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Research Article

On a Subclass of Analytic Functions Related to a Hyperbola

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The object of the present investigation is to solve Fekete-Szegö problem and determine the sharp upper bound to the second Hankel determinant for a new class $\widetilde{\mathscr{R}}(a, c, \rho)$ of analytic functions in the unit disk. We also obtain a sufficient condition for an analytic function to be in this class.

1. Introduction and Preliminaries

Let $\mathscr{A}$ be the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathscr{A}$ is said to be starlike function of order ρ and convex function of order ρ, respectively, if and only if $\mathrm{Re}\{zf'(z)/f(z)\} > \rho$ and $\mathrm{Re}\{1 + (zf''(z)/f'(z))\} > \rho$, for $0 \le \rho < 1$ and for all $z \in \mathscr{U}$. By usual notations, we denote these classes of functions by $\mathscr{S}^\star(\rho)$ and $\mathscr{K}(\rho)\,(0 \le \rho < 1)$, respectively. We write $\mathscr{S}^\star(0) = \mathscr{S}^\star$ and $\mathscr{K}(0) = \mathscr{K}$, the familiar subclasses of starlike functions and convex functions in $\mathscr{U}$.

Furthermore, a function $f \in \mathscr{A}$ is said to in the class $\mathscr{R}(\rho)$, if it satisfies the inequality:

$$\mathrm{Re}\left\{f'(z)\right\} > \rho \quad (0 \le \rho < 1; z \in \mathscr{U}). \quad (2)$$

Note that $\mathscr{R}(\rho)$ is a subclass of close-to-convex functions of order $\rho\,(0 \le \rho < 1)$ in $\mathscr{U}$.

Let $\mathscr{P}$ denote the class of analytic functions of the form:

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathscr{U}) \quad (3)$$

satisfying the condition $\mathrm{Re}\{\phi(z)\} > 0$ in $\mathscr{U}$.

Let the functions f and g be analytic in $\mathbb{U}$. We say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ and $f(z) = g(\omega(z)), z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [1]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \quad (4)$$

For the functions f, g analytic in $\mathscr{U}$ and given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (5)$$

their Hadamard product (or convolution), denoted by $f \star g$ is defined as

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g \star f)(z) \quad (z \in \mathscr{U}). \quad (6)$$

Note that $f \star g$ is analytic in $\mathscr{U}$.

The *Gauss hypergeometric function* ${}_2F_1$ is defined by the infinite series

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$
$$\left(a, b, c \in \mathbb{C}, c \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; z \in \mathscr{U}\right), \quad (7)$$

where $(\kappa)_n$ denotes the *Pochhammer symbol* (or *shifted factorial*) given, in terms of the Gamma function Γ, by

$$(\kappa)_n = \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \begin{cases} \kappa(\kappa+1)\cdots(\kappa+n-1), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases} \quad (8)$$

We note that the series, given by (7), converges absolutely for $z \in \mathscr{U}$ and hence the function ${}_2F_1$ represents an analytic function in the unit disc $\mathscr{U}$ [2].

We further observe that the *Gauss hypergeometric function* ${}_2F_1$ plays an important role in the study of various properties and characteristics of subclasses of univalent/multivalent functions in geometric function theory (cf., e.g. [3–5]). In our present investigation, we consider the incomplete beta function ψ, defined by

$$\psi(a,c;z) = z\ {}_2F_1(a,1;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (9)$$
$$\left(a,c \in \mathbb{C}, c \notin \mathbb{Z}_0^-; z \in \mathscr{U}\right).$$

By making use of the Hadamard product and the function ψ, Carlson and Shaffer [6] defined the linear operator $\mathscr{L}(a,c) : \mathscr{A} \to \mathscr{A}$ by

$$\mathscr{L}(a,c) f(z) = \psi(a,c;z) \star f(z) \quad \left(f \in \mathscr{A}; z \in \mathscr{U}\right). \quad (10)$$

If $f \in \mathscr{A}$ is given by (1), then it follows from (10) that

$$\mathscr{L}(a,c) f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \quad (z \in \mathscr{U}), \quad (11)$$

$$z(\mathscr{L}(a,c) f)'(z) = a\mathscr{L}(a+1,c) f(z) - (a-1)\mathscr{L}(a,c) f(z) \quad (z \in \mathscr{U}). \quad (12)$$

The operator $\mathscr{L}(a,c)$ extends several operators introduced and studied by earlier researchers in geometric function theory. For example, $\mathscr{L}(m+1,1)f(z) = \mathscr{D}^m f(z)$ $(f \in \mathscr{A}, m \in \mathbb{Z}, m > -1; z \in \mathscr{U})$, the well-known Ruscheweyh derivative operator [7] of f and $\mathscr{L}(2, 2-\lambda)f(z) = \Omega_z^{\lambda} f(z)$ $(f \in \mathscr{A}, 0 \le \lambda < 1; z \in \mathscr{U})$, the familiar Owa-Srivastava fractional differential operator [8] of f.

With the aid of the linear operator $\mathscr{L}(a,c)$, we introduce a subclass of $\mathscr{A}$ as follows.

