Research Article

On the Nusselt Solution of a Nonisothermal Two-Fluid Inclined Film Flow

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Nonisothermal viscous two-fluid flows occur in numerous kinds of coating devices. The corresponding mathematical models often represent two-dimensional free boundary value problems for the Navier-Stokes equations or their modifications. In the present paper we are concerned with a particular problem of coupled heat and mass transfer. Marangoni convection is incorporated, too. The solvability of a corresponding stationary problem is discussed. The obtained results generalize previous results for a similar isothermal problem.

1. Introduction

Thermocapillary convection describes a fluid motion driven by surface-tension gradients on a liquid-liquid interface, where these gradients arise from the temperature dependence of interface tension. This type of convection is quite important in several technological and scientific applications; interesting examples may be found in the field of materials science, particularly in coating and solidification processes or in crystal-growth processes (cf. [1–5]).

In this paper we study a problem for a 2D stationary flow with two viscous incompressible heat-conducting fluids (having kinematic viscosities $\nu_i > 0$, densities $\varrho_i > 0$, and thermal conductivities λ_i, $i = 1, 2$) down an inclined bottom S_0 having a slope α (cf. Figure 1). In fact, the bottom S_0 represents a perturbed plane. Assume that the bottom is given by the formula $S_0 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 = \varepsilon\varphi_0(x_1), \ -\infty < x_1 < +\infty\}$ with φ_0 having a compact support, that is, $\varphi_0(x_1) \equiv 0$ for $|x_1| \geq R_0 > 0$, and suppose that the direction $\mathbf{e}_g$ of the gravitational force is the vector $\mathbf{e}_g = (\sin\alpha, -\cos\alpha)^T$ which makes (with respect to the chosen coordinate system) an angle $\alpha^* := \pi/2 - \alpha$ ($0 < \alpha \leq \pi/2$) with the x_1-axis. Note that the corresponding problem will be formulated in dimensionless form. The concrete transition to that formulation can be found in [6].
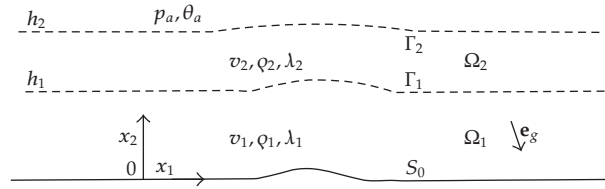
Figure 1: Flow domain of a nonisothermal two-fluid inclined film flow.

Let us formulate the problem. We consider the two-fluid flow down the inclined bottom S_0 caused by gravity $g\mathbf{e}_g$, only. This means mathematically that the positive (final) layer thickness in each liquid layer Ω_i $(i = 1, 2)$ is a priori prescribed. In slide coaters, such flows occur on some parts of the coater. The corresponding flow fields and layer profiles are essential there.

Suppose that the free interface Γ_1 separating the two fluid layers and the upper free surface Γ_2 admit the parametrizations $\Gamma_i = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \psi_i(x_1), -\infty < x_1 < +\infty\}$ $(i = 1, 2)$, where the functions ψ_i $(i = 1, 2)$ are a priori unknown and have to be found. Let $h_i > 0$ $(0 < h_1 < h_2)$ be the prescribed constant limits of $\psi_i(x_1)$ $(i = 1, 2)$, at infinity. The problem under consideration has the following form: to find a vector of velocity $\mathbf{v} = (v_1(x_1, x_2), v_2(x_1, x_2))^T$, a pressure $p(x_1, x_2)$, a temperature $\theta(x_1, x_2)$, and functions $\psi_i(x_1)$ $(i = 1, 2)$ satisfying in the domain $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 : \varepsilon\varphi_0(x_1) < x_2 < \psi_1(x_1), -\infty < x_1 < +\infty\}$ and $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^2 : \psi_1(x_1) < x_2 < \psi_2(x_1), -\infty < x_1 < +\infty\}$, then the following equations of a coupled heat and mass transfer are

