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FRACTIONAL DERIVATIVES OF HOLOMORPHIC FUNCTIONS ON BOUNDED SYMMETRIC DOMAINS OF C^n

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ABSTRACT. Let $f \in H(B_n)$ $f^{[\beta]}$ denotes the βth fractional derivative of f If $f^{[\beta]} \in A^{p,q,\alpha}(B_n)$, we show that

(I) If $\beta < \frac{\alpha+1}{p} + \frac{n}{q} = \delta$, then $f \in A^{s,t,\alpha}(B_n)$, and $\|f\|_{s,t,\alpha} \le C\left\|f^{[\beta]}\right\|_{p,q,\alpha}$, $s = \frac{\delta p}{\delta-\beta}$, $t = \frac{\delta q}{\delta-\beta}$

(II) If $\beta = \frac{\alpha+1}{p} + \frac{n}{q}$, then $f \in B(B_n)$ and $\|f\|_B \le C\left\|f^{[\beta]}\right\|_{p,q,\alpha}$

(III) If $\beta > \frac{\alpha+1}{p} + \frac{n}{q}$, then $f \in \wedge_{\beta-\frac{\alpha+1}{p}-\frac{n}{q}}(B_n)$ especially If $\beta = 1$ then $\|f\|_{\wedge_{1-\frac{\alpha+1}{p}-\frac{n}{q}}} \le C\|f^{[1]}\|_{p,q,\alpha}$ where B_n is the unit ball of C^n

KEY WORDS AND PHRASES. Fractional derivative, Bergman space, Bloch space, Lipschitz space
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Let Ω be a bounded symmetric domain in the complex vector space $C^n, o \in \Omega$, with Bergman-Silov boundary b, Γ the group of holomorphic automorphisms of Ω and Γ_0 its isotropy group. It is known that Ω is circular and star-shaped with respect to o and b is circular. The group Γ_0 is transitive on b and b has a unique normalized Γ_0-invariant measure σ with $\sigma(b) = 1$. Hua [2] constructed by group representation theory a system $\{\phi_{kv}\}$ of homogeneous polynomials, $k = 0, 1, \ldots$, $v = 1, \ldots, m_k, m_k = \binom{n+k-1}{k}$, complete and orthogonal on Ω and orthonormal on b.

By $H(\Omega)$ we denote the class of all holomorphic functions on Ω. Every $f \in H(\Omega)$ has a series expansion

$$f(z) = \sum_{k,v} a_{kv}\phi_{kv}(z)\,, \qquad a_{kv} = \lim_{r\to 1}\int_b f(r\xi)\,\overline{\phi_{kv}(\xi)}\,d\sigma(\xi) \tag{0}$$

where $\sum\limits_{k,v} = \sum\limits_{k=0}^{\infty}\sum\limits_{v=1}^{m_k}$ and the convergence is uniform on a compact subset of Ω.

Let $f \in H(\Omega)$ with the expansion (0) and $\beta > 0$. The βth fractional derivatives of f are defined, respectively, by

$$f^{[\beta]}(z) = \sum_{k,v} \frac{\Gamma(k+1+\beta)}{\Gamma(k+1)}\, a_{kv}\phi_{kv}(z)$$

$$f_{[\beta]}(z) = \sum_{k,v} \frac{\Gamma(k+1)}{\Gamma(k+1+\beta)}\, a_{kv}\phi_{kv}(z)$$

It is known that $f^{[\beta]}$, $f_{[\beta]} \in H(\Omega)$ and

$$f(r\xi) = \frac{1}{\Gamma(\beta)}\int_0^1 (1-\rho)^{\beta-1} f^{[\beta]}(r\rho\xi)d\rho\,. \tag{1}$$

Let $f \in H(\Omega)$. It will be said that f belongs to the Bergman spaces $A^{p,q,\alpha}(\Omega), 0 < p, q \le \infty, \alpha > -1$ if

$$\|f\|_{p,q,\alpha} = \begin{cases} \left(\int_0^1 (1-r)^\alpha M_q(r,f)^p dr\right)^{\frac{1}{p}}, & p < \infty \\ \sup\limits_{0<r<1} (1-r)^\alpha M_q(r,f)\,, & p = \infty \end{cases}$$

is finite, where

$$M_q(r,f) = \left(\int_b |f(r\xi)|^q d\sigma(\xi)^{\frac{1}{q}}\right), \qquad 0 < q < \infty$$

and

$$M_\infty(r,f) = \sup_{\xi \in b} |f(r\xi)|$$

see [1,3,5,6,7] for more on $A^{p,q,\alpha}(\Omega)$ For $0 < p \leq \infty$, let $A^p(\Omega)$denote $A^{p,p,o}(\Omega)$ (see [10,12]), $H^p(\Omega)$ denote $A^{\infty,p,0}(\Omega)$ (see [9])

Let B_n denote the unit ball in C^n A function $f \in H(B_n)$ is called a Bloch function, that is $f \in B(B_n)$, if ([8,11])

$$\|f\|_B = \sup_{z \in B_n} (1 - |z|)|f^{[1]}(z)| < \infty$$

For $0 < \alpha < \infty$, the definition of Lipschitz space $\wedge_\alpha(B_n)$ can be found in [4, §8 8]

In [10] and [12], Watanable and Stojan considered the problem If $f' \in A^p(D)$ (D is the unit disc of C^1), then $q = ?$ such that $f \in A^q(D)$ In this paper we consider and solve the same problem in $A^{p,q,\alpha}(\Omega)$

The main results of this paper are the following

THEOREM 1. Let $0 < p, q \leq \infty, \alpha > -1, 0 < \beta < \delta \leq \frac{\alpha+1}{p} + \frac{n}{q}$, if $f^{[\beta]} \in A^{p,q,\alpha}(\Omega)$ and $f^{[\beta]}(r\xi) = O\left(\|f^{[\beta]}\|_{p,q,\alpha}(1-r)^{-\delta}\right)$, then $f \in A^{s,t,\alpha}(\Omega)$ and $\|f\|_{s,t,\alpha} \leq C\|f^{[\beta]}\|_{p,q,\alpha}$, where $s = \frac{\delta p}{\delta - \beta}, t = \frac{\delta q}{\delta - \beta}$

THEOREM 2. Let $0 < p, q \leq \infty, \alpha > -1, 0 < \beta < \infty, f^{[\beta]} \in A^{p,q,\alpha}(B_n)$.

(I) If $\beta < \frac{\alpha+1}{p} + \frac{n}{q} = \delta$, then $f \in A^{s,t,\alpha}(B_n)$, and $\|f\|_{s,t,\alpha} \leq C\left\|f^{[\beta]}\right\|_{p,q,\alpha}$, where s, t are the same as above

(II) If $\beta = \frac{\alpha+1}{p} + \frac{n}{q}$, then $f \in B(B_n)$ and $\|f\|_B \leq C\left\|f^{[\beta]}\right\|_{p,q,\alpha}$

(III) If $\beta > \frac{\alpha+1}{p} + \frac{n}{q}$, then $f \in \wedge_{\beta - \frac{\alpha+1}{p} - \frac{n}{q}}(B_n)$, especially If $\beta = 1$, then $\|f\|_{\wedge_{1-\frac{\alpha+1}{p}-\frac{n}{q}}} \leq C\|f^{[1]}\|_{p,q,\alpha}$

REMARK. (i) Theorem 2(I) ($p = q, \alpha = 0, \beta = n = 1$) extends the results of Watanable's and Stojan's (ii) Theorem 1 ($p = \infty$) extends the results of Shi's ([9]) and Lou's ([6,7])

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