Definition 1. A function $f \in \mathscr{A}$ is said to be in the class $\widetilde{\mathscr{R}}(a,c,\rho)$, if it satisfies the following subordination relation:

$$\frac{\mathscr{L}(a,c) f(z)}{z} \prec \left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/2} \quad (0 \le \rho < 1; z \in \mathscr{U}), \quad (13)$$

where the power in the right hand side of (13) indicates the principal branch. Note that if $f \in \widetilde{\mathscr{R}}(a,c,\rho)$, then by (13)

$$\frac{\mathscr{L}(a,c) f(z)}{z} = \left\{\rho + (1-\rho)\phi(z)\right\}^{1/2} \quad (\phi \in \mathscr{P}; z \in \mathscr{U}). \quad (14)$$

We denote by $\widetilde{\mathscr{R}}(2,1,\rho) = \widetilde{\mathscr{R}}(\rho)$, the class of functions $f \in \mathscr{A}$ satisfying the subordination condition:

$$f'(z) \prec \left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/2} \quad (0 \le \rho < 1; z \in \mathscr{U}). \quad (15)$$

In fact, by suitably specializing the parameters a, c, and ρ in the class $\widetilde{\mathscr{R}}(a,c,\rho)$, we can obtain several subclasses of $\mathscr{A}$.

Remark 2. To bring out the geometrical significance of the class $\widetilde{\mathscr{R}}(a,c,\rho)$, we set

$$h_\rho(z) = \left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/2} \quad (0 \le \rho < 1; z \in \mathscr{U}) \quad (16)$$

and note that

$$\omega = h_\rho\left(e^{i\theta}\right) = \frac{1+(1-2\rho)e^{i\theta}}{1-e^{i\theta}} \quad (0 \le \theta \le 2\pi) \quad (17)$$

which gives $e^{i\theta}(\omega^2 - 2\rho + 1) = \omega^2 - 1$ or $|\omega^2 - 1| = |\omega^2 + 1 - 2\rho|$. Letting $\omega = u + iv$, we deduce that

$$1 - \left(u^2 - v^2\right)^2 + 4u^2v^2 = \left(u^2 - v^2 + 1 - 2\rho\right)^2 + 4u^2v^2, \quad (18)$$

which on simplification reduces to $u^2 - v^2 = \rho$. Thus, $h_\rho(\mathscr{U})$ is the interior of the right half branch of the hyperbola $u^2 - v^2 = \rho$. Hence, if $f \in \widetilde{\mathscr{R}}(a,c,\rho)$, then the set of values $\mathscr{L}(a,c)f(z)/z$ for $z \in \mathscr{U}$ lie in $h_\rho(\mathscr{U})$, where h_ρ is given by (16).

Fekete and Szegö [9] defined the Hankel determinant of a function f, given by (1) as

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad (a_1 = 1). \quad (19)$$

In our present investigation, we also consider the second Hankel determinant of f, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \quad (20)$$

It is known [10] that if f given by (1) is analytic and univalent in $\mathscr{U}$, then the sharp inequality $H_2(1) = |a_3 - a_2^2| \le 1$ holds. For a family $\mathscr{F}$ of functions in $\mathscr{A}$ of the form (1), the more general problem of finding the sharp upper bounds for the functionals $(a_3 - \mu a_2^2)$ $(\mu \in \mathbb{R}/\mathbb{C})$ is popularly known as Fekete-Szegö problem for the class $\mathscr{F}$. The Fekete-Szegö problem for the known classes of univalent functions, starlike functions, convex functions, and close-to-convex functions has been completely settled [9, 11–18]. Recently, Janteng et al. [19, 20] have obtained the sharp upper bounds to the second Hankel determinant $H_2(2)$ for the family $\mathscr{R}$ of functions in $\mathscr{A}$ whose derivatives have positive real part in $\mathscr{U}$. For initial work on the class $\mathscr{R}$, one may refer to the paper by MacGregor [21].

Our objective in the present paper is to solve the Fekete-Szegö problem and also to determine the sharp upper bound to the second Hankel determinant for the class $\widetilde{\mathscr{R}}(a, c, \rho)$ by following the techniques devised by Libera and Złotkiewicz [22, 23]. The criteria for functions in $\mathscr{A}$ to be in this class are also obtained.

To establish our main results, we will need the following results about the functions belonging to the class $\mathscr{P}$.

Lemma 3. *Let the function ϕ, given by* (3)*, be a member of the class $\mathscr{P}$. Then*

$$|p_k| \leq 2 \quad (k \geq 1), \tag{21}$$

$$\left|p_2 - \nu p_1^2\right| \leq 2 \max\{1, |2\nu - 1|\} \quad (\nu \in \mathbb{C}), \tag{22}$$

$$p_2 = \frac{1}{2}\left\{p_1^2 + \left(4 - p_1^2\right)x\right\}, \tag{23}$$

$$p_3 = \frac{1}{4}\left\{p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z\right\} \tag{24}$$

for some complex numbers x, z satisfying $|x| \leq 1$ and $|z| \leq 1$. The estimates in (21) *and* (22) *are sharp.*

We note that the estimate (21) is contained in [10]; the estimate (22) is obtained by Ma and Minda [24]; the results in (23) and (24) are due to Libera and Złotkiewicz [23] (see also [22]).

