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*Research Article*

# Distributed Robust $H_\infty$ Consensus Control for Uncertain Multiagent Systems with State and Input Delays

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Robust $H_\infty$ consensus control problem is investigated for multiagent systems. Each agent is tackled in a more generalized form, which includes parameter uncertainties, external disturbances, nonidentical time-varying state, and input delays. Firstly, a distributed control protocol based on state feedback of neighbors is designed. By a decoupling method, $H_\infty$ consensus control problem for multiagent systems is transformed into $H_\infty$ control problem for the decoupling subsystems. Then employing Lyapunov-Krasovskii functional and free-weighting matrices, a lower conservative bounded real lemma (BRL) is derived in terms of linear matrix inequalities (LMIs) such that a class of time-delay system is guaranteed to be globally asymptotically stable with the desired $H_\infty$ performance index. Extending BRL, a sufficient delay-dependent condition of lower complexity in terms of the matrix inequalities is obtained to make all agents asymptotically reach consensus with the desired $H_\infty$ performance index. Furthermore, an algorithm is elaborately designed to get feasible solution to this condition. Extending this algorithm, an optimization algorithm for control protocol parameter is proposed to improve the disturbance attenuation capacity or allowable delay bounds. Finally, simulation results are provided to illustrate the correctness of the theoretical results and the effectiveness of the algorithms.

## 1. Introduction

Consensus problem has attracted a great deal of attention due to its enormous potential applications in many areas such as flocking and swarming modeling [1], cooperative control of unmanned air vehicles [2], and formation control of multirobot systems [3]. In the past decade, a large number of results have been obtained for consensus problem of various multiagent systems, for example, consensus problems of multiagent systems with different dynamics such as double-integrator dynamics in [4] and Lipschitz nonlinear dynamics in [5], consensus problems of multiagent systems with the different network topologies such as random networks in [6] and switching topology in [7], and consensus problems of multiagent systems with the different time-delay such as nonuniform time-varying delays in [8] and input and communication delays in [9].

In practical applications, multiagent systems often have disturbances, uncertainties and time-delay such as communication noise, and uncertainties in network parameters and time-delay caused by communication or measuring. Moreover, the existence of these facts might destroy the convergence properties of multiagent systems. Therefore, it is significant to investigate robust consensus problems of multiagent systems, which reflects the effects of these facts on the behaviour of multiagent systems. In the past decade, some interesting results have been obtained for robust consensus problems. Lin et al. investigated robust $H_\infty$ consensus analysis of directed networks of first-order agents with time-delay and a class of second-order multiagent systems with uncertainty in [10] and [11], respectively. Li et al. investigated the distributed $H_2$ and $H_\infty$ control problems for multiagent systems with linear or linearized dynamics in [12]. Liu and Jia investigated $H_\infty$ consensus control problem for multiagent systems with linear coupling dynamics and communication delays in [13]. Wang et al. investigated distributed robust $H_\infty$ control problem synthesized with transient performance for a group of autonomous agents governed by uncertain general linear node dynamics in [14]. Hu et al. investigated the $H_\infty$ consensus problem for multiagent systems by means of a simultaneous stabilization approach in [15]. Shi and Qin studied rotational motion of multiagent systems with nonuniform time-delays in [16]. Li et al. investigated distributed robust $H_\infty$ rotating consensus control problem for directed networks of second-order agents with mixed uncertainties and time-delay in [17]. However, in [10, 11, 17], we could obtain the sufficient conditions where the parameters of the control protocols should be satisfied but their calculation method was not directly given. In [12, 13, 15], the parameter uncertainties and time-varying delay were not considered. In [16], uncertainty and external disturbance were not considered. In [14], time-delay was not considered. In addition, in practical applications, on one hand, each agent itself may be a time-delay system, which includes the state delay itself. On the other hand, due to the introduction of the control protocol, the system often contains the input delay. Furthermore, the state and input delays often are nonidentical and time variant, which makes the control problem more challenging. So far, there exists rare work dealing with robust $H_\infty$ consensus control problem for uncertain multiagent systems with nonidentical time-varying state and input delays.

