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*Research Article*

# On Some Integral Operators for Certain Classes of $p$-Valent Functions

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We study some generalized integral operators for the classes of $p$-valent functions with bounded radius and boundary rotation. Our work generalizes many previously known results. Many of our results are best possible.

## 1. Introduction

Let $A_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in N = \{1, 2, \ldots\}, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Let $f$ and $g$ be analytic functions in $U$ we say that $f$ is subordinate to $g$, written as

$$f \prec g; \tag{1.2}$$

if there exists a Schwarz function $w(z)$ in $U$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that

$$f(z) = g(w(z)). \tag{1.3}$$

In particular, when $g$ is univalent, then the above subordination is equivalent to

$$f(0) = 0, \qquad f(U) \subseteq g(U). \tag{1.4}$$

For functions $f, g \in A_p$, given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \quad z \in U, \tag{1.5}$$

we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.6}$$

Janowski [1] defined the class $P[A, B]$ as follows.

Let $h$ be a function, analytic in $U$, with $h(0) = 1$. Then $h$ is said to belong to the class $P[A, B]$, $-1 \le B < A \le 1$, if and only if, for $z \in U$,

$$h(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad \text{where } w(z) \text{ is a Schwarz function.} \tag{1.7}$$

Or equivalently, we can say that $h \in P[A, B]$, $-1 \le B < A \le 1$, if and only if,

$$h(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in U. \tag{1.8}$$

Geometrically, $h(z)$ is in the class $P[A, B]$, if and only if, $h(0) = 1$ and the image of $h(U)$ lies inside the open disc centered on the real axis with diameter end points,

$$D_1 = h(-1) = \frac{1 - A}{1 - B}, \quad D_2 = h(1) = \frac{1 + A}{1 + B}, \quad 0 < D_1 < 1 < D_2. \tag{1.9}$$

Clearly $P[A, B] \subset P((1 - A)/(1 - B))$.

In the recent paper, Noor [2] introduced the class $P_k(\alpha)$. We define it as follows. Let $P_k(\alpha), 0 \le \alpha < p$, be the class of functions $p(z)$ with $p(0) = 1$ and satisfying the property

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\alpha) z e^{-it}}{1 - z e^{-it}} d\mu(t), \tag{1.10}$$

where $\mu(t)$ is a real-valued function of bounded variation on $[0, 2\pi]$ and $\int_0^{2\pi} d\mu(t) = 2$ and $\int_0^{2\pi} |d\mu(t)| \le k$.

The classes $V_k(\alpha)$ and $R_k(\alpha)$ are related to the class $P_k(\alpha)$ and can be defined as

$$f \in V_k(\alpha), \quad \text{iff } \frac{(zf'(z))'}{pf'(z)} \in P_k(\alpha), \quad z \in U,$$
$$f \in R_k(\alpha), \quad \text{iff } \frac{zf'(z)}{pf(z)} \in P_k(\alpha), \quad z \in U. \tag{1.11}$$

We define a class $P_k[A,B]$ as follows.

Let $P_k[A,B]$, $k \geq 2$, $-1 \leq B < A \leq 1$, denote the class of $p$-valent analytic functions $h(z)$ that are represented by

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \quad z \in U, \tag{1.12}$$

where $h_1, h_2 \in P[A,B]$. For $A = 1 - 2\alpha$ $(0 \leq \alpha < p)$ and $B = -1$, it reduces to the class $P_k(\alpha)$ and $P_2(\alpha) = P(\alpha)$ is the class of $p$-valent analytic functions $h(z)$ with $\operatorname{Re} h(z) > \alpha$, $z \in U$. Taking $A = 1$, $B = -1$, and $p = 1$, we have $P_k[1,-1] = P_k$ (see [3]), and $P_2[1,-1] = P$ is the class of functions with positive real part.

*Definition 1.1.* A function $f$, analytic in $U$, and given by (1.1) is said to be in the class $R_k[A,B]$; $-1 \leq B < A \leq 1$, $k \geq 2$, if and only if,

$$\frac{zf'(z)}{pf(z)} \in P_k[A,B], \quad z \in U. \tag{1.13}$$

For $p = 1$, $R_k[A,B]$ is introduced and studied by Noor [4]. We note that

$$R_k[A,B] \subset R_k\left(\frac{1-A}{1-B}\right) \subset R_k, \tag{1.14}$$

where $R_k$ is the class of functions with bounded radius rotation (see [5]). For $k = 2$, we have

$$R_2[A,B] \equiv S_p^*[A,B] \subset S_p^*\left(\frac{1-A}{1-B}\right) \subset S_p^*, \tag{1.15}$$

where $S_p^*$ is the class of $p$-valent starlike functions. Similarly, we can define the class $V_k[A,B]$ as follows.

