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Research Article

Superconvergence of a New Nonconforming Mixed Finite Element Scheme for Elliptic Problem

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A new nonconforming mixed finite element scheme for the second-order elliptic problem is proposed based on a new mixed variational form. It has the lowest degrees of freedom on rectangular meshes. The superclose property is proven by employing integral identity technique. Then global superconvergence result is derived through interpolation postprocessing operators. At last, some numerical experiments are carried out to verify the theoretical analysis.

1. Introduction

Mixed finite element method (MFEM) is an important branch of FEMs and has been used widely in numerical computation of practical problems. A lot of studies on this aspect have been devoted to the second-order elliptic problems [1–5], in which two approximation spaces of MFEM should satisfy the famous B-B condition [6]. However, since the variable u and flux $\psi = \nabla u$ belong to $L^2(\Omega)$ and $H(\mathrm{div}, \Omega)$, respectively, it is not easy to construct a stable MFE space pair. In order to circumvent or ameliorate this deficiency, many approaches have been proposed, such as the least squares FEM [7], stabilization FEM [8], and H^1-Galerkin FEM [9]. Recently, [10] presented a new MFEM, in which B-B condition is automatically satisfied when M_h and H_h meet a relation of inclusion; that is, $\nabla M_h \subset H_h$, where M_h and H_h are finite element approximation spaces of u and flux ψ, respectively. This advantage makes the construction of stable MFE space pair extremely simple and convenient. A family of triangular and rectangular conforming MFE space pairs with lower degrees of freedom is constructed in [10], in which the total degrees of freedom of first-order and second-order MFE schemes are about $5N_p$ and $16N_p$, respectively; herein N_p denotes the number of nodal points in subdivision. Later, [11] derived the similar results with [10] for conforming MFEM and gave a numerical example.

In this paper, motivated by the idea of [10, 11], we first construct a new nonconforming MFEM (NMFEM). The original variable u is approximated by the constrained Q_1^{rot} element space [12] and the flux ψ by piecewise constant vectors space, respectively. Note that the total degrees of freedom of the NMFEM are only about $3N_p$, the lowest on rectangular meshes. We prove that it satisfies B-B condition. Then by the use of integral identity technique, we derive the superclose property for u in energy norm and flux ψ in L^2 norm. Furthermore, the global superconvergence result with order $O(h^2)$ is obtained through interpolation postprocessing operators. Finally, some numerical results are provided to verify the theoretical analysis. It is observed that, compared with FEM using 4-node quadrilateral (FEM-Q4) and MFEM (u and flux ψ are approximated by piecewise constants and the Raviart-Thomas element, resp.), NMFEM behaves well and has higher rates than MFEM, and it has almost the same rate as FEM-Q4 for u in L^2 norm, but a higher rate than FEM-Q4 for u in energy norm. Moreover, NMFEM is effective and accurate for the diffusion problem studied in [13].

The remainder of this paper is organized as follows. In Section 2, we introduce NMFEM and derive the superclose property and superconvergence results. In Section 3, we carry out some numerical experiments to show the performance of NMFEM.

We will use standard notations for the Sobolev spaces $H^m(\Omega)$ with norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$, $H^m(K)$ with norm $\|\cdot\|_{m,K}$ and seminorm $|\cdot|_{m,K}$, where $m > 0$ is an integer. Besides, let $\|\cdot\|_0$ and $\|\cdot\|_{0,K}$ be the $L^2(\Omega)$ norm and $L^2(K)$ norm, respectively. Throughout the paper, C denotes a positive constant independent of the mesh parameter h and may be different at each appearance.

2. Superconvergence Analysis for NMFEM

Consider the following elliptic problem:

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset R^2$ is a bounded convex polygon domain, $f \in L^2(\Omega)$.

Let $\psi = \nabla u$, and then problem (1) is equivalent to the following equations:

$$\begin{aligned} \psi - \nabla u &= 0, \quad \text{in } \Omega, \\ -\operatorname{div}\psi &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

We adopt a new mixed variational form in [10] of problem (2). Find $(\psi, u) \in H \times M$ such that

$$\begin{aligned} a(\psi,\varphi) + b(\varphi,u) &= 0, \quad \forall \varphi \in H, \\ b(\psi,v) &= G(v), \quad \forall v \in M, \end{aligned} \tag{3}$$

where $H = (L^2(\Omega))^2$, $M = H_0^1(\Omega)$, $a(\psi,\varphi) = \int_\Omega \psi\cdot\varphi\,dx\,dy$, $b(\varphi,v) = -\int_\Omega \varphi\cdot\nabla v\,dx\,dy$, $G(v) = -\int_\Omega f v\,dx\,dy$.

Obviously, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear functionals, $G(\cdot)$ is a continuous linear functional, and for all $\varphi \in H$, $a(\varphi,\varphi) = \int_\Omega \varphi\cdot\varphi\,dx\,dy = \|\varphi\|_0^2$. Moreover, for $v \in M$, we have $\nabla v \in H$. So there exists a constant $\beta > 0$ such that

$$\sup_{\varphi\in H} \frac{b(\varphi,v)}{\|\varphi\|_0} \ge \frac{b(-\nabla v, v)}{\|-\nabla v\|_0} = |v|_1 \ge \beta\|v\|_1, \tag{4}$$

that is, the B-B condition is satisfied, and therefore (3) has a unique solution (ψ,u).