$$
(\mathbf{v} \cdot \nabla)\mathbf{v} - \nu\nabla^2\mathbf{v} + \frac{1}{\varrho}\nabla p = g\mathbf{e}_g,
$$
$$
\nabla \cdot \mathbf{v} = 0, \tag{1.1}
$$
$$
(\mathbf{v} \cdot \nabla)\theta - \lambda\nabla^2\theta = 0,
$$

and the boundary and integral conditions are

$$
\mathbf{v}|_{S_0} = \mathbf{0}, \qquad \theta|_{S_0} = \theta_0, \tag{1.2}
$$

$$
[\theta]|_{\Gamma_1} = 0, \qquad \left[\lambda\frac{\partial\theta}{\partial n}\right]\Big|_{\Gamma_1} = 0, \qquad [\mathbf{v}]|_{\Gamma_1} = \mathbf{0},
$$

$$
\mathbf{v} \cdot \mathbf{n}|_{\Gamma_1^-} = 0, \quad [\boldsymbol{\tau} \cdot \mathbf{S}(\mathbf{v})\mathbf{n}]|_{\Gamma_1} = -b_1\frac{\partial\theta}{\partial\tau}\Big|_{\Gamma_1^-},
$$

$$
\frac{d}{dx_1}\frac{\psi_1'(x_1)}{\sqrt{1 + \psi_1'(x_1)^2}} = \frac{1}{\sigma_1(\theta)}\left[-p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v})\mathbf{n}\right]\Big|_{\Gamma_1}, \tag{1.3}
$$

$$
\lim_{|x_1| \to +\infty}\psi_1(x_1) = h_1,
$$

$$\left(\theta - \theta_a + \lambda_2 \frac{\partial \theta}{\partial n}\right)\Big|_{\Gamma_2} = 0,$$

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma_2} = 0, \quad \boldsymbol{\tau} \cdot \mathbf{S}(\mathbf{v})\mathbf{n}|_{\Gamma_2} = -b_2 \frac{\partial \theta}{\partial \tau}\Big|_{\Gamma_2},$$

$$\frac{d}{dx_1} \frac{\psi_2'(x_1)}{\sqrt{1 + \psi_2'(x_1)^2}} = \frac{1}{\sigma_2(\theta)} (p_a - p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v})\mathbf{n})\Big|_{\Gamma_2},$$

$$\lim_{|x_1| \to +\infty} \psi_2(x_1) = h_2.$$

(1.4)

In [1] it was shown that for a large number of liquids the surface tensions σ_i can be regarded as linear functions of the temperature θ along the free interface Γ_i $(i = 1, 2)$ (cf. also [3, 5]) as follows:

$$\sigma_i(\theta) = a_i - b_i\theta \quad (a_i, b_i > 0, \ i = 1, 2).$$

(1.5)

By λ_m we denote the thermal conductivity of the mth fluid $(m = 1, 2)$ in Problem (1.1)–(1.4). The symbol g means the acceleration of gravity. The value θ_0 is the (constant) given temperature of the wall S_0. Without loss of generality one can suppose that $\theta_0 = 0$ and that θ is in fact the difference between the physical temperature and θ_0. By p_a and θ_a we denote the given (constant) pressure and temperature of the ambient air, respectively.

Furthermore, the subsequent notations have been used: $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit vectors normal and tangential to Γ_1 and oriented as x_2, x_1, respectively. By $\mathbf{a} \cdot \mathbf{b}$ we mean the inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^T$ is the gradient operator, $\nabla p = \text{grad } p$, $\nabla \cdot \mathbf{v} = \text{div } \mathbf{v}$, $\varrho|_{\Omega_m} = \varrho_m$ $(m = 1, 2)$ is the restriction of ϱ to Ω_m (analogously for v and λ) and ∇^2 denotes the Laplace operator. By $\mathbf{S}(\mathbf{v})$ we denote the deviatoric stress tensor, that is, a matrix with elements $S_{ij}(\mathbf{v}) = \varrho v(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ $(i, j = 1, 2)$. The symbol $[w]|_{\Gamma_1}$ denotes the jump of w crossing the free interface Γ_1, that is,

$$[w(\mathbf{x}_0)]|_{\Gamma_1} := \lim_{\mathbf{y} \to \mathbf{x}_0} w(\mathbf{y}) - \lim_{\mathbf{x} \to \mathbf{x}_0} w(\mathbf{x}), \quad (\mathbf{x}_0 \in \Gamma_1, \ \mathbf{y} \in \Omega_1, \ \mathbf{x} \in \Omega_2),$$

