IJMMS 26:9 (2001) 525–537
PII. S0161171201006135
http://ijmms.hindawi.com

ON n-FOLD FUZZY POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRAS

YOUNG BAE JUN and KYUNG HO KIM

(Received 11 March 2000)

Abstract. We consider the fuzzification of the notion of an n-fold positive implicative ideal. We give characterizations of an n-fold fuzzy positive implicative ideal. We establish the extension property for n-fold fuzzy positive implicative ideals, and state a characterization of PI^n-Noetherian BCK-algebras. Finally we study the normalization of n-fold fuzzy positive implicative ideals.

2000 Mathematics Subject Classification. 06F35, 03G25, 03E72.

1. Introduction. For the general development of BCK-algebras, the ideal theory plays an important role. In 1999, Huang and Chen [1] introduced the notion of n-fold positive implicative ideals in BCK-algebras. In this paper, we consider the fuzzification of n-fold positive implicative ideals in BCK-algebras. We first define the notion of n-fold fuzzy positive implicative ideals of BCK-algebras, and then discuss the related properties. We give the relation between a fuzzy ideal and an n-fold fuzzy positive implicative ideal. We state a condition for a fuzzy ideal to be an n-fold fuzzy positive implicative ideal. Using level sets, we give a characterization of an n-fold fuzzy positive implicative ideal. We establish the extension property for an n-fold fuzzy positive implicative ideal. Using a family of n-fold fuzzy positive implicative ideals, we make a new n-fold fuzzy positive implicative ideal. We define the notion of PI^n-Noetherian BCK-algebras, and give its characterization. Furthermore, we study the normalization of an n-fold fuzzy positive implicative ideal.

2. Preliminaries. By a *BCK-algebra* we mean an algebra $(X;*,0)$ of type (2, 0) satisfying the axioms

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x,y,z\in X$. We can define a partial ordering $\le$ on X by $x\le y$ if and only if $x*y=0$. A BCK-algebra X is said to be *n-fold positive implicative* (see Huang and Chen [1]) if there exists a natural number n such that $x*y^{n+1}=x*y^n$ for all $x,y\in X$.

In any BCK-algebra X, the following hold:

(P1) $x*0=x$,
(P2) $x*y\le x$,
(P3) $(x*y)*z=(x*z)*y$,

(P4) $(x*z)*(y*z) \le x*y$,
(P5) $x \le y$ implies $x*z \le y*z$ and $z*y \le z*x$.

Throughout this paper X will always mean a BCK-algebra unless otherwise specified. A nonempty subset I of X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,
(I2) $x*y \in I$ and $y \in I$ imply $x \in I$.

A nonempty subset I of X is said to be a *positive implicative ideal* if it satisfies

(I1) $0 \in I$,
(I3) $(x*y)*z \in I$ and $y*z \in I$ imply $x*z \in I$.

Theorem 2.1 (see [3, Theorem 3]). *A nonempty subset I of X is a positive implicative ideal of X if and only if it satisfies*

(I1) $0 \in I$,
(I4) $((x*y)*y)*z \in I$ *and* $z \in I$ *imply* $x*y \in I$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$. For a fuzzy set μ in X and $t \in [0,1]$ define $U(\mu;t)$ to be the set $U(\mu;t) := \{x \in X \mid \mu(x) \ge t\}$.

A fuzzy set μ in X is said to be a *fuzzy ideal* of X if

(F1) $\mu(0) \ge \mu(x)$ for all $x \in X$,
(F2) $\mu(x) \ge \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in X$.

Note that every fuzzy ideal μ of X is order reversing, that is, if $x \le y$ then $\mu(x) \ge \mu(y)$.

A fuzzy set μ in X is called a *fuzzy positive implicative ideal* of X if it satisfies

(F1) $\mu(0) \ge \mu(x)$ for all $x \in X$,
(F3) $\mu(x*z) \ge \min\{\mu((x*y)*z), \mu(y*z)\}$ for all $x, y, z \in X$.

Theorem 2.2 (see [2, Proposition 1]). *For any fuzzy ideal μ of X, we have*

$$\mu(x*y) \ge \mu((x*y)*y) \Longleftrightarrow \mu((x*z)*(y*z)) \ge \mu((x*y)*z) \quad \forall x, y, z \in X. \tag{2.1}$$

3. n-fold fuzzy positive implicative ideals.

For any elements x and y of a BCK-algebra, $x*y^n$ denotes

$$(\cdots((x*y)*y)*\cdots)*y \tag{3.1}$$

in which y occurs n times. Using Theorem 2.1, Huang and Chen [1] introduced the concept of an n-fold positive implicative ideal as follows.

Definition 3.1. A subset A of X is called an *n-fold positive implicative ideal* of X if

(I1) $0 \in A$,
(I5) $x*y^n \in A$ whenever $(x*y^{n+1})*z \in A$ and $z \in A$ for every $x, y, z \in X$.

We try to fuzzify the concept of n-fold positive implicative ideal.

Definition 3.2. Let n be a positive integer. A fuzzy set μ in X is called an *n-fold fuzzy positive implicative ideal* of X if

(F1) $\mu(0) \ge \mu(x)$ for all $x \in X$, (F4) $\mu(x*y^n) \geq \min\{\mu((x*y^{n+1})*z), \mu(z)\}$ for all $x,y,z \in X$.