Let $Q((\psi,u),(\varphi,v)) = a(\psi,\varphi) + b(\varphi,u) - b(\psi,v)$, and then (3) can be written as

$$Q((\psi,u),(\varphi,v)) = -G(v), \quad \forall \varphi \in H, v \in M. \tag{5}$$

Let T_h be a rectangular partition of the domain Ω. For a given element $K \in T_h$, its four vertices are denoted by $a_1 = (x_K - h_x, y_K - h_y)$, $a_2 = (x_K + h_x, y_K - h_y)$, $a_3 = (x_K + h_x, y_K + h_y)$, $a_4 = (x_K - h_x, y_K + h_y)$, and four edges by $l_i = \overline{a_i a_{i+1}}$ $(i = 1,2,3,4 \mod 4)$, $h_K = \max_{K\in T_h}\{h_x,h_y\}$, $h = \max_{K\in T_h} h_K$. Let $\widehat{K} = [-1,1]\times[-1,1]$ be the reference element with nodes $\widehat{a}_1 = (-1,-1)$, $\widehat{a}_2 = (1,-1)$, $\widehat{a}_3 = (1,1)$, $\widehat{a}_4 = (-1,1)$ and edges $\widehat{l}_i = \overline{\widehat{a}_i\widehat{a}_{i+1}}$ $(i = 1,2,3,4 \mod 4)$.

Define the affine mapping $F : \widehat{K} \to K$ by

$$\begin{aligned} x &= x_K + h_x\xi, \\ y &= x_K + h_y\eta. \end{aligned} \tag{6}$$

Let $P_1(\widehat{K})$ be the space of polynomials with degrees ≤ 1 defined on $\widehat{K}$, and then the constrained Q_1^{rot} element space CR_0^h is defined by [12]:

$$\text{CR}_0^h = \Big\{ v;\ v|_K = \widehat{v}\circ F^{-1}, \widehat{v} \in P_1(\widehat{K}), \int_l [v]\,ds = 0, l \subset \partial K, \forall K \in T_h \Big\}, \tag{7}$$

where $[v]$ denotes the jump value of v across the boundary l, and $[v] = v$ if $l \subset \partial\Omega$.

Let N_i^V denote the number of interior nodes. It has been proven in [12] that $\dim(\text{CR}_0^h) = N_i^V$ and $P_1(\widehat{K}) = \text{span}\{\widehat{\phi}_i,\ i = 1 \sim 4\}$, where $\widehat{\phi}_i$ are defined associated with nodes $\widehat{a}_i$ $(i = 1,2,3,4)$ of $\widehat{K}$ as

$$\begin{aligned} \widehat{\phi}_1 &= \frac{1}{4}(1-\xi-\eta), \qquad \widehat{\phi}_2 = \frac{1}{4}(1+\xi-\eta), \\ \widehat{\phi}_3 &= \frac{1}{4}(1+\xi+\eta), \qquad \widehat{\phi}_4 = \frac{1}{4}(1-\xi+\eta). \end{aligned} \tag{8}$$

We choose the following FE spaces M_h and H_h to approximate M and H, respectively:

$$\begin{aligned} M_h &= \text{CR}_0^h, \\ H_h &= \left\{ q = (q_1,q_2);\ q|_K \in (Q_0(K))^2,\ \forall K \in T_h \right\}, \end{aligned} \tag{9}$$

where $Q_0(K)$ is the space of constants on K. Obviously, the total degree of freedoms of the nonconforming MFE space pair $H_h \times M_h$ is $3N_p$, and $\|\cdot\|_h = (\sum_{K\in T_h} |\cdot|_{1,K}^2)^{1/2}$ is a norm over M_h.

Then the MFE approximation of problem (3) is to find $(\psi_h, u_h) \in H_h \times M_h$ such that

$$\begin{aligned} a(\psi_h,\varphi_h) + b_h(\varphi_h,u_h) &= 0, \quad \forall \varphi_h \in H_h, \\ b_h(\psi_h,v_h) &= G(v_h), \quad \forall v_h \in M_h, \end{aligned} \tag{10}$$

where $b_h(\varphi_h,v_h) = -\sum_{K\in T_h}\int_K \varphi_h\cdot\nabla v_h\,dx\,dy$.

For $v_h \in M_h$, from the affine mapping F and the definition of M_h, we can get $v_h|_K \in P_1(K)$, and then $\nabla v_h|_K \in (Q_0(K))^2$, that is, $\nabla v_h \in H_h$. Thus

$$\begin{aligned} \sup_{\varphi_h\in H_h} \frac{b_h(\varphi_h,v_h)}{\|\varphi_h\|_0} &\ge \frac{b_h(-\nabla v_h, v_h)}{\|-\nabla v_h\|_0} \\ &= \frac{\sum_{K\in T_h}\int_K \nabla v_h\cdot\nabla v_h\,dx\,dy}{\|-\nabla v_h\|_0} = \frac{\|v_h\|_h^2}{\|v_h\|_h} = \|v_h\|_h, \end{aligned} \tag{11}$$

that is, the discrete B-B condition holds, and (10) has a unique solution (ψ_h, u_h).

For all $(q_h,\lambda_h) \in H_h \times M_h$, let $Q_h((q_h,\lambda_h),(\varphi_h,v_h)) = a(q_h,\varphi_h) + b_h(\varphi_h,\lambda_h) - b_h(q_h,v_h)$, and then we can prove the following two important lemmas.

Lemma 1. *For all $(q_h, \lambda_h) \in H_h \times M_h$, the following inequality holds:*

$$\sup_{(\varphi_h, v_h)\in H_h\times M_h} \frac{Q_h((q_h,\lambda_h),(\varphi_h,v_h))}{\|\varphi_h\|_0 + \|v_h\|_h} \ge C(\|q_h\|_0 + \|\lambda_h\|_h). \tag{12}$$

Proof. In order to prove (12), we will construct a pair $(\varphi_h, v_h) \in H_h \times M_h$ satisfying

$$Q_h((q_h,\lambda_h),(\varphi_h,v_h)) \ge C(\|q_h\|_0 + \|\lambda_h\|_h)(\|\varphi_h\|_0 + \|v_h\|_h). \tag{13}$$

Obviously,

$$Q_h((q_h,\lambda_h),(q_h,\lambda_h)) = \|q_h\|_0^2. \tag{14}$$

On the other hand, for a given arbitrary but fixed $\lambda_h \in M_h$, by the discrete B-B condition, there exists a $w_h \in H_h$ such that

$$\begin{gathered}\sum_{K\in T_h}\int_K w_h\cdot\nabla\lambda_h\,dx\,dy \ge \|w_h\|_0\|\lambda_h\|_h,\\ \|w_h\|_0 = \|\lambda_h\|_h.\end{gathered} \tag{15}$$

