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Research Article

Topological and Kinematic Singularities for a Class of Parallel Mechanisms

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We study singularities for a parallel mechanism with a planar moving platform in $\mathbb{R}^d$ $(d = 2, 3)$, with joints which are universal, spherical (spatial case), or rotational (planar case). For such mechanisms, we give a necessary condition for a topological singularity to occur, and describe the corresponding kinematic singularity. An example is provided.

1. Introduction

Kinematic singularities of parallel mechanisms have been studied extensively (cf. [1–4]), but less attention has been paid to topological singularities in this context. The topological theory of singularities is an extensive and rapidly growing field of mathematics, with connections to complex and real analysis, algebraic geometry, and differential topology (see [5–7]). In this note, we investigate the connection between the two types of singularities (for a certain class of mechanisms), thus illustrating their relevance to robotics.

1.1. Topological Singularities

By choosing appropriate local coordinates for a given mechanism Γ, we can think of the set of all its configurations as a topological space $\mathcal{C} = \mathcal{C}(\Gamma)$, called its *configuration space*, and try to endow it with the structure of a differentiable manifold. The points of $\mathcal{C}$ where this cannot be done constitute the *topological* (or differentiable) *singularities* of Γ (see Section 2).

These have been studied in detail mainly in the case of a single closed chain (see [8–11] and compare [12–14]). Here we consider parallel mechanisms with a planar moving platform, in any dimension. In the spatial case, we require the joints to be universal or spherical; in the planar case, all joints are rotational.

Evidently, topological singularities of $\mathcal{C}$ are an inherent feature of the mechanism, independent of the actuation scheme. However, at first glance they appear to have no mechanical significance. The main results of this paper areas follows:

(a) we give a necessary geometric condition for a topological singularity to arise in such a mechanism (Theorem 2.4);

(b) we show that topological singularities for these mechanisms always give rise to kinematic singularities (Proposition 3.1).

The occurence of such singularities is illustrated in a specific example in Section 4.

These results are intended to exemplify the power of (higher-dimensional) topological methods for the study of singularities in robotics. In the future we hope to show how they apply to more general types of mechanisms.

1.2. Kinematic Singularities

In general, a configuration $\mathcal{V}$ is naturally described by the vector $\mathbf{x} = (x_1, \ldots, x_N)$ whose coordinates x_i correspond to the positions of the various joints and links of the mechanism. In practice, we choose a subset $\mathbf{x}_{\text{in}}$ of *input* coordinates (corresponding to the actuated joints), which can serve as local coordinates for $\mathcal{C}$ around $\mathcal{V}$. In addition, we often focus on a subset $\mathbf{x}_{\text{out}}$ of *output* coordinates of interest (which may describe the position of the end effector). The remaining joints (if any) are *passive*.

The structure of the mechanism imposes relations which must hold among the coordinates x_i; in particular, we may assume that

$$F(\mathbf{x}_{\text{in}}, \mathbf{x}_{\text{out}}) = 0, \tag{1.1}$$

identically in a neighborhood of $\mathcal{V}$ in $\mathcal{C}$. The Jacobian $J := \partial F / \partial \mathbf{x}$ consists of two blocks $\partial F / \partial \mathbf{x}_{\text{in}}$ and $J_{\text{out}} := \partial F / \partial \mathbf{x}_{\text{out}}$, where J_{out} must be nonsingular if the $\mathbf{x}_{\text{in}}$ are to serve as local coordinates near $\mathcal{V}$ (cf. [15] or [16, Section 5]).

The kinematics of the mechanism are described by a time-dependent path $\mathbf{x}(t)$ in the configuration space. Differentiating (1.1) with respect to t we get

$$\frac{\partial F}{\partial \mathbf{x}} \dot{\mathbf{x}} = 0, \tag{1.2}$$

which can be written in the form

$$J_{\text{in}} \dot{\mathbf{x}}_{\text{in}} = J_{\text{out}} \dot{\mathbf{x}}_{\text{out}}, \tag{1.3}$$

where $J_{\text{in}} := -\partial F / \partial \mathbf{x}_{\text{in}}$.