Notice that the 1-fold fuzzy positive implicative ideal is a fuzzy positive implicative ideal.

Example 3.3. Let $X = \{0,a,b\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = t_0$, $\mu(a) = t_1$, and $\mu(b) = t_2$ where $t_0 > t_1 > t_2$ in $[0,1]$. Then μ is an n-fold fuzzy positive implicative ideal of X for every natural number n.

Proposition 3.4. *Every n-fold fuzzy positive implicative ideal is a fuzzy ideal for every natural number n.*

Proof. Let μ be an n-fold fuzzy positive implicative ideal of X. Then

$$\begin{aligned} \mu(x) = \mu(x*0^n) &\geq \min\{\mu((x*0^{n+1})*z), \mu(z)\} \\ &= \min\{\mu(x*z), \mu(z)\} \quad \forall x,z \in X. \end{aligned} \tag{3.2}$$

Hence μ is a fuzzy ideal of X. □

The following example shows that the converse of Proposition 3.4 may not be true.

Example 3.5. Let $X = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, in which the operation $*$ is defined by $x*y = \max\{0, x-y\}$ for all $x,y \in X$. Then X is a BCK-algebra [1, Example 1.3]. Let μ be a fuzzy set in X given by $\mu(0) = t_0 > t_1 = \mu(x)$ for all $x(\neq 0) \in X$. Then μ is a fuzzy ideal of X. But μ is not a 2-fold fuzzy positive implicative ideal of X because $\mu(5*2^2) = \mu(1) = t_1$ and $\mu((5*2^3)*0) = \mu(0) = t_0$, and so

$$\mu(5*2^2) \ngeq \min\{\mu((5*2^3)*0), \mu(0)\}. \tag{3.3}$$

Let X be an n-fold positive implicative BCK-algebra and let μ be a fuzzy ideal of X. For any $x,y,z \in X$ we have

$$\mu(x*y^n) = \mu(x*y^{n+1}) \geq \min\{\mu((x*y^{n+1})*z), \mu(z)\}. \tag{3.4}$$

Hence μ is an n-fold fuzzy positive implicative ideal of X. Combining this and Proposition 3.4, we have the following theorem.

Theorem 3.6. *In an n-fold positive implicative BCK-algebra, the notion of n-fold fuzzy positive implicative ideals and fuzzy ideals coincide.*

Proposition 3.7. *Let μ be a fuzzy ideal of X. Then μ is an n-fold fuzzy positive implicative ideal of X if and only if it satisfies the inequality $\mu(x*y^n) \geq \mu(x*y^{n+1})$ for all $x,y \in X$.*

Proof. Suppose that μ is an n-fold fuzzy positive implicative ideal of X and let $x, y \in X$. Then

$$\begin{aligned}\mu(x * y^n) &\geq \min\{\mu((x * y^{n+1}) * 0), \mu(0)\}\\ &= \min\{\mu(x * y^{n+1}), \mu(0)\}\\ &= \mu(x * y^{n+1}).\end{aligned} \tag{3.5}$$

Conversely, let μ be a fuzzy ideal of X satisfying the inequality

$$\mu(x * y^n) \geq \mu(x * y^{n+1}) \quad \forall x, y \in X. \tag{3.6}$$

Then

$$\mu(x * y^n) \geq \mu(x * y^{n+1}) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\} \quad \forall x, y, z \in X. \tag{3.7}$$

Hence μ is an n-fold fuzzy positive implicative ideal of X. □

Corollary 3.8. *Every n-fold fuzzy positive implicative ideal μ of X satisfies the inequality $\mu(x * y^n) \geq \mu(x * y^{n+k})$ for all $x, y \in X$ and $k \in \mathbb{N}$.*

Proof. Using Proposition 3.7, the proof is straightforward by induction. □

Lemma 3.9. *Let A be a nonempty subset of X and let μ be a fuzzy set in X defined by*

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in A, \\ t_2 & \text{otherwise,} \end{cases} \tag{3.8}$$

where $t_1 > t_2$ in $[0,1]$. Then μ is a fuzzy ideal of X if and only if A is an ideal of X.

Proof. Let A be an ideal of X. Since $0 \in A$, therefore $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$. Suppose that (F2) does not hold. Then there exist $a, b \in X$ such that $\mu(a) = t_2$ and $\min\{\mu(a * b), \mu(b)\} = t_1$. Thus $\mu(a * b) = t_1 = \mu(b)$, and so $a * b \in A$ and $b \in A$. It follows from (I2) that $a \in A$ so that $\mu(a) = t_1$. This is a contradiction. Suppose that μ is a fuzzy ideal of X. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) = t_1$ and hence $0 \in A$. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. Using (F2), we get $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = t_1$ and so $\mu(x) = t_1$, that is, $x \in A$. Consequently, A is an ideal of X. □

Proposition 3.10. *Let A be a nonempty subset of X, n a positive integer, and μ a fuzzy set in X defined as follows:*

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in A, \\ t_2 & \text{otherwise,} \end{cases} \tag{3.9}$$

where $t_1 > t_2$ in $[0,1]$. Then μ is an n-fold fuzzy positive implicative ideal of X if and only if A is an n-fold positive implicative ideal of X.