So, for $0 < r < 1$, there holds

$$\begin{aligned}&Q_h((q_h,\lambda_h),(-rw_h,0))\\ &\quad= -ra(q_h,w_h) + r\sum_{K\in T_h}\int_K w_h\cdot\nabla\lambda_h\,dx\,dy\\ &\quad\ge -r\|q_h\|_0\|\lambda_h\|_h + r\|\lambda_h\|_h^2\\ &\quad\ge -r\|q_h\|_0^2 + \frac{3}{4}r\|\lambda_h\|_h^2.\end{aligned} \tag{16}$$

As a result, setting $(\varphi_h, v_h) = (q_h - rw_h, \lambda_h)$, we have

$$\begin{aligned}&Q_h((q_h,\lambda_h),(q_h-rw_h,\lambda_h))\\ &\quad= Q_h((q_h,\lambda_h),(q_h,\lambda_h))\\ &\qquad+ Q_h((q_h,\lambda_h),(-rw_h,0))\\ &\quad\ge (1-r)\|q_h\|_0^2 + \frac{3}{4}r\|\lambda_h\|_h^2,\end{aligned} \tag{17}$$

which implies that

$$\begin{aligned}&Q_h((q_h,\lambda_h),(q_h-rw_h,\lambda_h))\\ &\quad\ge C_1(\|q_h\|_0^2 + \|\lambda_h\|_h^2)\\ &\quad\ge \frac{C_1}{2}(\|q_h\|_0 + \|\lambda_h\|_h)^2,\end{aligned} \tag{18}$$

where $C_1 = \min\{1-r, (3/4)r\}$.

Note that

$$\begin{aligned}&\|q_h - rw_h\|_0 + \|\lambda_h\|_h\\ &\quad\le \|q_h\|_0 + r\|w_h\|_0 + \|\lambda_h\|_h\\ &\quad= \|q_h\|_0 + (1+r)\|\lambda_h\|_h\\ &\quad\le (1+r)(\|q_h\|_0 + \|\lambda_h\|_h),\end{aligned} \tag{19}$$

and we have

$$\begin{aligned}&Q_h((q_h,\lambda_h),(q_h-rw_h,\lambda_h))\\ &\quad\ge \frac{C_1}{2(1+r)}(\|q_h\|_0 + \|\lambda_h\|_h)(\|q_h - rw_h\|_0 + \|\lambda_h\|_h),\end{aligned} \tag{20}$$

which follows the desired result (12).

Let Π_h and I_h denote the associated interpolation operators of M_h and conforming bilinear element space, respectively. Let π_h be the Q_1^{rot} element interpolation operator [14]; that is, for all $v \in H^1(K)$, $\pi_h v \in Q_1^{\text{rot}}$ satisfying $\int_l \pi_h v\,ds = \int_l v\,ds$, where $Q_1^{\text{rot}} = \{v_h; v_h|_K \in \text{span}\{1, x, y, x^2 - y^2\}, \int_l [v_h]ds = 0, l \subset \partial K\}$. It has been shown in [12] that, for all $v \in H^2(K)$, $\Pi_h v = \pi_h I_h v \in M_h$ because xy is a bubble function for π_h. □

Lemma 2. *Assume that $u \in H^3(\Omega)$, $\psi \in (H^2(\Omega))^2$, we have*

$$b_h(\psi - J_h\psi, v_h) = 0, \quad \forall v_h \in M_h, \tag{21}$$

$$\sum_{K\in T_h}\int_K \nabla(u - \Pi_h u)\cdot\varphi_h dx\,dy \le Ch^2\|u\|_3\|\varphi_h\|_0, \quad \forall\varphi_h \in H_h, \tag{22}$$

$$\sum_{K\in T_h}\int_{\partial K}\psi\cdot n v_h\,ds \le Ch^2|\psi|_2\|v_h\|_h, \quad \forall v_h \in M_h, \tag{23}$$

where $J_h : (L^2(\Omega))^2 \to H_h$ is a interpolation operator satisfying $\int_K(\psi - J_h\psi)dx\,dy = 0$.

Proof. Since (23) has been proven by one of the authors in [15], we only need to prove (21) and (22).

In fact, because $\partial v_h/\partial x$ and $\partial v_h/\partial y$ are constants on each $K \in T_h$, we have

$$b_h(\psi - J_h\psi, v_h) = -\sum_{K\in T_h}\int_K(\psi - J_h\psi)\cdot\nabla v_h\,dx\,dy = 0, \tag{24}$$

which is (21).

On the other hand, note that

$$\begin{aligned}&\int_K \nabla(u - \Pi_h u)\cdot\varphi_h dx\,dy\\ &\quad= \int_K \nabla(u - I_h u)\cdot\varphi_h dx\,dy\\ &\qquad+ \int_K \nabla(I_h u - \Pi_h u)\cdot\varphi_h dx\,dy.\end{aligned} \tag{25}$$

Since $\varphi_h = (\varphi_{1h}, \varphi_{2h})$ is a constant vector on K and $\Pi_h v = \pi_h I_h v$, it follows from integration by parts that

$$\begin{aligned}&\int_K \nabla(I_h u - \Pi_h u)\cdot\varphi_h dx\,dy\\ &\quad= \varphi_h|_K\int_{\partial K}(I_h u - \pi_h I_h u)\,ds\\ &\quad= \varphi_h|_K\int_{\partial K}(I_h u - I_h u)\,ds = 0.\end{aligned} \tag{26}$$