If J_{out} is of maximal rank and J_{in} is not, $\mathcal{V}$ is called a (instantaneous) kinematic singularity of *type I*; this means that not every infinitesimal change in output can be obtained by changing the actuated joints.

On the other hand, if J_{in} is of maximal rank and J_{out} is not, $\mathcal{V}$ is called singular of *of type II*; in this case the actuated joints do not determine uniquely the behavior of the outputs. Finally, if neither J_{out} nor J_{in} is of maximal rank, $\mathcal{V}$ is called singular *of type III* (cf. [17]). Gosselin and Angeles give examples of all three types of singularities for a parallel 3-RRR planar mechanism.

This classification is widely used in the robotics literature (see, e.g., [18, Section 6.2]). Note that in addition to the (somewhat arbitrary) choice of $\mathbf{x}_{\text{in}}$ and $\mathbf{x}_{\text{out}}$, the mechanism may have additional "passive coordinates" of interest. Thus, Zlatanov et al. provide a more detailed classification, listing six types of singular configurations (see [2]).

The kinematic singularities which arise in this paper are all of type I or III, corresponding to the *impossible output* (IO) of [2, Section 5, Definition 7], where there exists an infinitesimal output vector for which (1.2) cannot be satisfied with any combination of active and passive input vectors.

Other types of singularities for mechanisms have been considered—for example, *control*, *constraint*, and *architectural* (cf. [18, Section 6.2.1])—which we do not attempt to discuss here. Note, however, that there is no analogous classification of singularities for topological spaces, which can be extremely complicated in general (see [5]). See [19] for a general discussion of singularities in robotics.

1.3. Parallel Mechanisms with a Planar Platform

In this note we begin a study of the relationship between topological and kinematic singularities for a common class of mechanisms (which occur in applications).

These are *polygonal* mechanism Γ in $\mathbb{R}^d$ ($d = 2, 3$), consisting of a moving planar k-polygonal platform $\mathcal{P}$ with k chains attached to its vertices. The ith chain is a sequence of $n^{(i)}$ concatenated links of lengths $\ell_j^{(i)}$ ($j = 1, \ldots, n^{(i)}$), connected by spherical or universal joints, in the spatial case, and rotational joints, in the planar case. One end of the ith chain is attached to the vertex $\mathbf{p}^{(i)}$ of the moving platform, and the other is fixed at $\mathbf{x}^{(i)} \in \mathbb{R}^d$.

The restriction to these specific types of joints is intended to simplify the study of $\mathcal{C}(\Gamma)$, which can then be interpreted as a space of immersions of the corresponding metric graph. In future work, we hope to extend these results to more general parallel mechanisms.

Convention

For each component, we use parenthesized superscripts to indicate the chain number, and subscripts to indicate the link number. For example, $\ell_j^{(i)}$ denotes the length of the jth link of the ith chain.

2. Topological Singularities

From now on we consider a fixed polygonal mechanism Γ as in Section 1.3, and construct its configuration space $\mathcal{C}$ as follows.

Definition 2.1. A *chain configuration* for a single n-link chain consists of n vectors $V = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ in $\mathbb{R}^d$ of specified lengths: $\|\mathbf{v}_j\| = \ell_j$ for $j = 1, \ldots, n$.

A chain configuration V is said to be *aligned* if all the vectors $\mathbf{v}_j$ are scalar multiples of $\mathbf{v}$, which is called the *direction vector* of V. The *direction line* for V is $\text{Line} := \{\mathbf{x} + \tau\mathbf{v} \mid \tau \in \mathbb{R}\}$.