2. Main Results

Unless otherwise mentioned, we assume throughout the sequel that

$$a > 0, \qquad c > 0, \qquad 0 \leq \rho < 1. \tag{25}$$

Now, we determine the sharp upper bound for the functional $|a_3 - \mu a_2^2|$ $(\mu \in \mathbb{C})$ for functions of the form (1) belonging to the class $\widetilde{\mathscr{R}}(a, c, \rho)$.

Theorem 4. *Let $a > 0$ and $c > 0$. If the function f, given by* (1)*, belongs to the class $\widetilde{\mathscr{R}}(a, c, \rho)$, then for any $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(c)_2(1-\rho)}{(a)_2} \times \max\left\{1, \frac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)}\right\}. \tag{26}$$

The estimate in (26) *is sharp.*

Proof. Since $f \in \widetilde{\mathscr{R}}(a, c, \rho)$, by (14) we have

$$\frac{\mathscr{L}(a, c) f(z)}{z} = \{\rho + (1-\rho)\phi(z)\}^{1/2} \quad (z \in \mathscr{U}), \tag{27}$$

where $\phi \in \mathscr{P}$ is given by (3). It is easily seen that

$$\begin{aligned}\{\rho + (1-\rho)\phi(z)\}^{1/2} &= 1 + \frac{1}{2}(1-\rho)p_1 z + \frac{1-\rho}{2}\left\{p_2 - \frac{(1-\rho)}{4}p_1^2\right\}z^2 \\ &+ \frac{1-\rho}{2}\left\{p_3 - \frac{(1-\rho)}{2}p_1 p_2 + \frac{(1-\rho)^2}{8}p_1^3\right\}z^3 + \cdots \\ &(z \in \mathscr{U}).\end{aligned} \tag{28}$$

Writing the series expansion of $\mathscr{L}(a, c) f(z)$ given by (11), $\{\rho + (1-\rho)\phi(z)\}^{1/2}$, in (27) and equating the coefficients of z, z^2, z^3 in the resulting equation, we obtain

$$a_2 = \frac{(1-\rho)c}{2a}p_1, \tag{29}$$

$$a_3 = \frac{(1-\rho)(c)_2}{2(a)_2}\left\{p_2 - \frac{(1-\rho)}{4}p_1^2\right\}, \tag{30}$$

$$a_4 = \frac{(1-\rho)(c)_3}{2(a)_3}\left\{p_3 - \frac{(1-\rho)}{2}p_1 p_2 + \frac{(1-\rho)^2}{8}p_1^3\right\}. \tag{31}$$

Thus for any $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| = \frac{(c)_2(1-\rho)}{2(a)_2}\left|p_2 - \frac{(1-\rho)\{a(c+1) + 2(a+1)c\mu\}}{4a(c+1)}p_1^2\right| \tag{32}$$

and by using (22) in the above expression, we get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(c)_2(1-\rho)}{(a)_2} \times \max\left\{1, \left|\frac{(1-\rho)}{2a(c+1)}\{a(c+1) + 2(a+1)c\mu\} - 1\right|\right\} \tag{33}$$

which, upon simplification, gives the required assertion of Theorem 4.

Equality in (26) holds for the function f_0 defined in $\mathscr{U}$ by

$$f_0(z) = \begin{cases} \psi(c, a; z) \star z h_\rho(z), & \dfrac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)} \leq 1, \\ \psi(c, a; z) \star z h_\rho\left(z^2\right), & \dfrac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)} > 1, \end{cases} \tag{34}$$

where the function h_ρ is given by (16). This completes the proof of Theorem 4. ☐

Theorem 5. *Let $a > 0$, $c > 0$ and $\mu \in \mathbb{R}$. If the function f, given by* (1)*, belongs to the class $\widetilde{\mathcal{R}}(a, c, \rho)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{\{a(c+1)(1+\rho) - 2(a+1)c(1-\rho)\mu\} c(1-\rho)}{2a(a)_2}, & \mu < -\dfrac{a(c+1)}{2(a+1)c} \\ \dfrac{(c)_2(1-\rho)}{(a)_2}, & -\dfrac{a(c+1)}{2(a+1)c} \le \mu \le \dfrac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)} \\ \dfrac{\{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)\} c(1-\rho)}{2a(a)_2}, & \mu > \dfrac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)}. \end{cases} \tag{35}$$

The estimates are sharp.