Furthermore, let $F(y) = (1/2)((y - y_K)^2 - h_y^2)$. Note that $(u-I_h u)(x_K \pm h_x, y_K \pm h_y) = 0$ and $F(y_K \pm h_y) = 0$. Employing integral identity technique [16], we have

$$\begin{aligned}
&\int_K \frac{\partial (u - I_h u)}{\partial x} \varphi_{1h} dx\, dy \\
&= \varphi_{1h}|_K \int_K F''(y) \frac{\partial (u - I_h u)}{\partial x} dx\, dy \\
&= \varphi_{1h}|_K \left(F'(y_K + h_y) \int_{l_3} \frac{\partial (u - I_h u)}{\partial x} dx \right. \\
&\qquad - F'(y_K - h_y) \int_{l_1} \frac{\partial (u - I_h u)}{\partial x} dx \\
&\qquad \left. - \int_K F'(y) \frac{\partial^2 (u - I_h u)}{\partial x \partial y} dx\, dy \right) \\
&= -\varphi_{1h}|_K \left(F(y_K + h_y) \int_{l_3} \frac{\partial^2 (u - I_h u)}{\partial x \partial y} dx \right. \\
&\qquad - F(y_K - h_y) \int_{l_1} \frac{\partial^2 (u - I_h u)}{\partial x\, \partial y} dx \\
&\qquad \left. - \int_K F(y) \frac{\partial^3 (u - I_h u)}{\partial x \partial y^2} dx\, dy \right) \\
&= \varphi_{1h}|_K \int_K F(y) \frac{\partial^3 (u - I_h u)}{\partial x\, \partial y^2} dx\, dy \\
&\le C h^2 \|u\|_{3,K} \|\varphi_{1h}\|_{0,K}.
\end{aligned} \tag{27}$$

Similarly,

$$\int_K \frac{\partial (u - I_h u)}{\partial y} \varphi_{2h} dx\, dy \le C h^2 \|u\|_{3,K} \|\varphi_{2h}\|_{0,K}. \tag{28}$$

Therefore, from (25)–(28), we have

$$\begin{aligned}
&\sum_{K \in T_h} \int_K \nabla (u - \Pi_h u) \cdot \varphi_h dx\, dy \\
&= \sum_{K \in T_h} \int_K \nabla (u - I_h u) \cdot \varphi_h dx\, dy \\
&= \sum_{K \in T_h} \left(\int_K \frac{\partial (u - I_h u)}{\partial x} \varphi_{1h} dx\, dy + \int_K \frac{\partial (u - I_h u)}{\partial y} \varphi_{2h} dx\, dy \right) \\
&\le C h^2 \|u\|_3 \|\varphi_h\|_0.
\end{aligned} \tag{29}$$

The proof is completed. □

Now we start to state the following superclose property.

Theorem 3. *Assume that* (ψ, u) *and* (ψ_h, u_h) *are the solutions of* (3) *and* (10), *respectively;* $u \in H^3(\Omega)$, $\psi \in (H^2(\Omega))^2$, *there holds*

$$\|\psi_h - J_h \psi\|_0 + \|u_h - \Pi_h u\|_h \le C h^2 (\|u\|_3 + \|\psi\|_2). \tag{30}$$

Proof. For $(\varphi_h, v_h) \in H_h \times M_h$, from (2) and (10), we have

$$\begin{aligned}
&a(\psi, \varphi_h) + b_h(\varphi_h, u) = a(\psi_h, \varphi_h) + b_h(\varphi_h, u_h) = 0, \\
&b_h(\psi, v_h) - b_h(\psi_h, v_h) \\
&= b_h(\psi, v_h) - G(v_h) \\
&= -\sum_{K \in T_h} \int_K \psi \cdot \nabla v_h dx\, dy \\
&\quad + \sum_{K \in T_h} \int_K f v_h dx\, dy \\
&= -\sum_{K \in T_h} \int_{\partial K} \psi \cdot n v_h dx\, dy \\
&\quad + \sum_{K \in T_h} \int_K (\operatorname{div} \psi + f) v_h dx\, dy \\
&= -\sum_{K \in T_h} \int_{\partial K} \psi \cdot n v_h dx\, dy.
\end{aligned} \tag{31}$$

Applying (31) yields

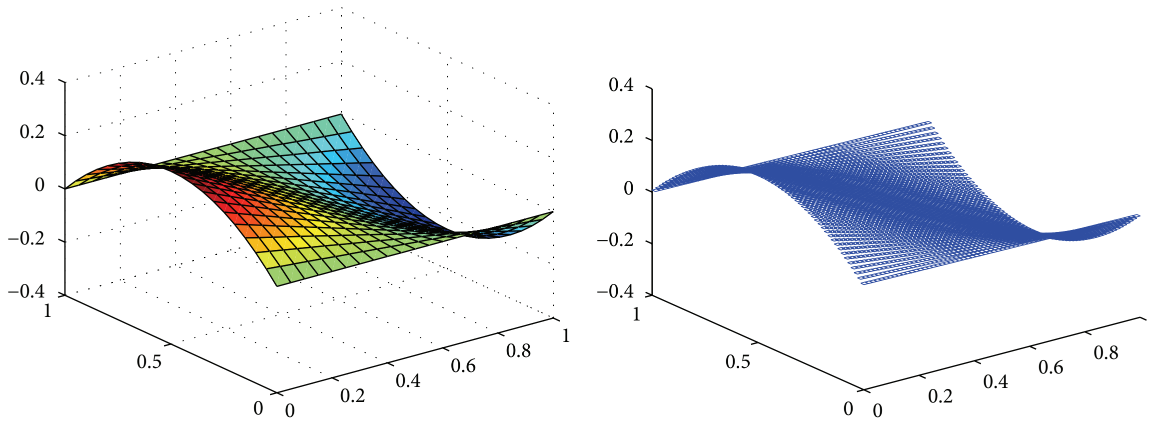
$$\begin{aligned}
&Q_h((\psi - \psi_h, u - u_h), (\varphi_h, v_h)) \\
&= \sum_{K \in T_h} \int_{\partial K} \psi \cdot n v_h dx\, dy, \quad \forall (\varphi_h, v_h) \in H_h \times M_h.
\end{aligned} \tag{32}$$