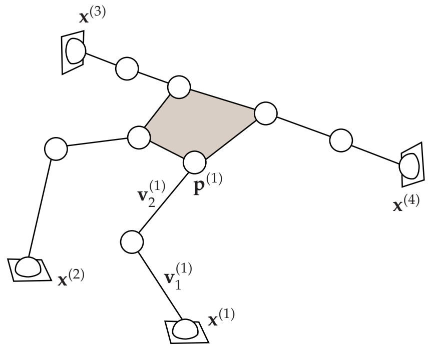

Figure 1: A singular configuration of type (a).

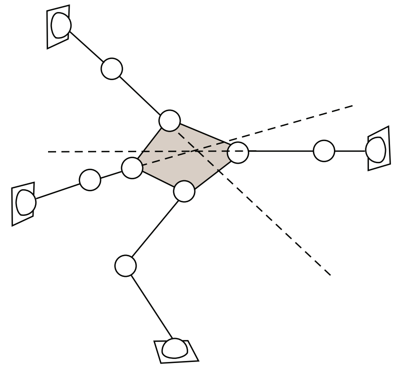

Figure 2: A singular configuration of type (b).

Definition 2.2. A configuration for Γ consists of a set $\mathcal{V} = (V^{(1)}, \ldots, V^{(k)})$ of chain configurations for each of the k chains, such that the k endpoints $\mathbf{p}^{(i)}$ of the corresponding chain configurations form a polygon congruent to the given moving platform $\mathcal{P}$. Here,

$$\mathbf{p}^{(i)} := \mathbf{x}^{(i)} + \sum_{j=1}^{n} \mathbf{v}_j^{(i)}, \tag{2.1}$$

for $i = 1, \ldots, k$, where the points $\mathbf{x}^{(i)}$ are the vertices of the fixed platform (see Figure 1).

The set $\mathcal{C} = \mathcal{C}(\Gamma)$ of all such configurations, topologized in the obvious way, is the *configuration space* of Γ.

Definition 2.3. A configuration $\mathcal{V}$ for Γ is called *singular of type (a)* if for two of its chains—say, numbers $i_1, i_2 \in \{1, \ldots, k\}$—the corresponding chain configurations $V^{(i_1)}$ and $V^{(i_2)}$ are aligned, with coinciding direction lines: $\text{Line}^{(i_1)} = \text{Line}^{(i_2)}$ (see Figure 1).

$\mathcal{V}$ is *singular of type (b)* if three of its chain configurations are aligned, with direction lines in the same plane meeting in a single point (this is referred to in literature as a *planar pencil*) (see Figure 2).

$\mathcal{V}$ is *singular of type (c)* if (at least) four of its chain configurations are aligned, with direction lines in the same plane (compare [18, Section 6.4.1, condition 3d]).

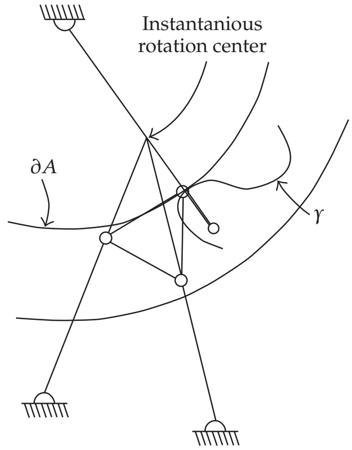

Figure 3: Coupler curve tangent to workspace boundary.

We can now formulate our main result.

Theorem 2.4. *A necessary condition for a configuration* $\mathcal{V} = (V^{(1)}, \ldots, V^{(k)})$ *of a polygonal mechanism* Γ *to be a topological singularity is that it be singular of type (a), (b), or (c).*

For the proof, see [20].

Remark 2.5. The three types of singular configurations defined above need not be topological singularities of $\mathcal{C}$; in particular, we know of no topological interpretation of the three different types (a)–(c). As noted above, in general the classification of topological or differentiable singularities is very difficult (see [6]).

Examples 2.6. Consider the following two examples.