Proof. Assume that μ is an n-fold fuzzy positive implicative ideal of X. Then μ is a fuzzy ideal of X. It follows from Lemma 3.9 that A is an ideal of X. Let $x, y \in X$ be such that $x * y^{n+1} \in A$. Using Proposition 3.7, we get $\mu(x * y^n) \geq \mu(x * y^{n+1}) = t_1$ and so

$\mu(x * y^n) = t_1$, that is, $x * y^n \in A$. Hence by [1, Theorem 1.5], we conclude that A is an n-fold positive implicative ideal of X. Conversely, suppose that A is an n-fold positive implicative ideal of X. Then A is an ideal of X (see [1, Proposition 1.2]). It follows from Lemma 3.9 that μ is a fuzzy ideal of X. For any $x, y \in X$, either $x * y^n \in A$ or $x * y^n \notin A$. The former induces $\mu(x * y^n) = t_1 \geq \mu(x * y^{n+1})$. In the latter, we know that $x * y^{n+1} \notin A$ by [1, Theorem 1.5]. Hence $\mu(x * y^n) = t_2 = \mu(x * y^{n+1})$. From Proposition 3.7 it follows that μ is an n-fold fuzzy positive implicative ideal of X. □

Proposition 3.11. *A fuzzy set μ in X is an n-fold fuzzy positive implicative ideal of X if and only if it satisfies*

(F1) $\mu(0) \geq \mu(x)$,

(F5) $\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y * z^n)\}$, *for all $x, y, z \in X$.*

Proof. Suppose that μ is an n-fold fuzzy positive implicative ideal of X and let $x, y, z \in X$. Then μ is a fuzzy ideal of X (see Proposition 3.4), and so μ is order reversing. It follows from (P3), (P4), and (P5) that

$$\mu((x * z^{2n}) * (y * z^n)) = \mu(((x * z^n) * (y * z^n)) * z^n) \geq \mu((x * y) * z^n). \tag{3.10}$$

Using (F2) and Corollary 3.8, we get

$$\begin{aligned} \mu(x * z^n) \geq \mu(x * z^{2n}) &\geq \min\{\mu((x * z^{2n}) * (y * z^n)), \mu(y * z^n)\} \\ &\geq \min\{\mu((x * y) * z^n), \mu(y * z^n)\}, \end{aligned} \tag{3.11}$$

which proves (F5). Conversely, assume that μ satisfies conditions (F1) and (F5). Taking $z = 0$ in (F5) and using (P1), we conclude that

$$\begin{aligned} \mu(x) = \mu(x * 0) &\geq \min\{\mu((x * y) * 0^n), \mu(y * 0^n)\} \\ &= \min\{\mu(x * y), \mu(y)\}. \end{aligned} \tag{3.12}$$

Hence μ is a fuzzy ideal of X. Putting $z = y$ in (F5) and applying (III), (IV), and (F1), we have

$$\begin{aligned} \mu(x * y^n) &\geq \min\{\mu((x * y) * y^n), \mu(y * y^n)\} \\ &= \min\{\mu(x * y^{n+1}), \mu(0)\} = \mu(x * y^{n+1}). \end{aligned} \tag{3.13}$$

By Proposition 3.7, we know that μ is an n-fold fuzzy positive implicative ideal of X. □

Now we give a condition for a fuzzy ideal to be an n-fold fuzzy positive implicative ideal.

Theorem 3.12. *A fuzzy set μ in X is an n-fold fuzzy positive implicative ideal of X if and only if μ is a fuzzy ideal of X in which the following inequality holds:*

(F6) $\mu((x * z^n) * (y * z^n)) \geq \mu((x * y) * z^n)$ *for all $x, y, z \in X$.*

Proof. Assume that μ is an n-fold fuzzy positive implicative ideal of X. By Proposition 3.4, it follows that μ is a fuzzy ideal of X. Let $a = x * (y * z^n)$ and $b = x * y$. Then

$$\begin{aligned} \mu((a * b) * z^n) &= \mu(((x * (y * z^n)) * (x * y)) * z^n) \\ &\geq \mu((y * (y * z^n)) * z^n) = \mu(0), \end{aligned} \tag{3.14}$$

and so $\mu((a*b)*z^n)=\mu(0)$. Using (F5) we obtain

$$\begin{aligned}\mu((x*z^n)*(y*z^n)) &= \mu((x*(y*z^n))*z^n)=\mu(a*z^n)\\ &\geq \min\{\mu((a*b)*z^n),\mu(b*z^n)\}\\ &= \min\{\mu(0),\mu(b*z^n)\}\\ &= \mu(b*z^n)=\mu((x*y)*z^n),\end{aligned} \tag{3.15}$$

which is condition (F6). Conversely, let μ be a fuzzy ideal of X satisfying condition (F6). It is sufficient to show that μ satisfies condition (F5). For any $x,y,z\in X$ we have

$$\begin{aligned}\mu(x*z^n) &\geq \min\{\mu((x*z^n)*(y*z^n)),\mu(y*z^n)\}\\ &\geq \min\{\mu((x*y)*z^n),\mu(y*z^n)\},\end{aligned} \tag{3.16}$$

which is precisely (F5). Hence μ is an n-fold fuzzy positive implicative ideal of X. □