Using (12) in Lemma 1 and (32), we can obtain

$$\begin{aligned}
&\|\psi_h - J_h \psi\|_0 + \|u_h - \Pi_h u\|_h \\
&\le C \sup_{(\varphi_h, v_h) \in H_h \times M_h} \frac{Q_h((\psi_h - J_h \psi, u_h - \Pi_h u), (\varphi_h, v_h))}{\|\varphi_h\|_0 + \|v_h\|_h} \\
&\le C \sup_{(\varphi_h, v_h) \in H_h \times M_h} \Bigg(\Bigg(Q_h((\psi - J_h \psi, u - \Pi_h u), (\varphi_h, v_h)) \\
&\qquad - \sum_{K \in T_h} \int_{\partial K} \psi \cdot n v_h dx\, dy \Bigg) \\
&\qquad \times (\|\varphi_h\|_0 + \|v_h\|_h)^{-1} \Bigg) \\
&= C \sup_{(\varphi_h, v_h) \in H_h \times M_h} \Bigg(\Bigg(a(\psi - J_h \psi, \varphi_h) + b_h(\varphi_h, u - \Pi_h u) \\
&\qquad - b_h(\psi - J_h \psi, v_h) \\
&\qquad - \sum_{K \in T_h} \int_{\partial K} \psi \cdot n v_h dx\, dy \Bigg) \\
&\qquad \times (\|\varphi_h\|_0 + \|v_h\|_h)^{-1} \Bigg).
\end{aligned} \tag{33}$$

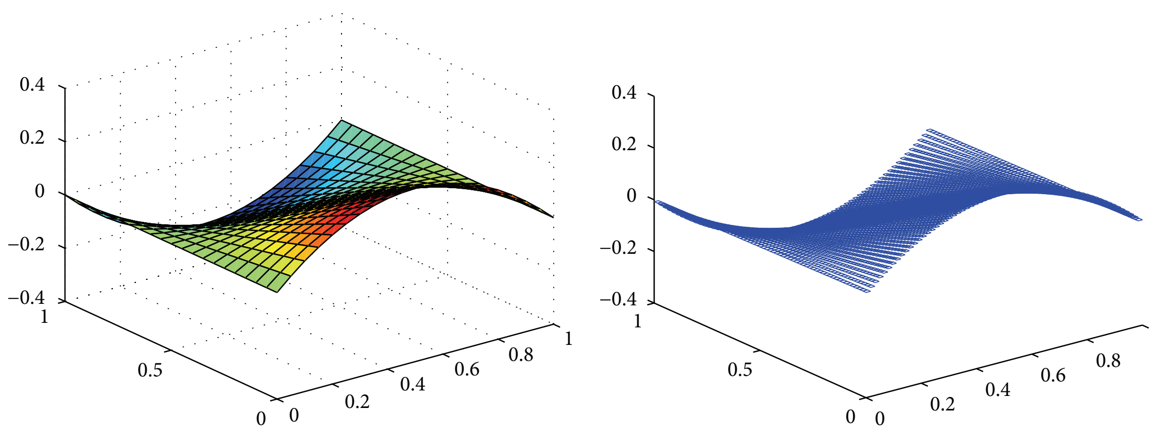
Hence the desired result follows from the interpolation theorem and Lemma 2.

In order to derive global superconvergence for u and flux ψ, we introduce the following postprocessing operators



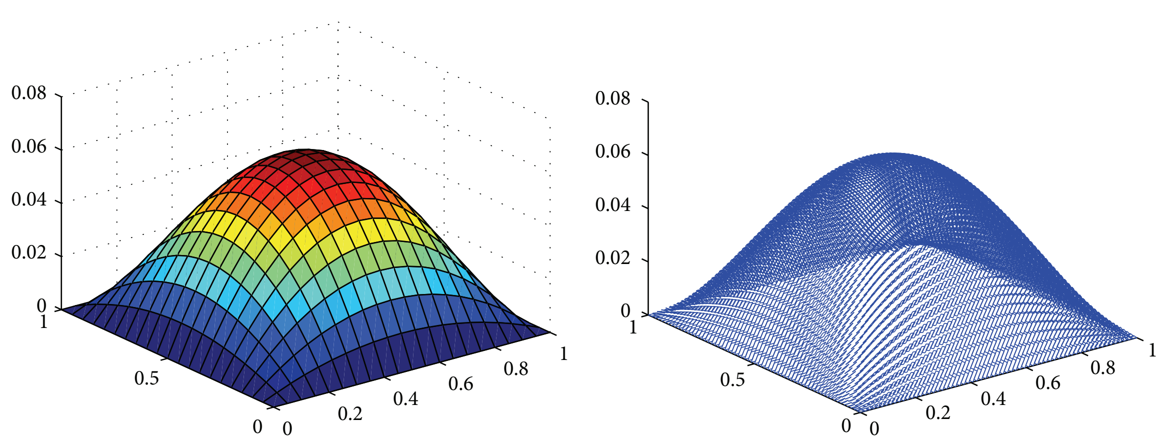
(a) (b)

Figure 1: The exact flux ψ_1 (a) Numerical approximation ψ_{1h} (b).



(a) (b)

Figure 2: The exact flux ψ_2 (a) and numerical approximation ψ_{2h} (b).



(a) (b)

Figure 3: The exact displacement u (a) and numerical approximation u_h (b).