Proof. First, we assume that $\mu < -\{a(c+1)\}/2(a+1)c$. Then

$$\frac{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)}{2a(c+1)} < -1 \tag{36}$$

so that by (26), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{\{a(c+1)(1+\rho) - 2(a+1)c(1-\rho)\mu\} c(1-\rho)}{2a(a)_2}. \tag{37}$$

Next, let

$$-\frac{a(c+1)}{2(a+1)c} \le \mu \le \frac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)}. \tag{38}$$

Then, a routine calculation yields

$$\frac{\left|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)\right|}{2a(c+1)} \le 1 \tag{39}$$

and by using (26) again, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{(c)_2(1-\rho)}{(a)_2}. \tag{40}$$

Finally, if $\{a(c+1)(3+\rho)\}/2(a+1)c(1-\rho) > 1$, then

$$\frac{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)}{2a(c+1)} > 1. \tag{41}$$

Thus, by (26), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{\{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)\} c(1-\rho)}{2a(a)_2}. \tag{42}$$

The estimates are sharp for the function f_0 defined in $\mathcal{U}$ by

$$f_0(z) = \begin{cases} \psi(c,a;z) \star zh_\rho(z), & \mu < -\dfrac{a(c+1)}{2(a+1)c} \ \text{ or } \ \mu > \dfrac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)} \\ \psi(c,a;z) \star zh_\rho(z^2), & -\dfrac{a(c+1)}{2(a+1)c} \le \mu \le \dfrac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)}, \end{cases} \tag{43}$$

where the function h_ρ is given by (16) and the proof of Theorem 5 is completed. □

Using (21) in (29) and putting $\mu = 0$ and $\mu = 1$, respectively, in Theorem 5, we get the following.

Corollary 6. *Let $a \ge c > 0$. If the function f, given by* (1)*, belongs to the class $\widetilde{\mathcal{R}}(a, c, \rho)$, then*

$$\left|a_2\right| \le \frac{c(1-\rho)}{a}, \tag{44}$$

$$\left|a_3\right| \le \frac{(c)_2(1-\rho)}{(a)_2}, \tag{45}$$

$$\left|a_3 - a_2^2\right| \le \frac{(c)_2(1-\rho)}{(a)_2}. \tag{46}$$

The estimates in (44) *and* (46) *are sharp for the function f_0 defined by*

$$f_0(z) = \psi(c,a;z) \star zh_\rho(z) \quad (z \in \mathcal{U}), \tag{47}$$

whereas the estimate in (45) *is sharp for the function f_0 given by*

$$f_0(z) = \psi(c,a;z) \star zh_\rho(z^2) \quad (z \in \mathcal{U}), \tag{48}$$

where the function h_ρ is given by (16).

Letting $a = 2$ and $c = 1$ in Theorem 8, we obtain the following.

Corollary 7. *If the function f, given by* (1)*, belongs to the class $\widetilde{\mathcal{R}}(\rho)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{\{2(1+\rho) - 3\mu(1-\rho)\}(1-\rho)}{12}, & \mu < -\dfrac{2}{3} \\ \dfrac{(1-\rho)}{3}, & -\dfrac{2}{3} \le \mu \le \dfrac{2(3+\rho)}{3(1-\rho)} \\ \dfrac{\{3\mu(1-\rho) - 2(1+\rho)\}(1-\rho)}{12}, & \mu > \dfrac{2(3+\rho)}{3(1-\rho)}. \end{cases} \tag{49}$$

The estimates are sharp for the function f_0 defined in $\mathcal{U}$ by

$$f_0(z)=\begin{cases}\displaystyle\int_0^z \frac{dt}{1-t}\star zh_\rho(z), & \mu<-\dfrac{2}{3}\quad or\quad \mu>\dfrac{2(3+\rho)}{3(1-\rho)}\\ \displaystyle\int_0^z \frac{dt}{1-t}\star zh_\rho(z^2), & -\dfrac{2}{3}\le\mu\le\dfrac{2(3+\rho)}{3(1-\rho)},\end{cases} \tag{50}$$

where h_ρ is given by (16).

Next, we find the sharp upper bound for the fourth coefficient of functions in the class $\widetilde{\mathcal{R}}(a,c,\rho)$.

Theorem 8. *Let the function f, given by* (1), *belong to the class $\widetilde{\mathcal{R}}(a,c,\rho)$. Then*

$$|a_4|\le\frac{(c)_3(1-\rho)}{(a)_3} \tag{51}$$

and the estimate in (51) *is sharp.*

Proof. From (31), we have

$$|a_4|=\frac{(c)_3(1-\rho)}{2(a)_3}\left|p_3-\frac{1-\rho}{2}p_1p_2+\frac{(1-\rho)^2}{8}p_1^3\right|. \tag{52}$$

Since the functions $\phi(z)$ and $\phi(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ are in the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $p_1>0$. For convenience of notation, we write $p_1=p$ $(0\le p\le 2)$. Now, by using (23) and (24) in (52), we deduce that

$$\begin{aligned}|a_4| &=\frac{(c)_3(1-\rho)}{2(a)_3}\left|\frac{1+\rho^2}{8}p^3+\frac{1+\rho}{4}(4-p^2)px\right.\\ &\quad\left.-\frac{1}{4}(4-p^2)px^2+\frac{1}{2}(4-p^2)(1-|x|^2)z\right|\end{aligned} \tag{53}$$

for some complex numbers x $(|x|\le 1)$ and z $(|z|\le 1)$.