Theorem 3.13. *Let μ be a fuzzy set in X and let n be a positive integer. Then μ is an n-fold fuzzy positive implicative ideal of X if and only if the nonempty level set $U(\mu;t)$ of μ is an n-fold positive implicative ideal of X for every $t\in[0,1]$.*

Proof. Assume that μ is an n-fold fuzzy positive implicative ideal of X and $U(\mu;t)\neq\emptyset$ for every $t\in[0,1]$. Then there exists $x\in U(\mu;t)$. It follows from (F1) that $\mu(0)\geq\mu(x)\geq t$ so that $0\in U(\mu;t)$. Let $x,y,z\in X$ be such that $(x*y^{n+1})*z\in U(\mu;t)$ and $z\in U(\mu;t)$. Then $\mu((x*y^{n+1})*z)\geq t$ and $\mu(z)\geq t$, which imply from (F4) that

$$\mu(x*y^n)\geq\min\left\{\mu\left(\left(x*y^{n+1}\right)*z\right),\mu(z)\right\}\geq t, \tag{3.17}$$

so that $x*y^n\in U(\mu;t)$. Therefore $U(\mu;t)$ is an n-fold positive implicative ideal of X. Conversely, suppose that $U(\mu;t)(\neq\emptyset)$ is an n-fold positive implicative ideal of X for every $t\in[0,1]$. For any $x\in X$, let $\mu(x)=t$. Then $x\in U(\mu;t)$. Since $0\in U(\mu;t)$, we get $\mu(0)\geq t=\mu(x)$ and so $\mu(0)\geq\mu(x)$ for all $x\in X$. Now assume that there exist $a,b,c\in X$ such that $\mu(a*b^n)<\min\{\mu((a*b^{n+1})*c),\mu(c)\}$. Selecting $s_0=(1/2)(\mu(a*b^n)+\min\{\mu((a*b^{n+1})*c),\mu(c)\})$, then

$$\mu(a*b^n)<s_0<\min\{\mu((a*b^{n+1})*c),\mu(c)\}. \tag{3.18}$$

It follows that $(a*b^{n+1})*c\in U(\mu;s_0)$, $c\in U(\mu;s_0)$, and $a*b^n\notin U(\mu;s_0)$. This is a contradiction. Hence μ is an n-fold fuzzy positive implicative ideal of X. □

Theorem 3.14. *If μ is an n-fold fuzzy positive implicative ideal of X, then the set*

$$X_\mu:=\{x\in X\mid\mu(x)=\mu(0)\} \tag{3.19}$$

is an n-fold positive implicative ideal of X.

Proof. Let μ be an n-fold fuzzy positive implicative ideal of X. Clearly $0\in X_\mu$. Let $x,y,z\in X$ be such that $(x*y^{n+1})*z\in X_\mu$ and $z\in X_\mu$. Then

$$\mu(x*y^n)\geq\min\{\mu((x*y^{n+1})*z),\mu(z)\}=\mu(0). \tag{3.20}$$

It follows from (F1) that $\mu(x*y^n)=\mu(0)$ so that $x*y^n\in X_\mu$. Hence X_μ is an n-fold positive implicative ideal of X. □

Theorem 3.15 (extension property for n-fold fuzzy positive implicative ideals). *Let μ and ν be fuzzy ideals of X such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If μ is an n-fold fuzzy positive implicative ideal of X, then so is ν.*

Proof. Using Proposition 3.7, it is sufficient to show that ν satisfies the inequality $\nu(x * y^n) \geq \nu(x * y^{n+1})$ for all $x, y \in X$. Let $x, y \in X$. Then

$$\begin{aligned}\nu(0) = \mu(0) &= \mu((x*(x*y^{n+1}))*y^{n+1}) \leq \mu((x*(x*y^{n+1}))*y^n) \\ &= \mu((x*y^n)*(x*y^{n+1})) \leq \nu((x*y^n)*(x*y^{n+1})).\end{aligned} \tag{3.21}$$

Since ν is a fuzzy ideal, it follows from (F1) and (F2) that

$$\begin{aligned}\nu(x*y^n) &\geq \min\{\nu((x*y^n)*(x*y^{n+1})), \nu(x*y^{n+1})\} \\ &\geq \min\{\nu(0), \nu(x*y^{n+1})\} = \nu(x*y^{n+1}).\end{aligned} \tag{3.22}$$

This completes the proof. □

4. PI^n-Noetherian BCK-algebras

Definition 4.1. A BCK-algebra X is said to satisfy the PI^n*-ascending* (resp., PI^n*-descending*) *chain condition* (briefly, PI^n-ACC (resp., PI^n-DCC)) if for every ascending (resp., descending) sequence $A_1 \subseteq A_2 \subseteq \cdots$ (resp., $A_1 \supseteq A_2 \supseteq \cdots$) of n-fold positive implicative ideals of X there exists a natural number r such that $A_r = A_k$ for all $r \geq k$. If X satisfies the PI^n-ACC, we say that X is a PI^n*-Noetherian BCK-algebra.*

Theorem 4.2. *Let $\{A_k \mid k \in \mathbb{N}\}$ be a family of n-fold positive implicative ideals of X which is nested, that is, $A_1 \supsetneqq A_2 \supsetneqq \cdots$. Let μ be a fuzzy set in X defined by*

$$\mu(x) = \begin{cases} \dfrac{k}{k+1} & \text{if } x \in A_k \setminus A_{k+1},\ k = 0,1,2,\ldots, \\ 1 & \text{if } x \in \cap_{k=0}^{\infty} A_k, \end{cases} \tag{4.1}$$

for all $x \in X$, where A_0 stands for X. Then μ is an n-fold fuzzy positive implicative ideal of X.