(1) If $\mathcal{V}$ is singular of type (a), as in Figure 1, consider the submechanism Γ' consisting of the two aligned chains of Γ, with the corresponding configuration $\mathcal{V}'$. If we assume that these two chains have one and two links, respectively, then $\mathcal{V}'$ has a neighborhood U' in the configuration space $\mathcal{C}(\Gamma')$ which is a one-point union of two 2-discs (see [14, Proposition 4.1])—so $\mathcal{V}'$ is singular.

On the other hand, if Γ'' is the mechanism obtained from Γ by omitting the two aligned chains, the configuration $\mathcal{V}''$ corresponding to $\mathcal{V}$ is nonsingular and has a Euclidean neighborhood U'' in $\mathcal{C}(\Gamma'')$. Since the $\mathcal{V}$ itself has a neighborhood in $\mathcal{C}(\Gamma)$ equivalent to $U' \times U''$, we see that $\mathcal{V}$ is a topological singularity.

(2) Consider a mechanism with a triangular platform, and two chains with one link each. In this case, the workspace for the third vertex of the platform is the coupler curve γ for the corresponding 4-chain mechanism (see [21, Chapter 4]), while the workspace for the third chain is an annulus A.

For suitable parameters, the boundary ∂A (where the third chain is aligned) will be tangent to γ. The configuration $\mathcal{V}$ corresponding to the point of tangency will be singular of type (b), as in Figure 3.

Note that near $\mathcal{V}$ each point in γ has two corresponding configurations, associated to "elbow up/down" positions of the third chain, which coalesce at $\mathcal{V}$ itself; thus $\mathcal{V}$ has a singular neighborhood in $\mathcal{C}$ consisting of two transverse intervals.

Remark 2.7. For simplicity, in the spatial case we restricted attention to mechanisms where all joints are universal. Replacing any such joint by a spherical one simply multiplies $\mathcal{C}$ by a circle, so it does not affect the topological singularities.

3. Kinematic Singularities

Theorem 2.4 gives a *necessary* condition for a configuration to be singular (topologically), namely, that some subset $\{i_1, \ldots, i_m\}$ of its chains be aligned, with direction lines $(\mathrm{Line}^{(i_j)})_{j=1}^{m}$, so that the Plücker vectors of these lines span certain types of *varieties*, of positive codimension in $\mathbb{R}^6$ (see [18, Section 5]). We now show how topological singularities give rise to kinematic singularities, for our class of polygonal mechanisms.

3.1. Architectures

Recall from Section 1.2 that kinematic singularities require an *actuation architecture*, that is, a choice of input coordinates $\mathbf{x}_{\mathrm{in}}$ (corresponding to the actuated joints) and output coordinates $\mathbf{x}_{\mathrm{out}}$ (corresponding to the end effector of the mechanism: in our case, the position and orientation of the moving platform).

The planar case (with more general joints) was analyzed by Bonev et al. in a series of papers, summarized in [22]. For a classification of the kinematic singularities of such mechanisms, using instantaneous centers of rotation, see [23].

For the spatial case (i.e., Γ embedded in $\mathbb{R}^3$), we assume for simplicity that all actuators are universal or spherical, while the passive joints (located at the end of the chain, say) are spherical. There are three architectures to consider.

(a) Six chains having $n^{(i)} - 1$ actuators for the ith chain ($i = 1, \ldots, 6$).

(b) The first chain having $n^{(1)}$ actuators; three more chains with $n^{(i)} - 1$ actuators for the ith chain ($i = 2, 3, 4$).

(c) Two chains having $n^{(i)}$ actuators ($i = 1, 2$); the third chain with $n^{(3)} - 1$ actuators.

3.2. Screw Theory

We use screw theory (see [24]) to describe the forces operating at each joint of our mechanism.