Applying the triangle inequality in the above expression followed by the replacement of $|x|$ with y in the resulting equation, we obtain

$$\begin{aligned}|a_4| &\le\frac{(c)_3(1-\rho)}{2(a)_3}\left\{\frac{1+\rho^2}{8}p^3+\frac{1+\rho}{4}(4-p^2)py\right.\\ &\quad\left.+\frac{1}{4}(4-p^2)(p-2)y^2+\frac{1}{2}(4-p^2)\right\}\\ &=G(p,y)\quad(0\le p\le 2,0\le y\le 1)\quad(\text{say}).\end{aligned} \tag{54}$$

We next maximize the function $G(p,y)$ on the closed rectangle $[0,2]\times[0,1]$. Since

$$\frac{\partial G}{\partial y}=\frac{1}{4}(4-p^2)\{p(1+\rho)-2(2-p)y\}, \tag{55}$$

we have $\partial G/\partial y<0$ for $0<p<2$ and $0<y<1$. Thus, $G(p,y)$ cannot have a maximum in the interior on the closed rectangle $[0,2]\times[0,1]$. Therefore, for fixed $p\in[0,2]$

$$\max_{0\le y\le 1}G(p,y)=G(p,0)=F(p)\quad(\text{say}), \tag{56}$$

where

$$F(p)=\frac{1+\rho^2}{8}p^3+\frac{1}{2}(4-p^2)\quad(0\le p\le 2). \tag{57}$$

A routine calculation yields

$$F'(p)=\frac{3(1+\rho^2)}{8}p^2-p=0 \tag{58}$$

for $p=0$ or $p=8/\{3(1+\rho^2)\}$. Since $F''(0)=-1<0$ and $F''(8/\{3(1+\rho^2)\})=1>0$, we conclude that the maximum of F is attained at $p=0$. Thus, the upper bound of the function G corresponds to $p=y=0$. Putting $p=y=0$ in (54), we get our desired estimate (51).

Equality in (51) holds for the function f_0 defined by

$$f_0(z)=\psi(c,a;z)\star zh_\rho(z^3)\quad(0\le\rho<1;z\in\mathcal{U}), \tag{59}$$

where h_ρ is given by (16). □

In the following theorem, we find the sharp upper bound to the second Hankel determinant for the class $\widetilde{\mathcal{R}}(a,c,\rho)$.

Theorem 9. *Let $a\ge c>0$ and $(a+2)(c+1)-3(a-c)>0$. If the function f, given by* (1), *belongs to the class $\widetilde{\mathcal{R}}(a,c,\rho)$, then*

$$\left|a_2a_4-a_3^2\right|\le\left\{\frac{(c)_2(1-\rho)}{(a)_2}\right\}^2. \tag{60}$$

The estimate in (60) *is sharp.*

Proof. From (29), (30), and (31), we deduce that

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &=\frac{c(c)_2(1-\rho)^2}{4a(a)_2}\\ &\quad\times\left|\left(\frac{c+2}{a+2}\right)p_1p_3-\left(\frac{c+1}{a+1}\right)p_2^2-\frac{(a-c)(1-\rho)}{2(a+1)(a+2)}p_1^2p_2\right.\\ &\quad\left.+\frac{\{(c+2)(a+1)+(a-c)\}(1-\rho)^2}{16(a+1)(a+2)}p_1^4\right|.\end{aligned} \tag{61}$$

As in Theorem 8, we assume without loss of generality that $p_1 > 0$ and for convenience of notation, we write $p_1 = p$ $(0 \leq p \leq 2)$. By using (23) and (24) in (61), we get

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \\ &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ &\quad\times\left|\frac{4(a-c)\rho + \{(c+2)(a+1)+(a-c)\}(1-\rho)^2}{16(a+1)(a+2)}p^4\right. \\ &\qquad + \frac{(a-c)(1+\rho)}{4(a+1)(a+2)}\left(4-p^2\right)p^2x \\ &\qquad - \frac{\left\{4(a+2)(c+1)+(a-c)p^2\right\}}{4(a+1)(a+2)}\left(4-p^2\right)x^2 \\ &\qquad \left. + \frac{c+2}{2(a+2)}\left(4-p^2\right)p\left(1-|x|^2\right)z\right|. \end{aligned} \tag{62}$$

Now, by applying the triangle inequality in (62) and replacing $|x|$ by y in the resulting equation, we get

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \\ &\leq \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ &\quad\times\left\{\frac{4(a-c)\rho + \{(a+1)(c+2)+(a-c)\}(1-\rho)^2}{16(a+1)(c+2)}p^4\right. \\ &\qquad + \frac{c+2}{2(a+2)(4-p^2)p} \\ &\qquad + \frac{(a-c)(1+\rho)}{4(a+1)(a+2)}\left(4-p^2\right)p^2y \\ &\qquad + \frac{\left\{(a-c)p^2 - 2(a+1)(c+2)p + 4(a+2)(c+1)\right\}}{4(a+1)(a+2)} \\ &\qquad \left.\times\left(4-p^2\right)y^2\right\} \\ &= \mathcal{G}(p,y) \quad (0 \leq p \leq 2, 0 \leq y \leq 1) \quad (\text{say}). \end{aligned} \tag{63}$$