Proof. Clearly $\mu(0) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Suppose that

$$(x*y^{n+1})*z \in A_k \setminus A_{k+1}, \quad z \in A_r \setminus A_{r+1} \tag{4.2}$$

for $k = 0,1,2,\ldots;\ r = 0,1,2,\ldots$. Without loss of generality, we may assume that $k \leq r$. Then obviously $z \in A_k$. Since A_k is an n-fold positive implicative ideal, it follows that $x * y^n \in A_k$ so that

$$\mu(x*y^n) \geq \frac{k}{k+1} = \min\{\mu((x*y^{n+1})*z), \mu(z)\}. \tag{4.3}$$

If $(x*y^{n+1})*z \in \cap_{k=0}^{\infty} A_k$ and $z \in \cap_{k=0}^{\infty} A_k$, then $x*y^n \in \cap_{k=0}^{\infty} A_k$. Hence

$$\mu(x*y^n) = 1 = \min\{\mu((x*y^{n+1})*z), \mu(z)\}. \tag{4.4}$$

If $(x * y^{n+1}) * z \notin \cap_{k=0}^{\infty} A_k$ and $z \in \cap_{k=0}^{\infty} A_k$, then there exists $i \in \mathbb{N}$ such that $(x * y^{n+1}) * z \in A_i \setminus A_{i+1}$. It follows that $x * y^n \in A_i$ so that

$$\mu(x * y^n) \geq \frac{i}{i+1} = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.5}$$

Finally, assume that $(x * y^{n+1}) * z \in \cap_{k=0}^{\infty} A_k$ and $z \notin \cap_{k=0}^{\infty} A_k$. Then $z \in A_j \setminus A_{j+1}$ for some $j \in \mathbb{N}$. Hence $x * y^n \in A_j$, and thus

$$\mu(x * y^n) \geq \frac{j}{j+1} = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.6}$$

Consequently, μ is an n-fold fuzzy positive implicative ideal of X. □

Theorem 4.2 tells that if every n-fold fuzzy positive implicative ideal of X has a finite number of values, then X satisfies the PI^n-DCC.

Now we consider the converse of Theorem 4.2.

Theorem 4.3. *Let X be a BCK-algebra satisfying PI^n-DCC and let μ be an n-fold fuzzy positive implicative ideal of X. If a sequence of elements of $\mathrm{Im}(\mu)$ is strictly increasing, then μ has a finite number of values.*

Proof. Let $\{t_k\}$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mu)$. Hence $0 \leq t_1 < t_2 < \cdots \leq 1$. Then $U(\mu; r) := \{x \in X \mid \mu(x) \geq t_r\}$ is an n-fold positive implicative ideal of X for all $r = 2, 3, \ldots$. Let $x \in U(\mu; r)$. Then $\mu(x) \geq t_r \geq t_{r-1}$, and so $x \in U(\mu; r-1)$. Hence $U(\mu; r) \subseteq U(\mu; r-1)$. Since $t_{r-1} \in \mathrm{Im}(\mu)$, there exists $x_{r-1} \in X$ such that $\mu(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in U(\mu; r-1)$, but $x_{r-1} \notin U(\mu; r)$. Thus $U(\mu; r) \subsetneq U(\mu; r-1)$, and so we obtain a strictly descending sequence

$$U(\mu; 1) \supsetneq U(\mu; 2) \supsetneq U(\mu; 3) \supsetneq \cdots \tag{4.7}$$

of n-fold positive implicative ideals of X which is not terminating. This contradicts the assumption that X satisfies the PI^n-DCC. Consequently, μ has a finite number of values. □

Theorem 4.4. *The following are equivalent.*

(i) *X is a PI^n-Noetherian BCK-algebra.*

(ii) *The set of values of any n-fold fuzzy positive implicative ideal of X is a well-ordered subset of $[0, 1]$.*

Proof. (i)⇒(ii). Let μ be an n-fold fuzzy positive implicative ideal of X. Assume that the set of values of μ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{t_k\}$ such that $\mu(x_k) = t_k$. It follows that

$$U(\mu; 1) \subsetneq U(\mu; 2) \subsetneq U(\mu; 3) \subsetneq \cdots \tag{4.8}$$

is a strictly ascending chain of n-fold positive implicative ideals of X, where $U(\mu; r) = \{x \in X \mid \mu(x) \geq t_r\}$ for every $r = 1, 2, \ldots$. This contradicts the assumption that X is PI^n-Noetherian.