*Definition 1.2.* A function $f$, analytic in $U$, and given by (1.1) is said to be in the class $V_k[A,B]$; $-1 \leq B < A \leq 1$, $k \geq 2$, if and only if,

$$\frac{(zf'(z))'}{pf'(z)} \in P_k[A,B], \quad z \in U. \tag{1.16}$$

It is clear that

$$f \in V_k[A,B], \quad \text{iff } \frac{zf'(z)}{p} \in R_k[A,B], \quad z \in U. \tag{1.17}$$

For $p = 1$, $V_k[A,B]$ is the class introduced and studied by Noor [4]. It is easy to see that,

$$V_k[A,B] \subset V_k\left(\frac{1-A}{1-B}\right) \subset V_k, \tag{1.18}$$

where $V_k$ is the class of functions with bounded boundary rotation see [5]. Also

$$V_2[A,B] \equiv C_p[A,B] \subset C_p\left(\frac{1-A}{1-B}\right) \subset C_p, \tag{1.19}$$

where $C_p$ is the class of $p$-valent convex functions.

Very recently, Frasin [6], introduced the following general integral operators for $p$-valent functions,

$$F_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1(t)}{t^p}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p}\right)^{\alpha_n} dt, \tag{1.20}$$

$$G_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1'(t)}{pt^{p-1}}\right)^{\alpha_1} \cdots \left(\frac{f_n'(t)}{pt^{p-1}}\right)^{\alpha_n} dt, \quad \text{where } \alpha_i \in \mathbb{C},\ z \in U. \tag{1.21}$$

Clearly, we may see that for $p = 1$, these operators become the general integral operators

$$F_1(z) = F_n(z), \qquad G_1(z) = F_{\alpha_1,\alpha_2,\ldots,\alpha_n}(z), \tag{1.22}$$

introduced and studied by Breaz and Breaz [7] and Breaz et al. [8], (see also [9, 10]).

For $p = n = 1$, $\alpha_1 = \alpha \in [0,1]$ in (1.20), we obtain the integral operator $\int_0^z (f(t)/t)^\alpha dt$ studied in [11] and for $p = n = 1$, $\alpha_1 = \delta \in \mathbb{C}$, $|\delta| < 1/4$ in (1.21), we obtain the integral operator $\int_0^z (f'(t))^\delta dt$, studied in [12].

## 2. Main Results

**Lemma 2.1.** *Let $\beta > 0$, $\beta + \gamma > 0$, $\alpha \in [\alpha_0, 1)$, with $\alpha_0 = \max\{(\beta - \gamma - 1)/2\beta, -\gamma/\beta\}$. If*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + (1-2\alpha)z}{1-z}, \tag{2.1}$$

*then*

$$p(z) \prec Q(z) \prec \frac{1 + (1-2\alpha)z}{1-z}, \tag{2.2}$$

*where* $Q(z) = 1/\beta G(z) - \gamma/\beta$,

$$G(z) = \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1} dt = {}_2F_1\left(2\beta(1-\alpha), 1, \beta+\gamma+1; \frac{z}{1-z}\right), \tag{2.3}$$

$$\rho = \rho(\alpha, \beta, \gamma) = \frac{\beta+\gamma}{\beta_2 F_1(2\beta(1-\alpha), 1, \beta+\gamma+1; 1/2)} - \frac{\gamma}{\beta}, \tag{2.4}$$

${}_2F_1$ *denotes the Gauss hypergeometric function. From* (2.2)*, we can deduce the sharp result* $p \in P(\rho)$*, where* $\rho$ *is defined in* (2.4)*. This result is a special case of one given in [11].*

*Proof.* To prove this Lemma we use Theorem 3.2j of [11, page 97]. Take $h(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \le \alpha < 1$ and

$$H(z) = \beta h(z) + \gamma = \frac{a + bz}{1 - z}, \tag{2.5}$$

where $a = \beta + \gamma$ and $b = \beta(1 - 2\alpha) + \gamma$.