(1.6)

and the symbol $w|_{\Gamma_1^-}$ denotes the limit from below at the interface Γ_1; more precisely

$$w(\mathbf{x}_0)|_{\Gamma_1^-} := \lim_{\mathbf{y} \to \mathbf{x}_0} w(\mathbf{y}), \quad (\mathbf{x}_0 \in \Gamma_1, \ \mathbf{y} \in \Omega_1).$$

(1.7)

Note that the left-hand side of (1.3)$_6$ (i.e., of the sixth equation in (1.3)) is equal to the curvature $K_1(x_1)$ of Γ_1. The same is true for $K_2(x_1)$ in case of Γ_2. Furthermore, (1.3)$_5$ represents the mathematical expression of the so-called Marangoni convection.

2. General Solution Technique

Mathematical problems for the stationary flows of a viscous incompressible fluid with a free boundary were studied by many authors. Numerous references on this topic can be found,

for example, in the bibliographies of [7–10]. In the analytical investigations [1–3, 5, 11], the temperature dependence was additionally taken into account. Numerical studies of nonisothermal free boundary problems can be found in the papers [1, 6].

For free boundary problems in which the unknown flow domain is unbounded in two directions as in Problem (1.1)–(1.4), a special linearization scheme is necessary (cf. [7, 8] and others).

In order to solve such kind of problems—in [7], and independently in [12], an appropriate scheme was proposed based on a linearization of the original problem on a corresponding exact solution in the unperturbed "uniform" flow domain, say: $\Pi = \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < h_1 \vee h_1 < x_2 < h_2\}$. The main difference of this scheme from previous applied ones is that on each step of iterations the determination of $\mathbf{v}, p, \theta$ is not separated from the determination of the free boundaries Γ_i $(i = 1, 2)$ (i.e., from the determination of the functions ψ_i describing Γ_i). For Problem (1.1)–(1.4) this scheme can be illustrated by the diagram

$$\left(\mathbf{v}^0, p^0, \theta^0, \psi_1^0, \psi_2^0\right) \longrightarrow \left(\mathbf{v}^1, p^1, \theta^1, \psi_1^1, \psi_2^1\right) \longrightarrow \cdots \longrightarrow (\mathbf{v}^m, p^m, \theta^m, \psi_1^m, \psi_2^m) \longrightarrow \cdots, \quad (2.1)$$

where on each step of iterations the linearized problem is solved in the same "strip-like" domain and the functions $\mathbf{v}, p, \theta$, and ψ_i $(i = 1, 2)$ are determined simultaneously.

A significant part in deriving the correct linearization takes the determination of exact solutions of the nonlinear problems in a "uniform" (not distorted) flow domain. These exact (basic) solutions in the uniform domain Π will be calculated in the appendix. They are also important for the numerical flow simulation: they can be used as inlet boundary data in more complicated problems. In [8] the analogous isothermal problem (without any inclusion of temperature) to Problem (1.1)–(1.4) was solved by numerical methods.

3. Function Spaces

When studying Problem (1.1)–(1.4), it is useful to work with weighted Sobolev spaces. Let Π_m $(m = 1, 2)$ be the strip-like domains

$$\Pi_1 := \left\{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < h_1, \ -\infty < x_1 < +\infty\right\},$$
$$\Pi_2 := \left\{\mathbf{x} \in \mathbb{R}^2 : h_1 < x_2 < h_2, \ -\infty < x_1 < +\infty\right\}, \quad (3.1)$$

and $\Pi = \Pi_1 \cup \Pi_2$ their union. We introduce the space $W_\beta^{l,2}(\Pi)$ of functions $\mathbf{u}$ on Π with restrictions $\mathbf{u}^{(m)} = \mathbf{u}|_{\Pi_m}$ belonging to $W_\beta^{l,2}(\Pi_m)$ $(m = 1, 2)$ having the finite norms