We next maximize the function $\mathcal{G}(p, y)$ on the closed rectangle $[0, 2] \times [0, 1]$. Since

$$\begin{aligned} \frac{\partial\mathcal{G}}{\partial y} &= \frac{(a-c)(1+\rho)}{4(a+1)(a+2)}\left(4-p^2\right)p^2 \\ &\quad + \frac{\{2(a+1)(c+2)-(2+p)(a-c)\}}{2(a+1)(a+2)} \\ &\quad \times\left(4-p^2\right)(2-p)y > 0 \end{aligned} \tag{64}$$

for $0 < p < 2$ and $0 < y < 1$, it follows that $\mathcal{G}(p, y)$ cannot have a maximum in the interior on the closed rectangle $[0, 2] \times [0, 1]$. Thus, for fixed $p \in [0, 2]$

$$\max_{0\leq y\leq 1}\mathcal{G}(p,y) = \mathcal{G}(p,1) = \mathcal{F}(p) \quad (\text{say}), \tag{65}$$

where

$$\begin{aligned} &\mathcal{F}(p) \\ &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ &\quad\times\left\{\frac{4(a-c)\rho + \{(a+1)(c+2)+(a-c)\}(1-\rho)^2}{16(a+1)(c+2)}p^4\right. \\ &\qquad + \frac{c+2}{2(a+2)(4-p^2)p} \\ &\qquad + \frac{(a-c)(1+\rho)}{4(a+1)(a+2)}\left(4-p^2\right)p^2 \\ &\qquad + \frac{\left\{(a-c)p^2 - 2(a+1)(c+2)p + 4(a+2)(c+1)\right\}}{4(a+1)(a+2)} \\ &\qquad \left.\times\left(4-p^2\right)\right\}, \end{aligned} \tag{66}$$

$0 \leq \rho < 1$. and $0 \leq p \leq 2$. Differentiating $\mathcal{F}$ with respect to p, we deduce that

$$\begin{aligned} &\mathcal{F}'(p) \\ &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ &\quad\times\left[\frac{\{(a+1)(c+2)+(a-c)\}(1-\rho)^2 - 8(a-c)}{4(a+1)(a+2)}p^3\right. \\ &\qquad \left. - \frac{2\{(a+2)(c+1)-(a-c)(2+\rho)\}}{(a+1)(a+2)}p\right] = 0 \end{aligned} \tag{67}$$

for $p = 0$ or

$$p^2 = \frac{8\{(a+2)(c+1)-(a-c)(2+\rho)\}}{\{(a+1)(c+2)+(a-c)\}(1-\rho)^2 - 8(a-c)}. \tag{68}$$

Since $p^2 > 4$ and

$$\mathcal{F}''(0) = \frac{2\{(a+2)(c+1)-(a-c)(2+\rho)\}}{(a+1)(a+2)} < 0 \tag{69}$$

by the hypothesis, we conclude that the maximum value of $\mathcal{F}$ is attained at $p = 0$ so that the upper bound of the function $\mathcal{G}$ corresponds to $p = 0$ and $y = 1$. Thus, by letting $p = 0$ and $y = 1$ in (63), we get the estimate (60).

The estimate in (60) is sharp for the function f_0 given by (48). This completes the proof of Theorem 9. □

Putting $a = 2$ and $c = 1$ in Theorem 9, we get the following.

Corollary 10. *If the function f, given by* (1) *belongs to the class $\widetilde{\mathscr{R}}(\rho)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(1-\rho)^2}{9} \tag{70}$$

and the estimate is sharp for the function f_0 defined by

$$f_0(z) = \int_0^z \frac{dt}{1-t} \star zh_\rho\left(z^2\right) \quad (z \in \mathscr{U}), \tag{71}$$

where the function h_ρ is given by (16).

Theorem 11. *Let $\gamma > 0$, $a \geq 1/(2\gamma)$, $c > 0$ and $1/2 \leq \rho < 1$. If $f \in \mathscr{A}$ satisfies the following inequality*

$$Re\left\{\frac{\mathscr{L}(a+1,c)f(z)}{\mathscr{L}(a,c)f(z)}\right\} > \frac{(2a\gamma+1)\rho - 1}{2a\gamma\rho} \quad (z \in \mathscr{U}), \tag{72}$$

then

$$\frac{\mathscr{L}(a,c)f(z)}{z} \prec \left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/\gamma} \quad (z \in \mathscr{U}). \tag{73}$$

The result is the best possible.