Since $H$ is convex to apply Theorem 3.2j of [11, page 97] we only need to determine condition $\operatorname{Re} H(z) > 0$.

The range of $|z| \le 1$ under $H(z)$ is a half plane. In order to satisfy the required condition this half plane needs to lie in the right half plane. This requirement will be satisfied if $\operatorname{Re} H(-1) = \operatorname{Re} H(i)$ and $\operatorname{Re} H(0) > \operatorname{Re} H(-1) \ge 0$. Or we can write it as

$$\beta(1-\alpha) > 0, \quad \beta\alpha + \gamma \ge 0. \tag{2.6}$$

When $\beta > 0$, $\beta + \gamma > 0$, these conditions imply that $\alpha \in [-\gamma/\beta, 1)$, and if $\beta + \gamma > 1$, then $\alpha \in [(\beta - \gamma - 1)/2\beta, 1)$. Hence all the conditions of Theorem 3.2j of [11, page 97] are satisfied for $\alpha \in [\alpha_0, 1)$, with $\alpha_0 = \max\{(\beta - \gamma - 1)/2\beta, -\gamma/\beta\}$, thus we have the required result. □

To show that the solution $Q(z)$ can be represented in terms of hypergeometric functions we take $A = 1 - 2\alpha$, $B = -1$, $n = 1$ in Theorem 3.3d of [11, page 109].

**Lemma 2.2.** *Let* $f \in V_k(\alpha)$, $0 \le \alpha < p$, $k \ge 2$*. Then* $f \in R_k(\rho)$ *in* $U$*, where*

$$\rho = \rho(\alpha, p) = \frac{1}{{}_2F_1(2p(1-\alpha), 1, p+1; 1/2)}. \tag{2.7}$$

*This result is sharp.*

*Proof.* Let for $k \ge 2$, $z \in U$, we have

$$\frac{zf'(z)}{pf(z)} = h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \tag{2.8}$$

where $h$, $h_i$ are analytic in $U$ with $h(0) = 1$, $h_i(0) = 1$, $i = 1, 2$.

We define

$$\phi_p(z) = z + \sum_{n=1}^{\infty} \frac{1}{p(1+(n-1))} z^n, \qquad z \in U. \tag{2.9}$$

By using (2.8), with convolution technique, see [13], we have

$$\frac{\phi_p(z)}{z} * h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{\phi_p(z)}{z} * h_1(z)\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{\phi_p(z)}{z} * h_2(z)\right). \tag{2.10}$$

This implies that,

$$h(z) + \frac{zh'(z)}{ph(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{ph_1(z)}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{zh_2'(z)}{ph_2(z)}\right). \tag{2.11}$$

Logarithmic differentiation of (2.8) yields,

$$\frac{(zf'(z))'}{pf'(z)} = h(z) + \frac{zh'(z)}{ph(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{ph_1(z)}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{zh_2'(z)}{ph_2(z)}\right). \tag{2.12}$$

Since $(zf'(z))'/pf'(z) \in P_k(\alpha)$, $0 \le \alpha < p$, thus

$$h_i(z) + \frac{zh_i'(z)}{ph_i(z)} \in P(\alpha), \qquad i = 1, 2. \tag{2.13}$$

By using Lemma 2.1 (for $\beta = p$ and $\gamma = 0$), we deduce that $h_i \in P(\rho)$, where $\rho$ is given in (2.7). This estimate is best possible because of the best dominant property of function $Q(z)$, where

$$Q(z) = \frac{1}{{}_2F_1(2p(1-\alpha), 1, p+1; z/(1-z))}, \qquad z \in U. \tag{2.14}$$

□

For $p = 1$, we have the sharp result proved in [14].

We begin with the following theorem.