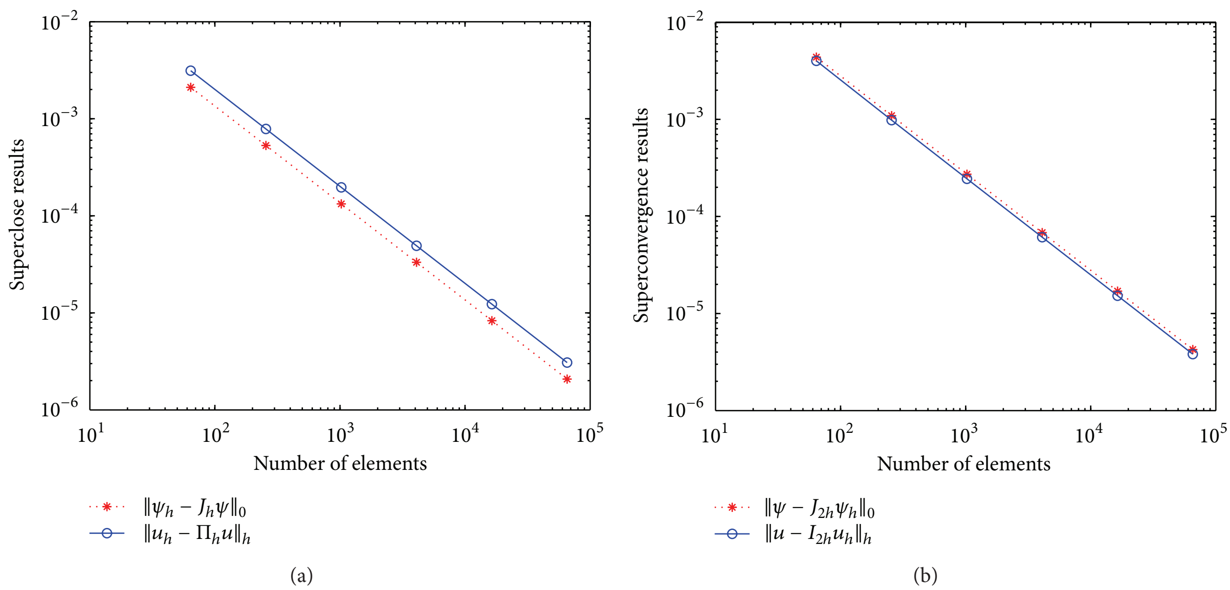


Figure 4: The superclose results (a) and the superconvergence results (b).

Table 1: Superclose results of ψ and u.

N	$\|\psi_h - J_h\psi\|_0$	Rate	$\|u_h - \Pi_h u\|_h$	Rate
8^2	0.00210961611276	/	0.00313196226739	/
16^2	$5.305340499025525e-004$	1.99146324331851	$7.857858400185900e-004$	1.99485875617198
32^2	$1.328284185506180e-004$	1.99788150355682	$1.966221092417072e-004$	1.99871061529750
64^2	$3.321927509970684e-005$	1.99947134660916	$4.916651922457212e-005$	1.99967742778650
128^2	$8.305579254650728e-006$	1.99986789716594	$1.229231701395274e-005$	1.99991934319524
256^2	$2.076442340906698e-006$	1.99996697808428	$3.073122207277826e-006$	1.99997983495344

$I_{2h} : u \in (H^2(\Omega) \cap H_0^1(\Omega)) \mapsto I_{2h}u \in Q_2(\kappa)$ and $J_{2h} : \psi \in (H^1(\Omega))^2 \mapsto J_{2h}\psi \in (Q_1(\kappa))^2$ as $I_{2h}u(a_i) = u(a_i)$, $i = 1, \ldots, 9$, and $\int_{K_j}(\psi - J_{2h}\psi) = 0$, $j = 1, 2, 3, 4$, where $u(a_i)$ are the value of u on the nodes a_i and a_i are nodes of T_h on macroelement κ, while $\kappa \in T_{2h}$ consists of the four small elements K_j in T_h $(j = 1, 2, 3, 4)$, and $Q_1(\kappa)$ and $Q_2(\kappa)$ are bilinear and biquadratic piecewise polynomials spaces, respectively. It can be checked that the following properties hold:

$$\begin{gathered} I_{2h}\Pi_h u = I_{2h}u, \qquad \|I_{2h}u - u\|_h \le Ch^2 |u|_3, \\ \|I_{2h}v_h\|_h \le C\|v_h\|_h, \quad \forall v_h \in M_h, \\ J_{2h}J_h\psi = J_{2h}\psi, \qquad \|J_{2h}\psi - \psi\|_0 \le Ch^2 |\psi|_2, \\ \|J_{2h}\varphi_h\|_0 \le C\|\varphi_h\|_0, \quad \forall \varphi_h \in H_h. \end{gathered} \tag{34}$$

Then we can have the following superconvergence result. □

Theorem 4. *Under the assumptions in Theorem 3, there holds*

$$\|u - I_{2h}u_h\|_h + \|\psi - J_{2h}\psi_h\|_0 \le Ch^2 \left(\|u\|_3 + \|\psi\|_2\right). \tag{35}$$

Proof. It follows from (30) in Theorem 3, (34), and the triangle inequality that

$$\begin{aligned} &\|u - I_{2h}u_h\|_h + \|\psi - J_{2h}\psi_h\|_0 \\ &\quad \le \|u - I_{2h}\Pi_h u\|_h + \|I_{2h}\Pi_h u - I_{2h}u_h\|_h \\ &\qquad + \|\psi - J_{2h}J_h\psi\|_0 + \|J_{2h}J_h\psi - J_{2h}\psi_h\|_0 \\ &\quad \le \|u - I_{2h}\Pi_h u\|_h + \|I_{2h}(\Pi_h u - u_h)\|_h \\ &\qquad + \|\psi - J_{2h}J_h\psi\|_0 + \| \, \|J_{2h}(J_h\psi - \psi_h)\|_0 \\ &\quad \le \|u - I_{2h}u\|_h + C\|\Pi_h u - u_h\|_h \\ &\qquad + \|\psi - J_{2h}\psi\|_0 + C\|J_h\psi - \psi_h\|_0 \\ &\quad \le Ch^2 \left(\|u\|_3 + \|\psi\|_2\right), \end{aligned} \tag{36}$$

which is the desired result. □

3. Numerical Experiments

In this section, some numerical examples and comparison with other methods are presented to confirm theoretical analysis and good performance of NMFEM.

Table 2: Superconvergence results of ψ and u.