A *screw* \$ is a Plücker vector in $\mathbb{R}P^5$, describing a line—or equivalently, the position and direction of a vector—in $\mathbb{R}^3$ (cf. [18, Section 5]). Thus Figure 4 depicts the equivalent kinematic chain of an arbitrary chain i of a mechanism, where the movement of each spherical joint j is described by three unit screws $\$_{j,1}^{(i)}$, $\$_{j,2}^{(i)}$, and $\$_{j,3}^{(i)}$ attached to its center. For a universal joint (or just before a passive joint), we can make do with two screws.

Considering the ith chain as an open chain, we can express the instantaneous twist of the end-effector as

$$\$_p = \sum_{j=0}^{n^{(i)}} \sum_{k=1}^{3} \$_{j,k}^{(i)} \cdot \dot{\theta}_{j,k}^{(i)}, \tag{3.1}$$

where $\theta_{j,k}^{(i)}$ is the input coordinate for the $\$_{j,k}^{(i)}$ screw (cf. [16, Section 5.6]).

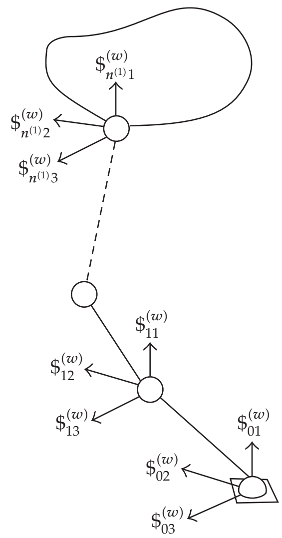

Figure 4: Equivalent kinematic structure of a chain.

In order to eliminate the passive joints from (3.1), for the ith chain, we must multiply both sides by appropriate reciprocal screws. More precisely, choose a basis $\{\$_t^{(i)\perp}\}_{t=1}^{\lambda^{(i)}}$ for the space $V^{(i)}$ of common reciprocals of the screws of all passive joints for this chain.

(a) If the chain has $n^{(i)} - 1$ actuators, at all joints but the last two, the reciprocal screw for these two joints corresponds to the line passing through them (i.e., $V^{(i)}$ is one-dimensional).

(b) If the chain has $n^{(i)}$ actuators, at all but the last joint, $V^{(i)}$ is 3-dimensional, with the corresponding lines all passing through the last joint (see [16, Chapter 5]).

For the ith chain we obtain a system of $\lambda^{(i)}$ linear equations for $\$_p$:

$$\$_t^{(i)\perp} \cdot \$_p = \sum_{j=0}^{n^{(i)}} \sum_{k=1}^{3} \$_t^{(i)\perp} \cdot \$_{j,k}^{(i)} \cdot \dot{\theta}_{j,k}^{(i)}, \tag{3.2}$$

$(t = 1, \ldots, \ell^{(i)})$, in which of course the unactuated inputs $\dot{\theta}_{j,k}^{(i)}$ have zero coefficient.

Combining the $\lambda = \sum_{i=1}^{k} \lambda^{(i)}$ equations (3.2) for all k chains, we obtain (1.3) for the chosen architecture ($\lambda = 6$ for the first two and $\lambda = 7$ for the third).

The $\lambda \times 6$ matrix J_{out} will take the form

$$J_{\text{out}} = \begin{pmatrix} \$_1^{(1)\perp} \\ \vdots \\ \$_{\lambda^{(1)}}^{(1)\perp} \\ \vdots \\ \$_{\lambda^{(k)}}^{(k)\perp} \end{pmatrix}, \tag{3.3}$$

whose rows are the reciprocal screws of all actuated chains.