**Theorem 2.3.** (i) *Let $\alpha_i > 0$, $f_i \in R_k[A, B]$ for all $i = 1, 2, \ldots, n$, and, $\sum_{i=1}^{n} \alpha_i = 1$. Then the integral operator $F_p \in V_k[A, B]$ in $U$, where $-1 \le B < A \le 1$, $k \ge 2$.*

(ii) *Let $f_i \in R_k(\alpha)$, $\alpha_i > 0$ for all $i = 1, 2, \ldots, n$ with $\alpha = (1-A)/(1-B)$, $k \ge 2$. If $\sum_{i=1}^{n} \alpha_i = 1$, then the integral operator $F_p$ defined by* (1.20) *also belongs to the class $R_k(\rho)$ in $U$, where $\rho = \rho(\alpha, p)$ is defined by* (2.7). *This result is sharp.*

*Proof (i).* From (1.20), we can see that $F_p \in A_p$ in $U$, and

$$F_p'(z) = pz^{p-1}\left[\left(\frac{f_1(z)}{z^p}\right)^{\alpha_1}\cdots\left(\frac{f_n(z)}{z^p}\right)^{\alpha_n}\right]. \tag{2.15}$$

Differentiating logarithmically and multiplying by $z$, we obtain,

$$\frac{zF_p''(z)}{F_p'(z)} = (p-1) + \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i'(z)} - p\right), \quad z \in U. \tag{2.16}$$

Thus, we have

$$1 + \frac{zF_p''(z)}{F_p'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i'(z)}\right), \tag{2.17}$$

or

$$\begin{aligned}\frac{\left(zF_p'(z)\right)'}{pF_p'(z)} &= \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{pf_i'(z)}\right)\\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i p_i(z)\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i h_i(z)\right),\end{aligned} \tag{2.18}$$

where $h_i, p_i \in P[A,B]$, for all $i = 1,2,\ldots,n$.

Since $P[A,B]$ is a convex set, see [15], it follows that,

$$\frac{\left(zF_p'(z)\right)'}{pF_p'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z), \tag{2.19}$$

where $H_1, H_2 \in P[A,B]$ and therefore,

$$\frac{\left(zF_p'(z)\right)'}{pF_p'(z)} \in P_k[A,B], \quad z \in U. \tag{2.20}$$

This proves the result. □

Substituting $p = 1$, in Theorem 2.3(i), we have the following corollary.

**Corollary 2.4.** *Let $\alpha_i > 0$, $f_i \in R_k[A,B]$ for all $i = 1,2,\ldots,n$, $-1 \le B < A \le 1$, $k \ge 2$. Then the integral operator $F_p \in V_k[A,B]$ in $U$.*

*Remark 2.5.* Letting $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and $n = 2$ in Corollary 2.4, we obtain a result due to Noor [4].

For $n = 1$, $\alpha_1 = \alpha = 1$, and $f_1 = f$ in Theorem 2.3(i), we have the following.

**Corollary 2.6.** *Let $f \in R_k[A,B]$ in $U$, $-1 \leq B < A \leq 1$, $k \geq 2$. Then the integral operator $\int_0^z p(f(t)/t)dt \in V_k[A,B]$, $z \in U$.*

*Proof (ii).* Taking $A = 1 - 2\alpha$, $B = -1$, with $\alpha = (1-A)/(1-B)$, we have for all $i = 1,2,\ldots,n$,

$$f_i \in R_k[1-2\alpha,-1] = R_k(\alpha), \tag{2.21}$$

using part (i) of Theorem 2.3, we have

$$F_p \in V_k[1-2\alpha,-1] = V_k(\alpha) \quad \text{in } U. \tag{2.22}$$

Now using Lemma 2.2 for $F_p \in V_k(\alpha)$, $\alpha = (1-A)/(1-B)$ implies that

$$F_p \in R_k(\rho), \qquad \text{where } \rho = \rho(\alpha,p) \text{ is defined in (2.7).} \tag{2.23}$$

The sharpness of the result is clear from the function $Q(z)$ defined by (2.14). □

For $p = 1$, we have the following corollary.

**Corollary 2.7.** *Let $\alpha_i > 0$, $f_i \in R_k(\alpha)$ for all $i = 1,2,\ldots,n$, with $\alpha = (1-A)/(1-B)$ and $A = 1-2\alpha$, $B = -1$. Then the integral operator $F_p$ defined by* (1.20) *also belongs to the class $R_k(\rho)$ in $U$, where*

$$\rho = \rho(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2 - 2^{2(1-\alpha)}}, & \text{if } \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2\ln 2}, & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{2.24}$$

*Remark 2.8.* Letting $\alpha_1 = \mu$, $\alpha_2 = \eta$, and $n = 2$ in Corollary 2.7, we have the sharp result proved in [14].