(ii)⇒(i). Assume that condition (i) is satisfied and X is not PI^n-Noetherian. Then there exists a strictly ascending chain

$$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \tag{4.9}$$

of n-fold positive implicative ideals of X. Let $A = \cup_{k\in\mathbb{N}} A_k$. Then A is an n-fold positive implicative ideal of X. Define a fuzzy set ν in X by

$$\nu(x) := \begin{cases} 0 & \text{if } x \notin A_k, \\ \dfrac{1}{r} & \text{where } r = \min\{k \in \mathbb{N} \mid x \in A_k\}. \end{cases} \tag{4.10}$$

We claim that ν is an n-fold fuzzy positive implicative ideal of X. Since $0 \in A_k$ for all $k = 1,2,\ldots$, we have $\nu(0) = 1 \geq \nu(x)$ for all $x \in X$. Let $x,y,z \in X$. If $(x * y^{n+1}) * z \in A_k \setminus A_{k-1}$ and $z \in A_k \setminus A_{k-1}$ for $k = 2,3,\ldots$, then $x * y^n \in A_k$. It follows that

$$\nu(x * y^n) \geq \frac{1}{k} = \min\{\nu((x * y^{n+1}) * z), \nu(z)\}. \tag{4.11}$$

Suppose that $(x * y^{n+1}) * z \in A_k$ and $z \in A_k \setminus A_r$ for all $r < k$. Since A_k is an n-fold positive implicative ideal, it follows that $x * y^n \in A_k$. Hence

$$\nu(x * y^n) \geq \frac{1}{k} \geq \frac{1}{r+1} \geq \nu(z), \quad \nu(x * y^n) \geq \min\{\nu((x * y^{n+1}) * z), \nu(z)\}. \tag{4.12}$$

Similarly for the case $(x * y^{n+1}) * z \in A_k \setminus A_r$ and $z \in A_k$, we have

$$\nu(x * y^n) \geq \min\{\nu((x * y^{n+1}) * z), \nu(z)\}. \tag{4.13}$$

Thus ν is an n-fold fuzzy positive implicative ideal of X. Since the chain (4.9) is not terminating, ν has a strictly descending sequence of values. This contradicts the assumption that the value set of any n-fold fuzzy positive implicative ideal is well ordered. Therefore X is PI^n-Noetherian. This completes the proof. □

We note that a set is well ordered if and only if it does not contain any infinite descending sequence.

Theorem 4.5. *Let $S = \{t_k \mid k = 1,2,\ldots\} \cup \{0\}$ where $\{t_k\}$ is a strictly descending sequence in $(0,1)$. Then a BCK-algebra X is PI^n-Noetherian if and only if for each n-fold fuzzy positive implicative ideal μ of X, $\mathrm{Im}(\mu) \subseteq S$ implies that there exists a natural number k such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.*

Proof. Assume that X is a PI^n-Noetherian BCK-algebra and let μ be an n-fold fuzzy positive implicative ideal of X. Then by Theorem 4.4 we know that $\mathrm{Im}(\mu)$ is a well-ordered subset of $[0,1]$ and so the condition is necessary.

Conversely, suppose that the condition is satisfied. Assume that X is not PI^n-Noetherian. Then there exists a strictly ascending chain of n-fold positive implicative ideals

$$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots. \tag{4.14}$$

Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in A_1, \\ t_k & \text{if } x \in A_k \setminus A_{k-1},\ k = 2,3,\ldots, \\ 0 & \text{if } x \in X \setminus \cup_{k=1}^{\infty} A_k. \end{cases} \tag{4.15}$$

Since $0 \in A_1$, we have $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$. If either $(x * y^{n+1}) * z$ or z belongs to $X \setminus \cup_{k=1}^{\infty} A_k$, then either $\mu((x * y^{n+1}) * z)$ or $\mu(z)$ is equal to 0 and hence

$$\mu(x * y^n) \geq 0 = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.16}$$

If $(x * y^{n+1}) * z \in A_1$ and $z \in A_1$, then $x * y^n \in A_1$ and thus

$$\mu(x * y^n) = t_1 = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.17}$$

If $(x * y^{n+1}) * z \in A_k \setminus A_{k-1}$ and $z \in A_k \setminus A_{k-1}$, then $x * y^n \in A_k$. Hence

$$\mu(x * y^n) \geq t_k = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.18}$$

Assume that $(x * y^{n+1}) * z \in A_1$ and $z \in A_k \setminus A_{k-1}$ for $k = 2, 3, \dots$. Then $x * y^n \in A_k$ and therefore

$$\mu(x * y^n) \geq t_k = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.19}$$

Similarly for $(x * y^{n+1}) * z \in A_k \setminus A_{k-1}$ and $z \in A_1$, $k = 2, 3, \dots$, we obtain

$$\mu(x * y^n) \geq t_k = \min\{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.20}$$

Consequently, μ is an n-fold fuzzy positive implicative ideal of X. This contradicts our assumption. □

5. Normalizations of n-fold fuzzy positive implicative ideals

Definition 5.1. An n-fold fuzzy positive implicative ideal μ of X is said to be *normal* if there exists $x \in X$ such that $\mu(x) = 1$.

Example 5.2. Let $= \{0, a, b\}$ be a BCK-algebra in Example 3.3. Then the fuzzy set μ in X defined by $\mu(0) = 1$, $\mu(a) = 0.8$, and $\mu(b) = 0.5$ is a normal n-fold fuzzy positive implicative ideal of X.