The matrix J_{in} is block-diagonal:

$$J_{\text{in}} = \begin{pmatrix} A^{(i_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^{(i_k)} \end{pmatrix}, \tag{3.4}$$

with the ith bloc (for the ith actuated chain, with d passive joints) and a $\lambda^{(i)} \times (n^{(i)} - d)$ matrix of the form:

$$A^{(i)} := \begin{pmatrix} \$_1^{(i)\perp} \cdot \$_{0,1}^{(i)} & \cdots & \$_1^{(i)\perp} \cdot \$_{n^{(i)}-d,3}^{(i)} \\ \vdots & & \vdots \\ \$_{\lambda^{(i)}}^{(i)\perp} \cdot \$_{0,1}^{(i)} & \cdots & \$_{\lambda^{(i)}}^{(i)\perp} \cdot \$_{n^{(i)}-d,3}^{(i)} \end{pmatrix}. \tag{3.5}$$

Proposition 3.1. *For a polygonal mechanism Γ (with no unactuated chains), there is an instantaneous kinematic singularity of type I or III at any topological singularity.*

Proof. At a topological singularity at least two chains are aligned, so the reciprocals to the passive joint(s) are reciprocal to all screws of these chains, and thus $A^{(i)} = 0$ for these chains. Since $\lambda \leq 7$, J_{in} is singular.

Now by Theorem 2.4, a topological singularity can have the following:

(1) two coaligned chains, each with a pair of unactuated joints; they have a common reciprocal (and $\lambda = 6$), so J_{out} has rank ≤ 5. The same holds in the second architecture whenever two chains are coaligned;

(2) three aligned chains whose lines lie in a planar pencil, each with a pair of unactuated joints; in this case the corresponding screws are linearly dependent, so again J_{out} has rank ≤ 5;

(3) four aligned chains whose lines are in one plane; the lines of those with a pair of unactuated joints; each lying in a planar pencil (rank 3). In the first architecture, each of the last two lines adds at most 1 to the rank; in the second, adding the last line forms a degenerate congruence (total rank 4). Thus in any case J_{out} has rank ≤ 5; □

Remark 3.2. The sort of conditions in the Grassmann algebra used here to identify singularities is of course well known in literature (see, e.g., [18, Section 6.4]). Our point is that these are *necessary* conditions for topological singularities, and *sufficient* for kinematic singularities, providing an implication between two concepts of independent interest.

4. An Example

To round off the discussion we now present an example of a topological singularity of type (b) for a specific real-life mechanism Γ, namely, the 3-URU 3-DOF mechanism (in $\mathbb{R}^3$), introduced in [25]. Zlatanov et al. studied the constraint singularities of Γ extensively. Here we do not attempt a comprehensive study, since our goal is merely to illustrate topological methods.

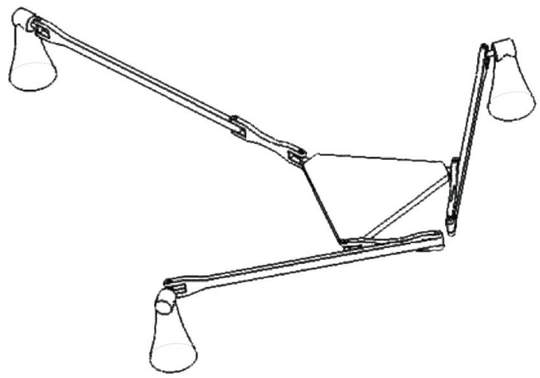

Figure 5: A singular configuration for the mechanism Γ.

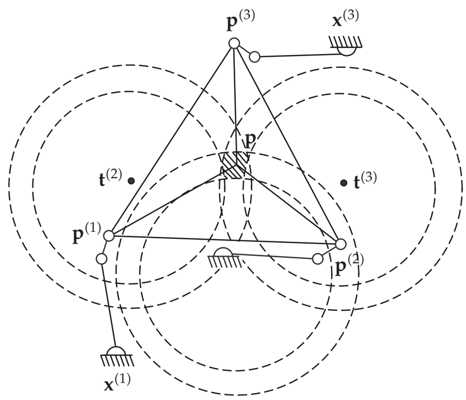

Figure 6: Constrained work space $\mathcal{W}_\theta$ for $\mathbf{p}$ with fixed θ.

