right) + O\left(\log r\right) \leq N\left(r, a, f\left(p\left(z\right)\right)\right)$$

$$\leq N\left(\lambda r^k, a, f\right) + O\left(\log r\right)$$

$$(1 - \varepsilon) T\left(\mu r^k, f\right) \leq T\left(r, f\left(p\left(z\right)\right)\right) \leq (1 + \varepsilon) T\left(\lambda r^k, f\right), \tag{30}$$

for all r *large enough.*

Lemma 6 (see [11]). *Let* $\phi : [r_0, +\infty) \to (0, +\infty)$ *be positive and bounded in every finite interval, and suppose that* $\phi(\mu r^m) \leq A\phi(r) + B$ *holds for all* r *large enough, where* $\mu > 0$, $m > 1$, $A > 1$ *and* B *are real constants. Then*

$$\phi\left(r\right) = O\left(\left(\log r\right)^\alpha\right), \tag{31}$$

where $\alpha = \log A / \log m$.

Lemma 7 (see [6]). *Let* $\phi : (r_0, \infty) \to (1, \infty)$, *where* $r_0 \geq 1$, *be a monotone increasing function. If for some real constant* $\alpha > 1$, *there exists a real number* $K > 1$ *such that* $\phi(\alpha r) \geq K\phi(r)$, *then*

$$\varliminf_{r \to \infty} \frac{\log \phi\left(r\right)}{\log r} \geq \frac{\log K}{\log \alpha}. \tag{32}$$

Lemma 8 (see [12]). *Let* $\phi : (1, \infty) \to (0, \infty)$ *be a monotone increasing function and let* f *be a nonconstant meromorphic function. If, for some real constant* $\alpha \in (0, 1)$, *there exist real constants* $K_1 > 0$ *and* $K_2 \geq 1$ *such that*

$$T\left(r, f\right) \leq K_1 \phi\left(\alpha r\right) + K_2 T\left(\alpha r, f\right) + S\left(\alpha r, f\right), \tag{33}$$

then

$$\rho\left(f\right) \leq \frac{\log K_2}{-\log \alpha} + \varlimsup_{r \to \infty} \frac{\log \phi\left(r\right)}{\log r}. \tag{34}$$

3. Proof of Theorems

Proof of Theorem 1. We assume $f(z)$ is a transcendental meromorphic solution of (10). Denoting $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. According to Lemmas 1, 2, and 3 and the last assertion of Lemma 5, we get that for any $\varepsilon_1 > 0$,

$$q\left(1 - \varepsilon_1\right) T\left(\mu r^k, f\right) + S\left(r, f\right)$$

$$\leq qT\left(r, f\left(p\left(z\right)\right)\right) + S\left(r, f\right)$$

$$= T\left(r, Q\left(z, f\left(p\left(z\right)\right)\right)\right)$$

$$= T\left(r, \frac{\sum_{\lambda \in I} \alpha_\lambda\left(z\right)\left(\prod_{\nu=1}^{n} f\left(z + c_\nu\right)^{l_{\lambda,\nu}}\right)}{\sum_{\mu \in J} \beta_\mu\left(z\right)\left(\prod_{\nu=1}^{n} f\left(z + c_\nu\right)^{m_{\mu,\nu}}\right)}\right)$$

$$\leq \sum_{\nu=1}^{n} \sigma_\nu T\left(r, f\left(z + c_\nu\right)\right) + S\left(r, f\right) \tag{35}$$

$$\leq \sum_{\nu=1}^{n} \sigma_\nu \left(1 + \varepsilon_1\right) T\left(r + C, f\left(z\right)\right) + S\left(r, f\right)$$

$$= \left(\sum_{\nu=1}^{n} \sigma_\nu\right)\left(1 + \varepsilon_1\right) T\left(r + C, f\left(z\right)\right) + S\left(r, f\right)$$

$$= \sigma\left(1 + \varepsilon_1\right) T\left(r + C, f\left(z\right)\right) + S\left(r, f\right),$$

where r is large enough and $\mu = |p_k| - \delta$ for some $0 < \delta < |p_k|$. Since $T(r + C, f) \leq T(\beta r, f)$ holds for r large enough for $\beta > 1$, we may assume r to be large enough to satisfy

$$q\left(1 - \varepsilon_1\right) T\left(\mu r^k, f\right) \leq \sigma\left(1 + \varepsilon_1\right) T\left(\beta r, f\right) \tag{36}$$

outside a possible exceptional set of finite linear measure. By Lemma 4, we know that whenever $\gamma > 1$,

$$q\left(1 - \varepsilon_1\right) T\left(\mu r^k, f\right) \leq \sigma\left(1 + \varepsilon_1\right) T\left(\gamma \beta r, f\right) \tag{37}$$

holds for all r large enough. Denote $t = \gamma \beta r$; thus the inequality (37) may be written in the form

$$T\left(\frac{\mu}{\left(\gamma \beta\right)^k} t^k, f\right) \leq \frac{\sigma\left(1 + \varepsilon_1\right)}{q\left(1 - \varepsilon_1\right)} T\left(t, f\right). \tag{38}$$

