

ON THE SHARP CONSTANT FOR STARLIKENESS

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ABSTRACT. We obtain a sharp constant of the sufficient condition for p -valently starlikeness, which had been studied by Nunokawa (1991), Obradović and Owa (1989), and Li (1993).

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1. Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N}) \quad (1.1)$$

which are analytic in $U = \{z : |z| < 1\}$. A function $f(z)$ in $A(p)$ is said to be p -valently starlike if and only if

$$\Re \left[\frac{zf'(z)}{f(z)} \right] > 0 \quad \text{in } U. \quad (1.2)$$

Let $S(p)$ denote the subclass of $A(p)$ consisting of all functions $f(z)$ which are p -valently starlike in U (cf. [1]). For a function $g(z)$ in $A(p)$, the interesting problem is to find the best constant A such that $g(z)$ is in $S(p)$ whenever

$$\left| 1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)} \right| < A \left| \frac{zg^{(p)}(z)}{g^{(p-1)}(z)} \right| \quad \text{in } U. \quad (1.3)$$

In 1989, Obradović and Owa [6] obtained that $A = 5/4$ for the case of $p = 1$. For the general case, Nunokawa [5] gained that $A = \log 4$. Recently, Li [2] improved these results and obtained that $A = 3/2$. In this paper, we will solve this problem completely and give the sharp constant $A = 1.80898\dots$, where A is the unique solution of the equation

$$xe^{1/(x^2-1)} = x + 1. \quad (1.4)$$

For proving our result, we should recall the concept of subordination between analytic functions. Given two analytic functions $f(z)$ and $F(z)$, the function $f(z)$ is said to be subordinate to $F(z)$ if $F(z)$ is univalent in U , $f(0) = F(0)$, and $f(U) \subset F(U)$. We denote this subordination by $f(z) \prec F(z)$ (see [7]).

Suppose that $h(z)$ is analytic in U , and that $\Phi(z)$ is analytic in an appropriate domain D , we consider the following first-order differential subordination

$$\beta + zp'(z)\Phi(p(z)) \prec h(z), \quad (1.5)$$

where $p(z)$ is analytic in \mathbf{U} , β is a complex constant. Changing the “ $<$ ” of (1.5) to “ $=$ ”, we get the corresponding first-order differential equation

$$\beta + zp'(z)\Phi(p(z)) = h(z). \tag{1.6}$$

2. Main results. Our results rest on the following lemma, which is the special case of [3, Theorem 3].

LEMMA 2.1. *Suppose that $h(z)$ is a starlike function in \mathbf{U} , $\Phi(z)$ is analytic in the domain \mathbf{D} and $p(z), q(z)$ are two analytic functions in \mathbf{U} . If $p(z)$ satisfies the relation (1.5), $q(z)$ is a univalent solution of the corresponding equation (1.6) and $p(0) = q(0)$, then $p(z) < q(z)$.*

THEOREM 2.2. *Let $g(z) \in A(p)$, and suppose that*

$$\left| 1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)} \right| < A \left| \frac{zg^{(p)}(z)}{g^{(p-1)}(z)} \right| \quad \text{in } \mathbf{U}, \tag{2.1}$$

where the constant A is given by (1.4). Then $g(z) \in S(p)$ and the result is sharp.

PROOF. Let

$$f(z) = \frac{g^{(p-1)}(z)}{p!}. \tag{2.2}$$

Then $f(z) \in A(1)$. From the assumption (2.1), $f(z)$ satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < A \left| \frac{zf'(z)}{f(z)} \right| \quad \text{in } \mathbf{U}. \tag{2.3}$$

By putting $p(z) = zf'(z)/f(z)$, equation (2.3) can be rewritten as

$$\left| 1 + \frac{zp'(z)}{p^2(z)} \right| < A. \tag{2.4}$$

Let $\varphi(z) = A(1 + Az)/(A + z)$ for $z \in \mathbf{U}$. Obviously $\varphi(z)$ is a conformal mapping from \mathbf{U} to $\Omega = \{w : |w| < A\}$ and $\varphi(0) = 1$. Combining (2.4) with the definition of subordination, we obtain

$$1 + \frac{zp'(z)}{p^2(z)} < \frac{A(1 + Az)}{A + z}. \tag{2.5}$$

Setting

$$q(z) = \frac{1}{1 + (A^2 - 1)\log A/(A + z)}, \tag{2.6}$$

we have

$$1 + \frac{zq'(z)}{q^2(z)} = \frac{A(1 + Az)}{A + z} \tag{2.7}$$

and $p(0) = q(0) = 1$. As $A > 1$, we can choose a uniform analytic branch of $\log(A + z)$ such that $q(z)$ is univalent on this branch. By taking the real part of the denominator of $q(z)$ and combining (1.4), we conclude that

$$\Re \left[1 + (A^2 - 1)\log \frac{A}{A + z} \right] > 1 + (A^2 - 1)\log \frac{A}{A + 1} = 0. \tag{2.8}$$