Proof. We define the function w by

$$\frac{\mathscr{L}(a,c)f(z)}{z} = \left\{\frac{1+(1-2\rho)w(z)}{1-w(z)}\right\}^{1/\gamma} \quad (z \in \mathscr{U}). \tag{74}$$

Choosing the principal branch in the right hand side in (74), we note that w is analytic in $\mathscr{U}$ with $w(0) = 0$. Furthermore, logarithmically differentiating (74) and using the identity (12) in the resulting equation, we find that

$$\frac{\mathscr{L}(a+1,c)f(z)}{\mathscr{L}(a,c)f(z)} = 1 + \frac{1-2\rho}{a\gamma}\frac{zw'(z)}{1+(1-2\rho)w(z)} + \frac{1}{a\gamma}\frac{zw'(z)}{1-w(z)} \quad (z \in \mathscr{U}). \tag{75}$$

We claim that $|w(z)| < 1$ for all $z \in \mathscr{U}$. If not, then there exists a point $z_0 \in \mathscr{U}$ such that

$$\max\{|w(z)| : |z| \leq |z_0|\} = |w(z_0)| = 1 \quad (w(z_0) \neq 1), \tag{76}$$

and let $w(z_0) = e^{i\theta}$. Now, by applying Jack's lemma [25], we have

$$z_0w'(z_0) = kw(z_0) \quad (k \geq 1). \tag{77}$$

From (75) and (77), we obtain

$$\begin{aligned} &\operatorname{Re}\left\{\frac{\mathscr{L}(a+1,c)f(z)}{\mathscr{L}(a,c)f(z)}\right\} \\ &= 1 + \frac{k}{a\gamma}\operatorname{Re}\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right) + \frac{(1-2\rho)k}{a\gamma}\operatorname{Re}\left\{\frac{e^{i\theta}}{1+(1-2\rho)e^{i\theta}}\right\} \\ &= 1 - \frac{c}{2a\gamma} + \frac{(1-2\rho)k}{a\gamma} \times \frac{1-2\rho+\cos\theta}{1+2(1-2\rho)\cos\theta+(1-2\rho)^2} \\ &\leq 1 - \frac{c}{2a\gamma} + \frac{(2\rho-1)k}{2a\gamma\rho} \\ &\leq \frac{(2a\gamma+1)\rho-1}{2a\gamma\rho}, \end{aligned} \tag{78}$$

which contradicts the hypothesis (72). Thus, we conclude that $|w(z)| < 1$ for all $z \in \mathscr{U}$ and (74) yields the required subordination relation (73).

To see that the result is the best possible, we consider the function $f_0 \in \mathscr{A}$ defined by

$$f_0(z) = \psi(c,a;z) \star z\left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/\gamma} \quad \left(\gamma > 0, a \geq \frac{1}{2\gamma}, c > 0, \frac{1}{2} \leq \rho < 1; z \in \mathscr{U}\right) \tag{79}$$

from which it follows that

$$\frac{\mathscr{L}(a,c)f_0(z)}{z} = \left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/\gamma} \quad (z \in \mathscr{U}). \tag{80}$$

Thus, f_0 satisfies the subordination relation (73). On differentiating the expression in (80) followed by the use of the identity (12) in the resulting equation, we deduce that

$$\frac{\mathscr{L}(a+1,c)f_0(z)}{\mathscr{L}(a,c)f_0(z)} = 1 + \left(\frac{1-2\rho}{a\gamma}\right)\frac{z}{1+(1-2\rho)z} + \left(\frac{1}{a\gamma}\right)\frac{z}{1-z} \quad (z \in \mathscr{U}). \tag{81}$$

This implies that

$$\frac{\mathscr{L}(a+1,c)f_0(z)}{\mathscr{L}(a,c)f_0(z)} \longrightarrow \frac{(2a\gamma+1)\rho-1}{2a\gamma\rho} \quad \text{as } z \longrightarrow -1. \tag{82}$$

and the proof of Theorem 11 is completed. □

In the special case $\gamma = 2$, we get the following sufficient condition for the class $\widetilde{\mathscr{R}}(a,c,\rho)$.

Corollary 12. *Let $a \geq 1/4, c > 0$ and $1/2 \leq \rho < 1$. If $f \in \mathcal{A}$ satisfies the following inequality:*

$$Re\left\{\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}\right\} > \frac{(4a+1)\rho-1}{4a} \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U}\right), \tag{83}$$

then $f \in \widetilde{\mathcal{R}}(a,c,\rho)$. The result is the best possible for the function f_0 given by (47).

Letting $a = 2, c = 1$ and $\gamma = 2$ in Theorem 11, we obtain the following.

Corollary 13. *If $1/2 \leq \rho < 1$ and $f \in \mathcal{A}$ satisfies*

$$Re\left\{1+\frac{f''(z)}{f'(z)}\right\} > \frac{5\rho-1}{4\rho} \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U}\right), \tag{84}$$

then $f \in \widetilde{\mathcal{R}}(\rho)$. The result is the best possible for the function f_0 defined by

$$f_0(z) = \int_0^z \frac{dt}{1-t} \star zh_\rho(z) \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U}\right), \tag{85}$$

where the function h_ρ is given by (16).