By Lemma 6, we have

$$T\left(r, f\right) = O\left(\left(\log r\right)^{\alpha_1}\right), \tag{39}$$

where

$$\alpha_1 = \frac{\log\left(\sigma\left(1+\varepsilon_1\right)/q\left(1-\varepsilon_1\right)\right)}{\log k}$$
$$= \frac{\log\sigma - \log q}{\log k} + \frac{\log\left(\left(1+\varepsilon_1\right)/\left(1-\varepsilon_1\right)\right)}{\log k}. \tag{40}$$

Denoting now $\alpha = (\log\sigma - \log q)/\log k$ and $\varepsilon = \log((1+\varepsilon_1)/(1-\varepsilon_1))/\log k$; thus we obtain the required form.

Finally, we show that $qk \leq \sigma$. If $qk > \sigma$, then we have $\alpha < 1$. For sufficiently small $\varepsilon > 0$, we have $\alpha + \varepsilon < 1$, which contradicts with the transcendency of f. Thus Theorem 1 is proved. □

Proof of Theorem 2. Suppose $f(z)$ is a transcendental meromorphic solution of (13). Denoting $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$.

(i) $0 < |a| < 1$. We may assume that $q > \sigma$, since the case $q \leq \sigma$ is trivial by the fact that $\mu(f) \geq 0$. By Lemmas 1–3, we have for any $\varepsilon > 0$ and $\beta > 1$,

$$qT\left(r, f\left(p\left(z\right)\right)\right) + S\left(r, f\right)$$
$$= T\left(r, Q\left(z, f\left(p\left(z\right)\right)\right)\right)$$
$$= T\left(r, \frac{\sum_{\lambda\in I}\alpha_\lambda\left(z\right)\left(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J}\beta_\mu\left(z\right)\left(\prod_{\nu=1}^{n}f(z+c_\nu)^{m_{\mu,\nu}}\right)}\right)$$
$$\leq \sum_{\nu=1}^{n}\sigma_\nu T\left(r, f\left(z+c_\nu\right)\right) + S\left(r, f\right)$$
$$\leq \sum_{\nu=1}^{n}\sigma_\nu\left(1+\varepsilon\right)T\left(r+C, f\left(z\right)\right) + S\left(r, f\right)$$
$$= \left(\sum_{\nu=1}^{n}\sigma_\nu\right)\left(1+\varepsilon\right)T\left(r+C, f\left(z\right)\right) + S\left(r, f\right)$$
$$= \sigma\left(1+\varepsilon\right)T\left(r+C, f\left(z\right)\right) + S\left(r, f\right)$$
$$\leq \sigma\left(1+\varepsilon\right)T\left(\beta r, f\right) + S\left(r, f\right), \tag{41}$$

where r is large enough.

By the last assertion of Lemma 5 and (41), we obtain that, for $\mu = |a| - \delta$ $(0 < \delta < |a|, 0 < \mu < 1)$, the following inequality

$$q\left(1-\varepsilon\right)T\left(\mu r, f\right) \leq \sigma\left(1+\varepsilon\right)T\left(\beta r, f\right) \tag{42}$$

holds, where r is large enough outside of a possible set of finite linear measure. By Lemma 4, we get that for any $\gamma > 1$ and sufficiently large r,

$$q\left(1-\varepsilon\right)T\left(\mu r, f\right) \leq \sigma\left(1+\varepsilon\right)T\left(\gamma\beta r, f\right). \tag{43}$$

Therefore,

$$\frac{q\left(1-\varepsilon\right)}{\sigma\left(1+\varepsilon\right)}T\left(r, f\right) \leq T\left(\frac{\gamma\beta}{\mu}r, f\right). \tag{44}$$

Since $\beta > 1$, $\gamma > 1$, $0 < \mu < 1$ and $q > \sigma$, we have $\beta\gamma/\mu > 1$ and $q(1-\varepsilon)/\sigma(1+\varepsilon) > 1$ when ε is small enough. Using Lemma 7, we see that

$$\mu\left(f\right) \geq \frac{\log q\left(1-\varepsilon\right) - \log\sigma\left(1+\varepsilon\right)}{\log\gamma\beta - \log\mu}. \tag{45}$$