For $A = 1$, $B = -1$, and $p = 1$, we have

$$f_i \in R_k(0) \quad \text{implies that } F_p \in V_k\left(\frac{1}{2}\right) \text{ in } U. \tag{2.25}$$

**Theorem 2.9.** (i) *Let $\alpha_i > 0, f_i \in V_k[A,B]$ for all $i = 1,2,\ldots,n$. If $\sum_{i=1}^n \alpha_i = 1$, then the integral operator $G_p$ defined by* (1.21)*, also belongs to the class $V_k[A,B]$ in $U$, where $-1 \leq B < A \leq 1$, $k \geq 2$.*

(ii) *Let for $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$ and $f_i \in V_k(\alpha)$, for all $i = 1,2,\ldots,n$ with $0 \leq \alpha < p$, $\alpha = (1-A)/(1-B)$, $k \geq 2$. Then the integral operator $G_p \in R_k(\rho)$ in $U$, where $\rho = \rho(\alpha,p)$ is defined by* (2.7). *This result is sharp.*

*Proof (i).* From definition (1.20), we have

$$\begin{aligned} 1+\frac{zG_p''(z)}{G_p'(z)} &= p+\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}-p+1\right) \\ &= \sum_{i=1}^{n}\alpha_i\frac{(zf_i'(z))'}{f_i'(z)}, \end{aligned} \tag{2.26}$$

or

$$\begin{aligned} \frac{\left(zG_p'(z)\right)'}{pG_p'(z)} &= \sum_{i=1}^{n}\alpha_i\frac{(zf_i'(z))'}{pf_i'(z)} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i p_i(z)\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i h_i(z)\right), \end{aligned} \tag{2.27}$$

where $h_i, p_i \in P[A,B]$, for all $i = 1,2,\ldots,n$.

Since $P[A,B]$ is a convex set, see [15], it follows that,

$$\frac{\left(zG_p'(z)\right)'}{pG_p'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)H_2(z), \quad z\in U, \tag{2.28}$$

where $H_1, H_2 \in P[A,B]$ and therefore,

$$\frac{\left(zG_p'(z)\right)'}{pG_p'(z)} \in P_k[A,B] \quad \text{in } U. \tag{2.29}$$

This implies that $G_p \in V_k[A,B]$. □

Letting $p = 1$ in Theorem 2.9(i), we have the following corollary.

**Corollary 2.10.** *Let $\alpha_i > 0$, $f_i \in V_k[A,B]$ for all $i = 1,2,\ldots,n$ and $-1 \le B < A \le 1$, $k \ge 2$. If $\sum_{i=1}^{n}\alpha_i = 1$, then $G_p \in V_k[A,B]$ in $U$.*

*Proof (ii).* Taking $A = 1-2\alpha$, $B = -1$, we have for all $i = 1,2,\ldots,n$

$$f_i \in V_k[1-2\alpha,-1] = V_k(\alpha), \quad \text{where } \alpha = \frac{1-A}{1-B}. \tag{2.30}$$

Now using part (i) of Theorem 2.9, we have

$$G_p \in V_k[1-2\alpha,-1] = V_k(\alpha) \quad \text{in } U. \tag{2.31}$$

Now using Lemma 2.2, for $\alpha = (1 - A)/(1 - B)$, we have

$$G_p \in V_k(\alpha) \text{ implies that } G_p \in R_k(\rho), \text{ in } U, \quad \text{where } \rho = \rho(\alpha, p) \text{ is defined in (2.7).} \tag{2.32}$$

The sharpness of the result is clear from the function $Q(z)$ defined by (2.14). □

For $p = 1$, we have the following corollary.

**Corollary 2.11.** (i) *Let $\alpha_i > 0$, $f_i \in V_k(\alpha)$, $i = 1, 2, \ldots, n$, with $\alpha = (1 - A)/(1 - B)$ and $A = 1 - 2\alpha$, $B = -1$. Then $G_p \in R_k(\rho)$ in $U$, where $\rho = \rho(\alpha)$ and defined in* (2.24).

*Also for $A = 1$, $B = -1$, we have.*

(ii) *If $f_i \in V_k(0)$ for all $i = 1, 2, \ldots, n$, then $G_p \in R_k(1/2)$ in $U$.*

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