The mechanism consists of three two-link chains, and both the base and moving platforms are equilateral triangles. It turns out that in a certain region $\mathcal{U}$ of the configuration space $\mathcal{C}$, Γ acts as a planar mechanism (see [25]). In particular, the three R-joint axes of the base platform meet in the base triangle but not in the center (otherwise mobility would be increased by the additional spin dexterity of the extended chain). Furthermore, in this region the three intermediate R-joints in each chain are parallel (see Figure 5).

Since the topological singularity in question is located in this region $\mathcal{U}$, for simplicity we may regard Γ as if it were a 3-RRR planar mechanism $\widehat{\Gamma}$. The pose of the equilateral moving platform $\mathcal{P}$ in $\mathcal{U}$ is determined by the two coordinates (x, y) of its barycenter $\mathbf{p}$, and the rotation θ of $\mathcal{P}$ in $\mathbb{R}^2$. Denote the common distance from the vertices to the barycenter by $r = d(\mathbf{p}, \mathbf{p}^{(i)})$ $(i = 1, 2, 3)$.

The work space for each vertex $\mathbf{p}^{(i)}$ of $\mathcal{P}$ is an annulus $\mathcal{A}^{(i)}$ centered at the fixed endpoint $\mathbf{x}^{(i)}$ of the ith chain, with boundary radii $\ell_1^{(i)} \pm \ell_2^{(i)}$. Thus, if we fix the orientation θ of $\mathcal{P}$, the resulting constrained work space $\mathcal{W}_\theta$ for $\mathbf{p}$ (the shaded area in Figure 6) is the intersection of three annuli (with centers at $\mathbf{t}^{(i)}$), namely, the displacements $\widehat{\mathcal{A}}^{(i)}$ $(i = 1, 2, 3)$ of $\mathcal{A}^{(i)}$ by a vector $\mathbf{p}^{(i)}\mathbf{p} = \mathbf{x}^{(i)}\mathbf{t}^{(i)}$ of length r.

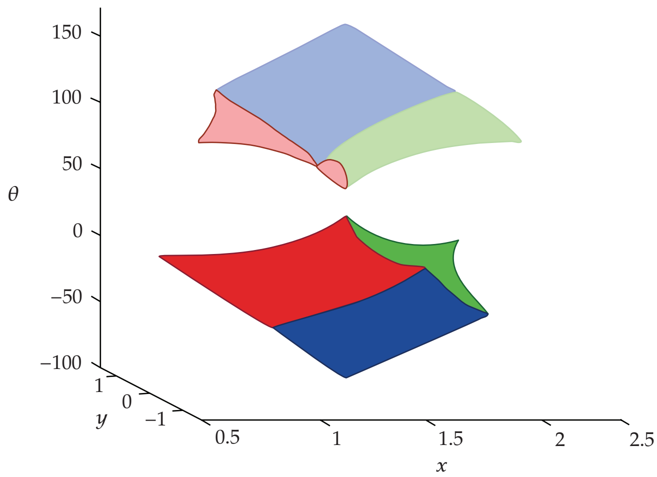

Figure 7: Configuration space for 3-RRR mechanism.

The configuration space $\mathcal{C}$ is described in a neighborhood $\mathcal{U}$ of any configuration $\mathcal{V}$ by

(a) discrete data on the elbow up/down position of each chain at $\mathcal{V}$;

(b) the orientation θ (which takes value in an open interval $I \subseteq \mathbb{R}$);

(c) the location of $\mathbf{p}$ in $\mathcal{W}_\theta$.

The work spaces $\mathcal{W}_\theta$ will generically all be homeomorphic to a fixed curvilinear polygonal region $\mathcal{W}' = \mathcal{W}_{\theta_0}$ (for θ near θ_0). Thus topologically $\mathcal{U}$ will be a cube, that is, a product $I \times \mathcal{W}'$. The discrete data yield eight identical copies of $\mathcal{U}$ identified along their boundaries (which represent chain alignments).