It follows that $\Re[q(z)] > 0$, so $q(z)$ is analytic and univalent. Let $\mathbf{D} = \mathbb{C} \setminus \{0\}$, $\Phi(z) = 1/z^2$, $\beta = 1$, and $h(z) = A(1 + Az)/(A + z)$, where \mathbb{C} is the complex plane. It is clear that $h(z)$ is a starlike function. From [Lemma 2.1](#), we deduce that $p(z) < q(z)$. Hence

$$\Re \left[\frac{zf'(z)}{f(z)} \right] = \Re[p(z)] \geq \min_{|z|=r<1} \Re[q(z)] > 0. \tag{2.9}$$

This is equivalent to

$$\Re \left[\frac{zg^{(p)}(z)}{g^{(p-1)}(z)} \right] = \Re \left[\frac{zf'(z)}{f(z)} \right] > 0 \quad \text{in } \mathbf{U}. \tag{2.10}$$

From [\[4, Theorem 5\]](#), we have

$$\Re \left[\frac{zg'(z)}{g(z)} \right] > 0 \quad \text{in } \mathbf{U}. \tag{2.11}$$

This proves $g(z) \in S(p)$.

For any $A_1 > A = 1.80898\dots$, we get a function $q_1(z)$ by replacing A in [\(2.6\)](#) with A_1 and choosing an appropriate branch of $\log(A_1 + z)$. We can easily observe that the real part of $q_1(z)$ is not always positive. Through the relations $q_1(z) = zf'(z)/f(z)$ and $f(z) = g^{(p-1)}(z)/p!$, we can construct an analytic function $g(z)$ which belongs to $A(p)$ and satisfies [\(2.1\)](#), but it is not in $S(p)$. This completes the proof. \square

Taking $p = 1$ in [Theorem 2.2](#), we easily have the following corollary.

COROLLARY 2.3. *If $f(z) \in A(1)$ and it satisfies the condition*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < A \left| \frac{zf'(z)}{f(z)} \right| \quad \text{in } \mathbf{U}, \tag{2.12}$$

where the constant A is given by [\(1.4\)](#), then $f(z)$ is univalent and starlike in \mathbf{U} .

The problem that Nunokawa proposed in [\[5\]](#) has been solved completely, but the converse proposition of [Theorem 2.2](#) is not true. We find a simple example $f(z) = z/(1 - z)$ which belongs to $S(1)$, but it does not satisfy [\(2.12\)](#). The following theorem is better than [\(2.1\)](#) because it includes at least this example.

THEOREM 2.4. *Let $g(z) \in A(p)$, and suppose that*

$$\left| 1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)} - \frac{zg^{(p)}(z)}{g^{(p-1)}(z)} \right| < \left| \frac{zg^{(p)}(z)}{g^{(p-1)}(z)} \right| \quad \text{in } \mathbf{U}. \tag{2.13}$$

Then $g(z) \in S(p)$.

PROOF. Let

$$f(z) = \frac{g^{(p-1)}(z)}{p!}. \tag{2.14}$$

Then $f(z) \in A(1)$. From the assumption [\(2.13\)](#), $f(z)$ satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \left| \frac{zf'(z)}{f(z)} \right| \quad \text{in } \mathbf{U}. \tag{2.15}$$

By setting $p(z) = zf'(z)/f(z)$, equation (2.15) can be rewritten as

$$\left| \frac{zp'(z)}{p^2(z)} \right| < 1. \quad (2.16)$$

From the definition of subordination, we obtain

$$\frac{zp'(z)}{p^2(z)} < z. \quad (2.17)$$

Let $q(z) = 1/(1-z)$, we observe that $zq'(z)/q^2(z) = z$, $p(0) = q(0) = 1$, and $\Re[q(z)] > 0$. From Lemma 2.1, we know that $p(z) < 1/(1-z)$. Therefore

$$\Re \left[\frac{zf'(z)}{f(z)} \right] = \Re[p(z)] \geq \min_{|z|=r < 1} \Re[q(z)] > 0. \quad (2.18)$$

This is equivalent to

$$\Re \left[\frac{zg^{(p)}(z)}{g^{(p-1)}(z)} \right] = \Re \left[\frac{zf'(z)}{f(z)} \right] > 0 \quad \text{in } \mathbf{U}. \quad (2.19)$$

From [4, Theorem 5], we have

$$\Re \left[\frac{zg'(z)}{g(z)} \right] > 0 \quad \text{in } \mathbf{U}. \quad (2.20)$$

This completes the proof. \square

Taking $p = 1$ in Theorem 2.4, we obviously have the following corollary.

COROLLARY 2.5. *If $f(z) \in A(1)$ and it satisfies the condition*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \left| \frac{zf'(z)}{f(z)} \right| \quad \text{in } \mathbf{U}, \quad (2.21)$$

then $f(z) \in S(1)$.

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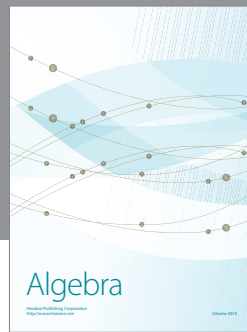
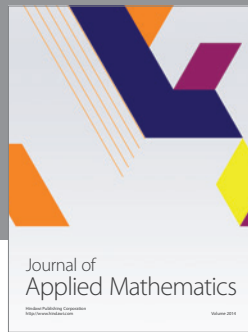
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