N	$\|\psi - J_{2h}\psi_h\|_0$	Rate	$\|u - I_{2h}u_h\|_h$	Rate
8^2	0.00438686071544	/	0.00401808068890	/
16^2	0.00109136640979	2.00705335488651	$9.832140356840235e-004$	2.03092911988780
32^2	$2.724757835149567e-004$	2.00193562594041	$2.445206258258704e-004$	2.00754934612234
64^2	$6.809546234729317e-005$	2.00049744480245	$6.105100787093085e-005$	2.00186914715453
128^2	$1.702238617992853e-005$	2.00012537844145	$1.525782546054283e-005$	2.00046574790862
256^2	$4.255503868189354e-006$	2.00003141881610	$3.814148839722483e-006$	2.00011631625361

Table 3: Superclose results of ψ and u.

N	$\|\psi_h - J_h\psi\|_0$	Rate	$\|u_h - \Pi_h u\|_h$	Rate
8^2	0.02493802431889	/	0.03360261961598	/
16^2	0.00619633909611	2.00885917091448	0.00820721453088	2.03360913804615
32^2	0.00154620480519	2.00268467156460	0.00203799536238	2.00974189629474
64^2	$3.863629284378830e-004$	2.00070284735607	$5.086068699851279e-004$	2.00252791412788
128^2	$9.657883351384945e-005$	2.00017773032582	$1.270955080172888e-004$	2.00063790775102
256^2	$2.414396265195381e-005$	2.00004455935162	$3.177035666820934e-005$	2.00015984995534

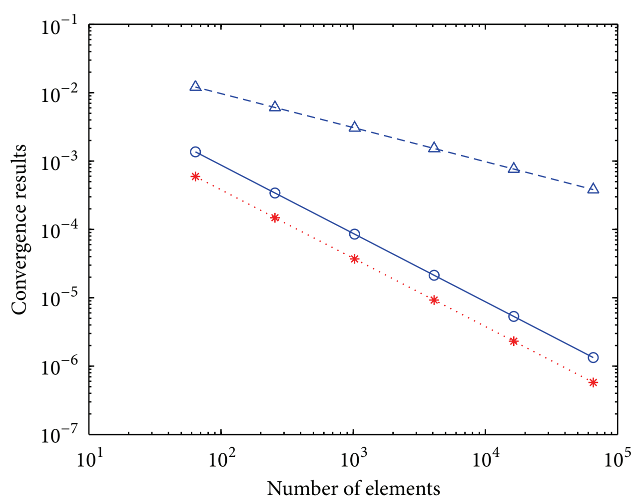


Figure 5: Convergence rates of u in L^2 norm.

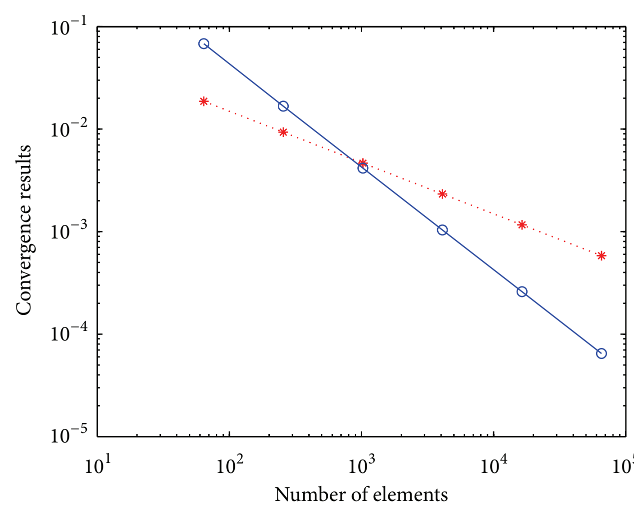


Figure 6: Convergence rates of u in energy norm.

We consider the problem (1) with $\Omega = [0,1]^2$ and the exact solution is $u = xy(1-x)(1-y)$, and then the flux field $\psi = (\psi_1, \psi_2)$ can be expressed as $\psi_1 = y(1-2x)(1-y)$, $\psi_2 = x(1-x)(1-2y)$. We divide the domain Ω into a family of quasiuniform rectangles with number of N. The figures of exact solution (u, ψ) of problem (1) and finite element approximation (u_h, ψ_h) with $N = 64^2$ are plotted in Figures 1, 2, and 3, respectively.

In Tables 1 and 2, we present the superclose and superconvergence results of the original variable u in energy norm and flux ψ in L^2 norm with $N = 8^2;\ 16^2;\ 32^2;\ 64^2;\ 128^2;\ 256^2$, respectively. It is clearly that $\|\psi_h - J_h\psi\|_0$, $\|u_h - \Pi_h u\|_h$, $\|\psi - J_{2h}\psi_h\|_0$, and $\|u - I_{2h}u_h\|_h$ are converged at order 2 with respect to h, which coincide with our theoretical analysis in Theorems 3 and 4. In order to describe the results more intuitively, we plot the errors in the logarithm scales in Figure 4.

Moreover, we compare the results of NMFEM with those of FEM using 4-node quadrilateral (FEM-Q4) and MFEM (u and flux ψ are approximated by piecewise constants and the Raviart-Thomas element, resp.).

The convergence rates of errors of u in L^2 and energy norm are shown in Figures 5 and 6. The comparison of the flux ψ in L^2 norm of NMFEM with MFEM is also given in Figure 7. It is observed that (a) the convergence rates of u and flux ψ in L^2 norm of NMFEM are better than those of MFEM;

Table 4: Superconvergence results of ψ and u.