$$\left\|\mathbf{u}^{(m)}; W_\beta^{l,2}(\Pi_m)\right\| = \left\|\mathbf{u}^{(m)} \exp\left(\beta\sqrt{1 + x_1^2}\right); W^{l,2}(\Pi_m)\right\|, \quad (m = 1, 2), \quad (3.2)$$

where $W^{l,2}(\Pi_m)$ is the usual Sobolev space. The norm in $W_\beta^{l,2}(\Pi)$ is given by

$$\left\|\mathbf{u}; W_\beta^{l,2}(\Pi)\right\| = \sum_{m=1}^{2} \left\|\mathbf{u}^{(m)} \exp\left(\beta\sqrt{1 + x_1^2}\right); W^{l,2}(\Pi_m)\right\|. \quad (3.3)$$

If $\beta > 0$, then elements of $W_\beta^{l,2}(\Pi)$ vanish exponentially as $|x_1| \to \infty$, and if $\beta < 0$, then elements $\mathbf{u} \in W_\beta^{l,2}(\Pi)$ might exponentially increase as $|x_1| \to \infty$.

The spaces $W_\beta^{l-1/2,2}(\mathbb{R})$ of functions defined on $\mathbb{R}$ can be introduced analogously. Let $S = \{x \in \Pi : x_1 \in \mathbb{R}, \ x_2 = h \in \{0, h_1, h_2\}\}$ be a line. Denote by $W_\beta^{l-1/2,2}(S)$ the spaces of traces on S of functions from $W_\beta^{l,2}(\Pi)$. Then $W_\beta^{l-1/2,2}(\mathbb{R})$ coincides with $W_\beta^{l-1/2,2}(S)$, that is, if $\mathbf{u} \in W_\beta^{l-1/2,2}(\Pi)$, then $\mathbf{u}(\cdot, h) \in W_\beta^{l-1/2,2}(\mathbb{R})$.

In the paper the spaces of scalar and vector-valued functions are not distinguished in notations. The norm for vector-valued functions is then the sum of the norms of the corresponding coordinate functions.

4. Solvability Results

Problem (1.1)–(1.4) can be handled by the same methods as in [8, 13]. Let us start with the main result about this problem.

Theorem 4.1. *Let $S_0 = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \varepsilon\varphi_0(x_1), -\infty < x_1 < +\infty\}$, $\varphi_0 \in W_\beta^{l+5/2,2}(\mathbb{R})$ with $l \geq 0$, and $\beta = |\beta_0|\sin\alpha > 0$, where α denotes the slope of the inclined bottom S_0. Assume that α is sufficiently small. Then there exist positive numbers $\widehat{\varepsilon}, \widehat{r}$ such that for every $\varepsilon \in (0, \widehat{\varepsilon})$. Problem (1.1)–(1.4) has a unique solution $(\mathbf{v}, p, \theta, \psi_1, \psi_2)^T$. The solution admits the representation*

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^0(\mathbf{x}) + \varepsilon\mathbf{u}(\mathbf{x}), \qquad p(\mathbf{x}) = p^0(\mathbf{x}) + \varepsilon q(\mathbf{x}), \qquad \theta(\mathbf{x}) = \theta^0(\mathbf{x}) + \varepsilon\vartheta(\mathbf{x}), \tag{4.1}$$

$$\psi_1(x_1) = h_1 + \varepsilon\Psi_1(x_1), \qquad \psi_2(x_1) = h_2 + \varepsilon\Psi_2(x_1), \tag{4.2}$$

where $\{\mathbf{v}^0, p^0, \theta^0\}$ are the functions of the basic solution from (A.3)–(A.6), while

$$\mathbf{U} := (\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2)^T \in \left[W_\beta^{l+2,2}(\Pi)\right]^2 \times W_\beta^{l+1,2}(\Pi) \times W_\beta^{l+2,2}(\Pi) \times \left[W_\beta^{l+5/2,2}(\mathbb{R})\right]^2$$
$$\equiv \mathfrak{D}_\beta^{l,2}W(\Pi), \tag{4.3}$$

and the following inequalities hold:

$$\left\|\mathbf{U}; \mathfrak{D}_\beta^{l,2}W(\Pi)\right\| \leq \widehat{r}, \qquad \widehat{\varepsilon} \leq \mathrm{const} \cdot \sin^2\alpha. \tag{4.4}$$