Note that if μ is a normal n-fold fuzzy positive implicative ideal of X, then clearly $\mu(0) = 1$, and hence μ is normal if and only if $\mu(0) = 1$.

Proposition 5.3. *Given an n-fold fuzzy positive implicative ideal μ of X let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in X$. Then μ^+ is a normal n-fold fuzzy positive implicative ideal of X which contains μ.*

Proof. We have $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \geq \mu(x)$ for all $x \in X$. For any $x, y, z \in X$, we have

$$\begin{aligned}
&\min\{\mu^+((x * y^{n+1}) * z), \mu^+(z)\} \\
&\quad = \min\{\mu((x * y^{n+1}) * z) + 1 - \mu(0), \mu(z) + 1 - \mu(0)\} \\
&\quad = \min\{\mu((x * y^{n+1}) * z), \mu(z)\} + 1 - \mu(0) \\
&\quad \leq \mu(x * y^n) + 1 - \mu(0) = \mu^+(x * y^n).
\end{aligned} \tag{5.1}$$

Hence μ^+ is a normal n-fold fuzzy positive implicative ideal of X, and obviously $\mu \subseteq \mu^+$. □

Noticing that $\mu \subseteq \mu^+$, we have the following corollary.

Corollary 5.4. *If there is $x \in X$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.*

Using Proposition 3.10, we know that for any n-fold positive implicative ideal A of X, the characteristic function χ_A of A is a normal n-fold fuzzy positive implicative ideal of X. It is clear that μ is a normal n-fold fuzzy positive implicative ideal of X if and only if $\mu^+ = \mu$.

Proposition 5.5. *If μ is an n-fold fuzzy positive implicative ideal of X, then $(\mu^+)^+ = \mu^+$.*

Proof. The proof is straightforward. □

Corollary 5.6. *If μ is a normal n-fold fuzzy positive implicative ideal of X, then $(\mu^+)^+ = \mu$.*

Proposition 5.7. *Let μ and ν be n-fold fuzzy positive implicative ideals of X. If $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$, then $X_\mu \subseteq X_\nu$.*

Proof. If $x \in X_\mu$, then $\nu(x) \geq \mu(x) = \mu(0) = \nu(0)$ and so $\nu(x) = \nu(0)$, that is, $x \in X_\nu$. Therefore $X_\mu \subseteq X_\nu$. □

Proposition 5.8. *Let μ be an n-fold fuzzy positive implicative ideal of X. If there is an n-fold fuzzy positive implicative ideal ν of X satisfying $\nu^+ \subseteq \mu$, then μ is normal.*

Proof. Assume that there is an n-fold fuzzy positive implicative ideal ν of X such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(0) \leq \mu(0)$, and so $\mu(0) = 1$. Hence μ is normal. □

Given an n-fold fuzzy positive implicative ideal, we construct a new normal n-fold fuzzy positive implicative ideal.

Theorem 5.9. *Let μ be an n-fold fuzzy positive implicative ideal of X and let $f : [0,\mu(0)] \to [0,1]$ be an increasing function. Let $\mu_f : X \to [0,1]$ be a fuzzy set in X defined by $\mu_f(x) = f(\mu(x))$ for all $x \in X$. Then μ_f is an n-fold fuzzy positive implicative ideal of X. In particular, if $f(\mu(0)) = 1$ then μ_f is normal; and if $f(t) \geq t$ for all $t \in [0,\mu(0)]$, then $\mu \subseteq \mu_f$.*

Proof. Since $\mu(0) \geq \mu(x)$ for all $x \in X$ and since f is increasing, we have $\mu_f(0) = f(\mu(0)) \geq f(\mu(x)) = \mu_f(x)$ for all $x \in X$. For any $x, y, z \in X$ we get

$$\begin{aligned} \min\{\mu_f((x*y^{n+1})*z), \mu_f(z)\} &= \min\{f(\mu((x*y^{n+1})*z)), f(\mu(z))\} \\ &= f(\min\{\mu((x*y^{n+1})*z), \mu(z)\}) \leq f(\mu(x*y^n)) = \mu_f(x*y^n). \end{aligned} \tag{5.2}$$

Hence μ_f is an n-fold fuzzy positive implicative ideal of X. If $f(\mu(0)) = 1$, then clearly μ_f is normal. Assume that $f(t) \geq t$ for all $t \in [0,\mu(0)]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves $\mu \subseteq \mu_f$. □

Let $\mathcal{N}(X)$ denote the set of all normal n-fold fuzzy positive implicative ideals of X.