Letting $\varepsilon \to 0$, $\delta \to 0$, $\beta \to 1$ and $\gamma \to 1$, we have

$$\mu\left(f\right) \geq \frac{\log q - \log\sigma}{-\log|a|}. \tag{46}$$

(ii) $|a| > 1$. By the similar reasoning as is (i), we easily obtain that

$$q\left(1-\varepsilon\right)T\left(\mu r, f\right) \leq qT\left(r, f\left(p\left(z\right)\right)\right)$$
$$\leq \sigma\left(1+\varepsilon\right)T\left(r+C, f\left(z\right)\right) + S\left(r, f\right) \tag{47}$$

for all r large enough. We may select sufficiently small numbers $\delta > 0$ and $\varepsilon > 0$, such that $\mu = |a| - \delta > 1$ and $(1/\mu) + \varepsilon < 1$. Thus we have

$$T\left(\mu r, f\right) \leq \frac{\sigma\left(1+\varepsilon\right)}{q\left(1-\varepsilon\right)}T\left(r+C, f\left(z\right)\right) + S\left(r, f\right); \tag{48}$$

namely,

$$T\left(\mu r, f\right) \leq \frac{\sigma\left(1+\varepsilon\right)}{q\left(1-\varepsilon\right)}T\left(r+C, f\left(z\right)\right), \tag{49}$$

where r is large enough possibly outside of a set of finite linear measure. By Lemma 4, we have for any $1 < \gamma < \mu$,

$$T\left(\mu r, f\right) \leq \frac{\sigma\left(1+\varepsilon\right)}{q\left(1-\varepsilon\right)}T\left(\gamma r, f\left(z\right)\right); \tag{50}$$

that is,

$$T\left(r, f\right) \leq \frac{\sigma\left(1+\varepsilon\right)}{q\left(1-\varepsilon\right)}T\left(\frac{\gamma}{\mu}r, f\left(z\right)\right) \tag{51}$$

holds for all sufficiently large r. By Lemma 8, we obtain

$$\rho\left(f\right) \leq \frac{\log\sigma - \log q + \log\left(1+\varepsilon\right) - \log\left(1-\varepsilon\right)}{-\log\left(\gamma/\mu\right)}. \tag{52}$$

Letting $\varepsilon \to 0$, $\delta \to 0$ and $\gamma \to 1$, we have

$$\rho\left(f\right) \leq \frac{\log\sigma - \log q}{\log|a|}. \tag{53}$$

(iii) $|a| = 1$ and $q > \sigma$. The proof of this case is completely similar as in the case in (i). In fact, we set $\mu = |a| - \delta = 1 - \delta$ $(0 < \delta < 1, 0 < \mu < 1)$. Similarly, we can get

$$\mu\left(f\right) \geq \frac{\log q - \log\sigma}{-\log|a|}. \tag{54}$$

Since $|a| = 1$, we have $\mu(f) = \rho(f) = \infty$. □

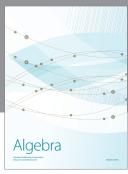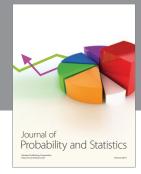## Conflict of Interests

Acknowledgments

References

[1] W. Cherry and Z. Ye, *Nevanlinna's Theory of Value Distribution*, Springer Monographs in Mathematics, Springer, Berlin, Germany, 2001.

[2] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.

[3] Y. Z. He and X. Z. Xiao, *Algebroid Functions and Ordinary Differential Equations*, Beijing, China, 1988.

[4] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, vol. 15 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1993.

[5] I. Laine, J. Rieppo, and H. Silvennoinen, "Remarks on complex difference equations," *Computational Methods and Function Theory*, vol. 5, no. 1, pp. 77–88, 2005.

[6] J. Rieppo, "On a class of complex functional equations," *Annales Academiæ Scientiarum Fennicæ*, vol. 32, no. 1, pp. 151–170, 2007.

[7] X.-M. Zheng, Z.-X. Chen, and J. Tu, "Growth of meromorphic solutions of some difference equations," *Applicable Analysis and Discrete Mathematics*, vol. 4, no. 2, pp. 309–321, 2010.

[8] A. Z. Mokhon'ko, "The Nevanlinna characteristics of certain meromorphic functions," *Teorija Funkciĭ, Funkcional'nyĭ Analiz i ih Priloženija*, vol. 14, pp. 83–87, 1971 (Russian).

[9] A. A. Mokhon'ko and V. D. Mokhon'ko, "Estimates of the Nevanlinna characteristics of certain classes of meromorphic functions, and their applications to differential equations," *Akademija Nauk SSSR*, vol. 15, pp. 1305–1322, 1974.

[10] R. Goldstein, "Some results on factorisation of meromorphic functions," *Journal of the London Mathematical Society*, vol. 4, pp. 357–364, 1971.

[11] R. Goldstein, "On meromorphic solutions of certain functional equations," *Aequationes Mathematicae*, vol. 18, no. 1-2, pp. 112–157, 1978.

[12] G. G. Gundersen, J. Heittokangas, I. Laine, J. Rieppo, and D. Yang, "Meromorphic solutions of generalized Schröder equations," *Aequationes Math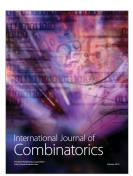ematicae*, vol. 63, no. 1-2, pp. 110–135, 2002.