We would like to present a short sketch of the proof of this theorem by successive approximations. First, the original (perturbed) and unknown flow domain Ω (cf. Figure 1) is transformed onto the uniform (strip-like) domain Π. Then, using this transformation mapping, the original flow Problem (1.1)–(1.4) is linearized over the basic solution (A.3)–(A.6) (see the appendix) in domain Π. By $\mathfrak{N}$ we denote the operator of the left-hand side of that linearized auxiliary problem. In virtue of a corresponding theorem for the linear auxiliary problem (see, e.g., [13]), there exists a bounded inverse operator $\mathfrak{N}^{-1}$ such that

$$\mathfrak{N}^{-1} : \quad \mathcal{R}_\beta^{l,2}W(\Pi) \longmapsto \mathfrak{D}_\beta^{l,2}W(\Pi), \tag{4.5}$$

with $\beta = |\beta_0| \sin\alpha$. Herein β_0 is independent of α and depends on eigenvalues of the operator pencils associated with the corresponding linear problem (cf. [13]). Also, the multidimensional space $\mathcal{R}_\beta^{l,2}W(\Pi)$ to which the right-hand side of the linearized problem belongs is introduced in a similar way as that of the space $\mathfrak{D}_\beta^{l,2}W(\Pi)$ (see above). Moreover, one can show that there holds the estimate

$$\left\| \mathfrak{N}^{-1} \right\| \leq \frac{C}{\sin\alpha}, \tag{4.6}$$

with a constant C being independent of α. Therefore, Problem (1.1)–(1.4) is equivalent to the following operator equation in the space $\mathfrak{D}_\beta^{l,2}W(\Pi)$:

$$\mathbf{U} = \mathfrak{N}^{-1}\mathfrak{F}(\mathbf{U}) \equiv \mathfrak{K}(\mathbf{U}), \tag{4.7}$$

where $\mathbf{U} = (\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2)^T$, and $\mathfrak{F}(\mathbf{U}) = (f_1(\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2), f_2(\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2), 0, f_3(\mathbf{u}, q, \vartheta, \Psi_1, \Psi_2), \dots)^T$ denotes the long vector of right-hand side after the linearization. The elements of the right-hand side vector depend on ε via the transformation mapping of the original flow domain. In order to show the convergence of the successive approximations $(\mathbf{u}^{(n)}, q^{(n)}, \vartheta^{(n)}, \Psi_1^{(n)}, \Psi_2^{(n)})^T$, it is sufficient to show that the operator $\mathfrak{K}$ is a contraction mapping in a ball of the space $\mathfrak{D}_\beta^{l,2}W(\Pi)$ for small values of ε and α.

Note that the corresponding isothermal problem to Problem (1.1)–(1.4) (i.e., without any inclusion of temperature) was analytically examined in details in the papers [8, 13]. In order to prove Theorem 4.1 in details, one has to repeat and to modify all the investigations from those papers. Since the temperature equation is also nonlinear elliptic there, are not essential changes in the proof. Thus we omit the detailed proof here.