Theorem 5.10. *Let $\mu \in \mathcal{N}(X)$ be nonconstant such that it is a maximal element of the poset $(\mathcal{N}(X), \subseteq)$. Then μ takes only the values 0 and 1.*

Proof. Since μ is normal, we have $\mu(0) = 1$. Let $x \in X$ be such that $\mu(x) \neq 1$. It is sufficient to show that $\mu(x) = 0$. If not, then there exists $a \in X$ such that $0 < \mu(a) < 1$. Define a fuzzy set ν in X by $\nu(x) = (1/2)\{\mu(x)+\mu(a)\}$ for all $x \in X$. Clearly, ν is well defined, and we get

$$\nu(0) = \frac{1}{2}\{\mu(0)+\mu(a)\} = \frac{1}{2}\{1+\mu(a)\} \geq \frac{1}{2}\{\mu(x)+\mu(a)\} = \nu(x) \quad \forall x \in X. \tag{5.3}$$

Let $x, y, z \in X$. Then

$$\begin{aligned} \nu(x * y^n) &= \frac{1}{2}\{\mu(x*y^n)+\mu(a)\} \geq \frac{1}{2}\{\min\{\mu((x*y^{n+1})*z),\mu(z)\}+\mu(a)\} \\ &= \min\left\{\frac{1}{2}\{\mu((x*y^{n+1})*z)+\mu(a)\}, \frac{1}{2}\{\mu(z)+\mu(a)\}\right\} \\ &= \min\{\nu((x*y^{n+1})*z),\nu(z)\}. \end{aligned} \tag{5.4}$$

Hence ν is an n-fold fuzzy positive implicative ideal of X. By Proposition 5.3, ν^+ is a maximal n-fold fuzzy positive implicative ideal of X, where ν^+ is defined by $\nu^+(x) = \nu(x)+1-\nu(0)$ for all $x \in X$. Note that

$$\begin{aligned} \nu^+(a) = \nu(a)+1-\nu(0) &= \frac{1}{2}\{\mu(a)+\mu(a)\}+1-\frac{1}{2}\{\mu(0)+\mu(a)\} \\ &= \frac{1}{2}\{\mu(a)+1\} > \mu(a) \end{aligned} \tag{5.5}$$

and $\nu^+(a) < 1 = \nu^+(0)$. It follows that ν^+ is nonconstant, and μ is not a maximal element of $(\mathcal{N}(X), \subseteq)$. This is a contradiction. □

Definition 5.11. An n-fold fuzzy positive implicative ideal μ of X is said to be *fuzzy maximal* if μ is nonconstant and μ^+ is a maximal element of the poset $(\mathcal{N}(X), \subseteq)$.

For any positive implicative ideal I of X let μ_I be a fuzzy set in X defined by

$$\mu_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Theorem 5.12. *Let μ be an n-fold fuzzy positive implicative ideal of X. If μ is fuzzy maximal, then*

(i) *μ is normal,*

(ii) *μ takes only the values 0 and 1,*

(ii) *$\mu_{X_\mu} = \mu$,*

(iv) *X_μ is a maximal n-fold positive implicative ideal of X.*

Proof. Let μ be an n-fold fuzzy positive implicative ideal of X which is fuzzy maximal. Then μ^+ is a nonconstant maximal element of the poset $(\mathcal{N}(X), \subseteq)$. It follows from Theorem 5.10 that μ^+ takes only the values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$, and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$. By Corollary 5.4, we have $\mu(x) = 0$, and so $\mu(0) = 1$. Hence μ is normal and $\mu^+ = \mu$. This proves (i) and (ii).

(iii) Obviously $\mu_{X_\mu} \subset \mu$ and μ_{X_μ} takes only the values 0 and 1. Let $x \in X$. If $\mu(x) = 0$, then $\mu \subseteq \mu_{X_\mu}$. If $\mu(x) = 1$, then $x \in X_\mu$ and so $\mu_{X_\mu}(x) = 1$. This shows that $\mu \subseteq \mu_{X_\mu}$.

(iv) Since μ is nonconstant, X_μ is a proper n-fold positive implicative ideal of X. Let J be an n-fold positive implicative ideal of X containing X_μ. Then $\mu = \mu_{X_\mu} \subseteq \mu_J$. Since μ and μ_J are normal n-fold fuzzy positive implicative ideals of X and since $\mu = \mu^+$ is a maximal element of $\mathcal{N}(X)$, we have that either $\mu = \mu_J$ or $\mu_J = \mathbf{1}$ where $\mathbf{1} : X \to [0,1]$ is a fuzzy set defined by $\mathbf{1}(x) = 1$ for all $x \in X$. The later case implies that $J = X$. If $\mu = \mu_J$, then $X_\mu = X_{\mu_J} = J$. This shows that X_μ is a maximal n-fold positive implicative ideal of X. This completes the proof. □

Acknowledgement. The first author was supported by Korea Research Foundation Grant (KRF-2000-005-D00003).

References

[1] Y. Huang and Z. Chen, *On ideals in BCK-algebras*, Math. Japon. **50** (1999), no. 2, 211–226. CMP 1 718 851. Zbl 938.06018.

[2] Y. B. Jun, S. M. Hong, J. Meng, and X. L. Xin, *Characterizations of fuzzy positive implicative ideals in BCK-algebras*, Math. Japon. **40** (1994), no. 3, 503–507. CMP 1 305 546. Zbl 814.06012.

[3] J. Meng, *On ideals in BCK-algebras*, Math. Japon. **40** (1994), no. 1, 143–154. MR 95e:06049. Zbl 807.06012.

Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea

E-mail address: ybjun@nongae.gsnu.ac.kr

Kyung Ho Kim: Department of Mathematics, Chungju National University, Chungju 380-702, Korea

E-mail address: ghkim@gukwon.chungju.ac.kr