For some values of θ, $\mathcal{W}_\theta$ may be empty, so in fact $\mathcal{U}$ may split up into two disjoint cubes as above (see Figure 7), where the vertical gap between them represents values of θ for which $\mathcal{W}_\theta = \varnothing$. This corresponds to a situation where the platform $\mathcal{P}$ cannot rotate continuously between two orientations, each of which is feasible in itself. No singularities arise in this case.

More care is required for the analysis of the full configuration space $\mathcal{C}(\widehat{\Gamma})$ near $\mathcal{V}_0$, because there are actually eight regions $\mathcal{U}_1, \ldots, \mathcal{U}_8$, corresponding to the eight possible choices of "elbow up/down" for the three chains. The boundaries of $\mathcal{W}_\theta$ are arcs of the boundary circles of the annuli $\widehat{\mathcal{A}}^{(i)}$, where the links of the ith chain are aligned. Therefore (as noted above), the boundaries of the "cubes" $\mathcal{U}_j$ $(j = 1, \ldots, 8)$ are glued together in $\mathcal{C}(\widehat{\Gamma})$ according to a combinatorial pattern represented by the colors in the two figures. For example, gluing faces for the situation depicted in Figure 7 yields two disjoint 3-dimensional tori.

Of course, this is only true in the region where the original mechanism Γ is planar (and thus equivalent to $\widehat{\Gamma}$); all we can conclude about $\mathcal{C}(\Gamma)$ in the region corresponding to Figure 7 is that it has two connected components, locally isomorphic to $\mathbb{R}^3$.

However, as the parameters for Γ (and thus $\widehat{\Gamma}$) vary, we find that in certain cases the two connected components of $\mathcal{U}$ approach each other, and, for an appropriate Γ, they actually touch at one point $\mathcal{V}_0 \in \mathcal{C}$ (see Figure 8).

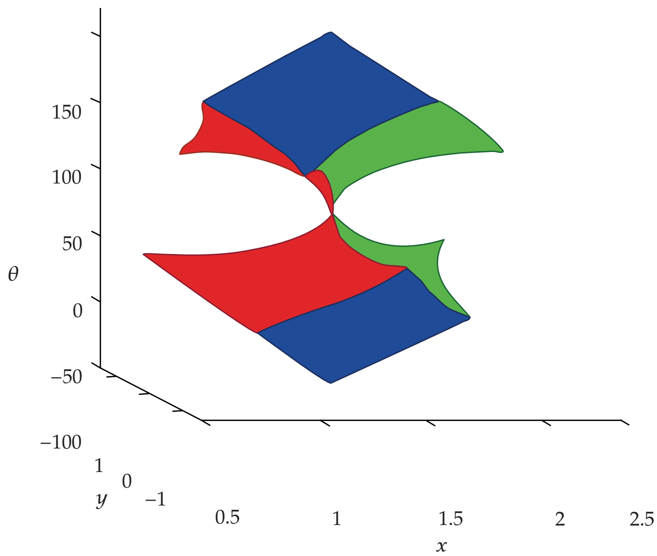

Figure 8: Configuration space is locally $\mathbb{R}^3 \bigvee_c \mathbb{R}^3$.

In this case, $\mathcal{U}$ is homeomorphic to $\mathbb{R}^3 \bigvee_{\mathcal{V}_0} \mathbb{R}^3$, so that $\mathcal{V}_0$ is topologically singular in $\mathcal{U}$. In $\mathcal{C}(\widehat{\Gamma})$—and therefore, in $\mathcal{C}(\Gamma)$ too, at least locally—we obtain a one-point union of two 3-tori.

The aligned poses can be calculated analytically using the algorithm in Gosselin and Merlet (cf. [26]), since each of the extreme situations can be treated as an equivalent 3-RPR robot, whose link lengths are fixed and known.

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