Appendix

On the Nonisothermal Base Flow

Let us now investigate the two-fluid flow in the "uniform" two-fluid domain Π down the really inclined plane, that is, in the case where S_0 is replaced by the inclined straight line $S_* := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \equiv 0, -\infty < x_1 < +\infty\}$. We suppose that the (unknown) free interface and the free surface Γ_i permit the representations $\Gamma_i = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \psi_i(x_1) = h_i = \text{const.}, -\infty < x_1 < +\infty\}$, and we are looking for stationary unidirectional flows. Such flows fulfil the assumptions

$$v_2 \equiv 0, \qquad \frac{\partial v_1}{\partial x_1} \equiv 0, \qquad \frac{\partial \theta}{\partial x_1} \equiv 0. \tag{A.1}$$

As a consequence we obtain that the pressure gradient downstream is a constant $\partial p/\partial x_1 = p_0 = \text{const.}$ Since the motion is generated by gravity, only the pressure gradient downstream

p_0 must be equal to 0. Under these assumptions, the nonisothermal Navier-Stokes equations reduce to

$$-\nu\varrho\nabla^2 v_1 = \varrho g \sin\alpha,$$

$$\frac{\partial p}{\partial x_2} = -\varrho g \cos\alpha, \qquad -\lambda\nabla^2\theta = 0, \tag{A.2}$$

and the equation of continuity $(1.1)_2$ is automatically fulfilled. Problem (1.1)–(1.4) can now be transformed to three systems of equations containing the unknowns $v_1(x_2), p(x_2)$, and $\theta(x_2)$, independently. These three systems have the following solutions:

$$v_1^0(x_2) = \begin{cases} g \sin\alpha\left[-\dfrac{1}{2\nu_1}x_2^2 + \left(\dfrac{h_2 - h_1}{r\nu_2} + \dfrac{h_1}{\nu_1}\right)x_2\right], & 0 \le x_2 \le h_1, \\ g \sin\alpha\left[-\dfrac{1}{2\nu_2}(h_2-x_2)^2 + \dfrac{1}{2\nu_2}(h_2-h_1)^2 + \dfrac{1}{2\nu_1}h_1^2 + \dfrac{1}{r\nu_2}(h_2-h_1)h_1\right], & h_1 \le x_2 \le h_2, \end{cases} \tag{A.3}$$

$$v_2^0(x_2) \equiv 0, \quad \text{with } r := \frac{\nu_1\varrho_1}{\nu_2\varrho_2}, \tag{A.4}$$

while

$$p^0(x_2) = \begin{cases} (h_1 - x_2)\varrho_1 g \cos\alpha + (h_2 - h_1)\varrho_2 g \cos\alpha + p_a, & 0 \le x_2 \le h_1, \\ (h_2 - x_2)\varrho_2 g \cos\alpha + p_a, & h_1 \le x_2 \le h_2, \end{cases} \tag{A.5}$$

$$\theta^0(x_2) = \begin{cases} \dfrac{\theta_a\lambda_2}{\lambda_2 h_1 + \lambda_1(h_2 - h_1) + \lambda_1\lambda_2}x_2, & 0 \le x_2 \le h_1, \\ \dfrac{\theta_a[\lambda_1 x_2 + h_1(\lambda_2 - \lambda_1)]}{\lambda_2 h_1 + \lambda_1(h_2 - h_1) + \lambda_1\lambda_2}, & h_1 \le x_2 \le h_2. \end{cases} \tag{A.6}$$

The so-called fluxes F_i $(i = 1, 2)$ for both liquid layers that are the integrals

$$F_1 = \int_0^{h_1} v_1^{(1)}(x_2)dx_2, \qquad F_2 = \int_{h_1}^{h_2} v_1^{(2)}(x_2)dx_2 \tag{A.7}$$

play an important role in the theoretical proof of the solvability to Problem (1.1)–(1.4). Their values can be calculated as

$$F_1 = g \sin\alpha\left[\frac{1}{3\nu_1}h_1^3 + \frac{1}{2r\nu_2}h_1^2(h_2 - h_1)\right],$$

$$F_2 = g \sin\alpha\left[\frac{1}{3\nu_2}(h_2 - h_1)^3 + (h_2 - h_1)\left(\frac{1}{2\nu_1}h_1^2 + \frac{1}{r\nu_2}h_1(h_2 - h_1)\right)\right]. \tag{A.8}$$

Recall that if $\sin\alpha$ is small, then for the fluxes F_i the same is true. Furthermore, let us emphasize that solution (A.3)–(A.6) to problem (A.2) is frequently called *Nusselt* solution of the inclined film flow in the fluid mechanical literature.

Acknowledgment

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