Theorem 14. *Let $a > 0$, $c > 0$ and $\gamma > 0$. If $f \in \mathcal{A}$ satisfies the following subordination relation:*

$$\frac{\mathcal{L}(a,c)f(z)}{z} \prec \left\{\frac{1+(1-2\rho)z}{1-z}\right\}^{1/\gamma} \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U}\right), \tag{86}$$

then

$$Re\left(\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}\right) > \rho \quad (|z| < r_0(a,\gamma,\rho)), \tag{87}$$

where

$$r_0(a,\gamma,\rho) = \begin{cases} \dfrac{(1+a\gamma\rho)-\sqrt{(1+a\gamma\rho)^2-(a\gamma)^2(2\rho-1)}}{a\gamma(2\rho-1)}, & \dfrac{1}{2} < \rho < 1 \\ \dfrac{a\gamma}{2+a\gamma}, & \rho = \dfrac{1}{2}. \end{cases} \tag{88}$$

The bound $r_0(a,\gamma,\rho)$ in (88) *is the best possible.*

Proof. From (86), we get

$$\left(\frac{\mathcal{L}(a,c)f(z)}{z}\right) = \{\rho+(1-\rho)\phi(z)\}^{1/\gamma} \quad (\phi \in \mathcal{P}; z \in \mathcal{U}), \tag{89}$$

where we choose the principal branch in (89). Taking logarithmic differentiation in (89) and using the identity (12) in the resulting equation, we deduce that

$$\operatorname{Re}\left\{\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}\right\} - \rho \geq (1-\rho)\left[1-\frac{|z\phi'(z)|}{a\gamma\{|\rho+(1-\rho)\phi(z)|\}}\right] \quad (z \in \mathcal{U}). \tag{90}$$

Using the following well-known estimates [21]

$$\frac{|z\phi'(z)|}{\operatorname{Re}\{\phi(z)\}} \leq \frac{2r}{1-r^2}, \quad |\phi(z)| \leq \frac{1+r}{1-r} \quad (|z| = r < 1) \tag{91}$$

in (90), we get

$$\begin{aligned}\operatorname{Re}\left\{\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}\right\} - \rho &\geq (1-\rho)\left[1-\frac{2r}{a\gamma\{\rho(1-r)^2+(1-\rho)(1-r^2)\}}\right] \\ &\geq (1-\rho)\left[1-\frac{2r}{a\gamma\{(2\rho-1)r^2-2\rho r+1\}}\right]\end{aligned} \tag{92}$$

which is certainly positive for $|z| < r_0(a,\gamma,\rho)$, where $r_0(a,\gamma,\rho)$ is given by (88).

To show that the result is the best possible, we consider the function f_0 defined by

$$f_0(z) = \psi(c,a;z) \star z\left\{\rho+(1-\rho)\frac{1+z}{1-z}\right\}^{1/\gamma} \quad \left(\frac{1}{2} \leq \rho < 1, 0 < \gamma; z \in \mathcal{U}\right). \tag{93}$$

Noting that

$$\left\{\frac{\mathcal{L}(a+1,c)f_0(z)}{\mathcal{L}(a,c)f_0(z)}\right\} - \rho = (1-\rho)\left[1+\frac{2z}{a\gamma\{\rho(1-z)^2+(1-\rho)(1-z^2)\}}\right] = 0 \tag{94}$$

for $z = -r_0(a,\gamma,\rho)$, we conclude that the bound is the best possible. This proves Theorem 14. □

Taking $\gamma = 2$ in Theorem 14, we get the following.

Corollary 15. *If $a > 0$, $c > 0$, $1/2 \leq \rho < 1$ and $f \in \widetilde{\mathcal{R}}(a,c,\rho)$, then*

$$Re\left(\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}\right) > \rho \quad (|z| < \kappa(a,\rho)), \tag{95}$$

where

$$\kappa(a,\rho) = \begin{cases} \dfrac{(1+2a\rho) - \sqrt{4a^2(1-\rho)^2 + 4a\rho + 1}}{2a(2\rho - 1)}, & \dfrac{1}{2} < \rho < 1 \\ \dfrac{a}{1+a}, & \rho = \dfrac{1}{2}. \end{cases} \quad (96)$$

The bound $\kappa(a,\rho)$ *is the best possible for the function* f_0, *given by* (47).

Setting $a = 2, c = 1$ and $\gamma = 1$ in Theorem 14, we get the following.

Corollary 16. *If* $f \in \mathcal{A}$ *satisfies*

$$Re\left\{f'(z)\right\} > \rho \quad \left(\frac{1}{2} \le \rho < 1; z \in \mathcal{U}\right), \quad (97)$$

then

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 2\rho - 1 \quad (|z| < \varkappa(\rho)), \quad (98)$$

where

$$\varkappa(\rho) = \begin{cases} \dfrac{(1+2\rho) - \sqrt{4\rho^2 - 4\rho + 5}}{2(2\rho - 1)}, & \dfrac{1}{2} < \rho < 1 \\ \dfrac{1}{2}, & \rho = \dfrac{1}{2}. \end{cases} \quad (99)$$

The bound $\varkappa(\rho)$ *is the best possible for the function* f_0, *given in Corollary 13.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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