N	$\|\psi - J_{2h}\psi_h\|_0$	Rate	$\|u - I_{2h}u_h\|_h$	Rate
8^2	0.07704230005041	/	0.06805851358971	/
16^2	0.01905713318621	2.01531966736879	0.01670893020781	2.02615627537278
32^2	0.00474917374465	2.00458266513090	0.00415565843089	2.00747038382646
64^2	0.00118603209709	2.00153348351974	0.00103721683378	2.00235955252584
128^2	$2.964169140807930e-004$	2.00044337559203	$2.591844848686839e-004$	2.00066626183030
256^2	$7.409813186219603e-005$	2.00011869741141	$6.478819154402888e-005$	2.00017656612731

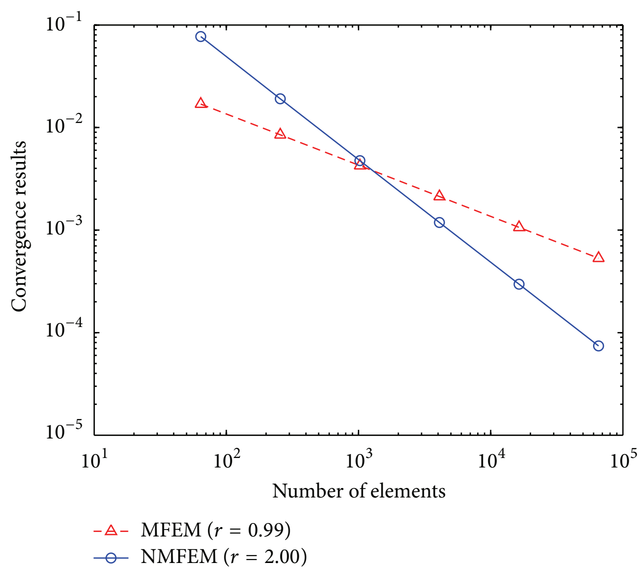


Figure 7: Convergence rates of the flux ψ in L^2 norm.

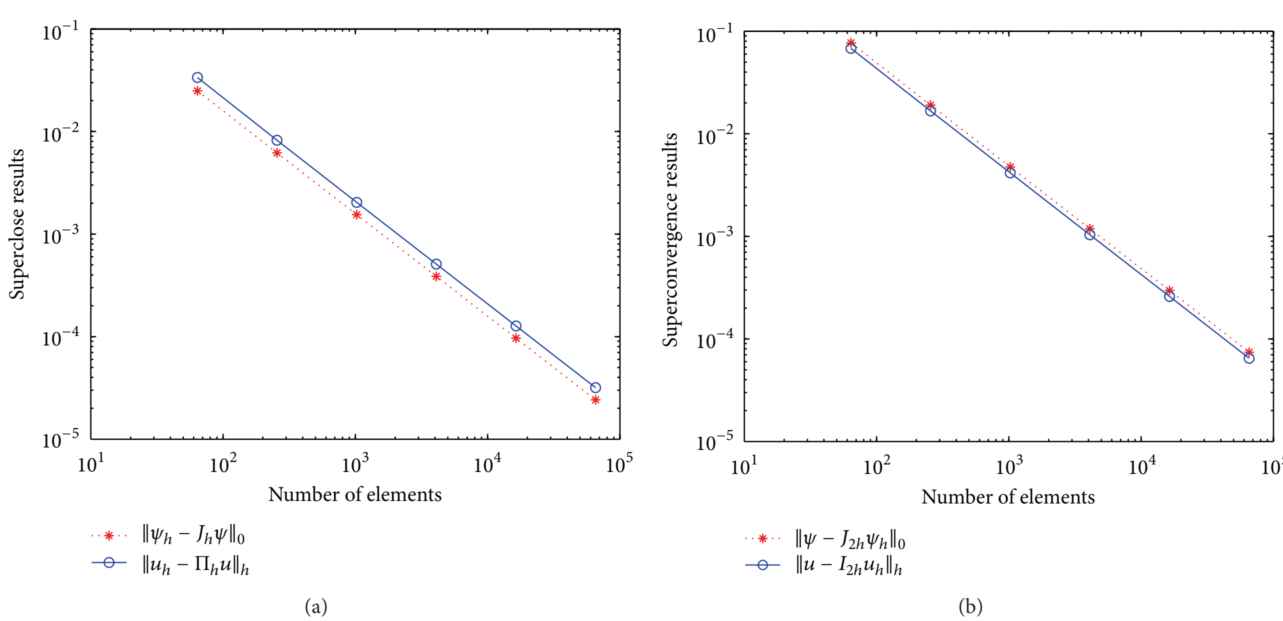


Figure 8: The superclose results (a) and the superconvergence results (b).

(b) NMFEM has almost the same rate as FEM-Q4 for u in L^2 norm, but a higher rate than FEM-Q4 for u in energy norm.

Furthermore, a numerical experiment is carried out to demonstrate the effectiveness and accuracy of NMFEM for the following diffusion problem:

$$\begin{aligned} -\operatorname{div}(D\nabla u) &= f, \quad \text{in } \Omega, \\ u &= u_\partial, \quad \text{on } \partial\Omega, \end{aligned} \tag{37}$$

where f is the source term and $f \in L^2(\Omega)$, u_∂ is the boundary data, the permeability D is a symmetric tensor-valued function such that (a) D is piecewise Lipschitz-continuous on Ω and (b) the set of the eigenvalues of D is included in $[\lambda_{\min}, \lambda_{\max}]$ (with $\lambda_{\min} > 0$) for all $(x, y) \in \Omega$.

As [13], we consider the problem (37) with $\Omega = [0, 1]^2$,

$$\begin{aligned} D &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } x \le 0.5, \\ D &= \begin{pmatrix} 100 & 0 \\ 0 & 0.01 \end{pmatrix} \quad \text{if } x > 0.5, \end{aligned} \tag{38}$$

and the analytical solution

$$\begin{aligned} u &= \cos(\pi x)\sin(\pi y) \quad \text{if } x \le 0.5, \\ u &= 0.01\cos(\pi x)\sin(\pi y) \quad \text{if } x > 0.5. \end{aligned} \tag{39}$$

Let $\psi = D\nabla u$, we can get the superclose and superconvergence results of u in energy norm and flux ψ in L^2 norm. The errors are listed in Tables 3 and 4 and plotted in the logarithm scales in Figure 8, respectively.

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