Motivated by the previous observations, this paper investigates robust $H_\infty$ consensus control problem for uncertain multiagent systems, which is subject to parameter uncertainties, external disturbances, and nonidentical time-varying state and input delays. Firstly, a distributed control protocol based on state feedback of neighbors is designed and the closed-loop dynamics is built. By applying matrix theory tools, multiagent system is decoupled and $H_\infty$ consensus control problem for multiagent systems is transformed into $H_\infty$ control problem for the decoupling subsystems. Then employing Lyapunov-Krasovskii functional, Jensen integral inequality, and Newton-Leibnitz formula with free-weighting matrices, a lower conservative bounded real lemma (BRL) is derived in terms of LMIs such that a class of time-delay system is guaranteed to be globally asymptotically stable with the desired $H_\infty$ performance index. Extending BRL, a sufficient delay-dependent condition in terms of the matrix inequalities, whose complexity is lower because the system is decoupled, is obtained to make all agents asymptotically reach consensus with the desired $H_\infty$ performance index. Furthermore, an algorithm is elaborately designed to get feasible solution to this condition by the cone-complementary method. Extending this algorithm, an optimization algorithm for control protocol parameter is proposed to improve the disturbance attenuation capacity or allowable delay bounds. At last, simulation results are provided to illustrate the correctness of the theoretical results and the effectiveness of the algorithms.

The following notations will be used throughout this paper. $\mathbf{0}_n$ denotes the corresponding $n \times 1$ column vectors whose elements are all zeros. $I$ and $0$ denote identity matrix and zero value or zero matrix with appropriate dimensions, respectively. $I_n$ and $0_n$ ($0_{n\times m}$) denote the $n \times n$ identity matrix and the $n \times n$ ($n \times m$) zero matrix, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the $n$-dimensional and the $(n\times m)$-dimensional Euclidean spaces, respectively. $\|\cdot\|$ refers to the standard Euclidean norm for vectors. The notations $\times$ and $\otimes$ denote the vector product and the Kronecker product, respectively. The superscript $T$ and $-1$ stand for matrix transposition and matrix inverse, respectively. In symmetric block matrices, $*$ is used as an ellipsis for terms induced by symmetry. The notation $\mathrm{diag}\{M_1, \ldots, M_n\}$ denotes a block diagonal matrix whose diagonal blocks are given by $M_1, \ldots, M_n$.

## 2. Preliminaries

In this section, some preliminary knowledge of graph theory is introduced for the following analysis (referring to [18]). Let $\mathscr{G}(\mathscr{V}, \mathscr{E}, \mathscr{A})$ be an undirected graph of order $n$, where $\mathscr{V} = \{s_1, \ldots, s_n\}$ is the set of nodes, $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of edges, and $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a weighted adjacency matrix. The node indexes belong to a finite index set $\mathscr{I} = \{1, 2, \ldots, n\}$. An edge of $\mathscr{G}$ is denoted by $e_{ij} = (s_i, s_j)$, where the first element $s_i$ of $e_{ij}$ is said to be the tail of the edge and the other $s_j$ to be the head. The adjacency elements are defined as $a_{ii} = 0$ and $a_{ij} = a_{ji} \geqslant 0$. $a_{ij} > 0$ if and only if there is an edge between node $v_i$ and node $v_j$. The Laplacian of the undirected graph is defined as $L = \Delta - \mathscr{A} \in \mathbb{R}^{n\times n}$, where $\Delta = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ii} = \sum_{j=1}^{n} a_{ij}$. The set of neighbors of node $s_i$ is denoted by $N_i = \{s_j \in \mathscr{V} : (s_i, s_j) \in \mathscr{E}\}$. If there is a path from every node to every other node, the graph is said to be connected.

## 3. Problem Statement

Consider a multiagent system consisting of $n$ agents. Each agent is regarded as a node in a graph $\mathscr{G}$. Each edge $(s_i, s_k) \in \mathscr{E}$ corresponds to an available information channel between agents $s_i$ and $s_k$. Suppose that the $i$th agent $s_i$ ($i \in \mathscr{I}$, $\mathscr{I} \triangleq \{1, 2, \ldots, n\}$) has the dynamics as follows:

$$\begin{aligned} \dot{x}_i(t) &= (A + \Delta A(t))\, x_i(t) \\ &\quad + \left(A_d + \Delta A_d(t)\right) x_i\left(t - d_1(t)\right) \\ &\quad + B_1 u_i\left(t - d_2(t)\right) + B_2 w_i(t), \end{aligned} \tag{1}$$

$$\begin{gathered} 0 < d_1(t) \leqslant \tau_1, \\ \left|\dot{d}_1(t)\right| \leqslant \mu_1, \end{gathered} \tag{2}$$

$$\begin{gathered} 0 < d_2(t) \leqslant \tau_2, \\ \left|\dot{d}_2(t)\right| \leqslant \mu_2, \end{gathered} \tag{3}$$

where $x_i(t) = [x_{i1}, \ldots, x_{im}]^T \in \mathbb{R}^m$ denotes the state of the $i$th agent $s_i$, $u_i(t) = [u_{i1}, \ldots, u_{im_1}]^T \in \mathbb{R}^{m_1}$ denotes the control input (or control protocol) of the $i$th agent $s_i$, and $A$, $A_d$, $B_1$, and $B_2$ are constant matrices with compatible dimensions. $d_1(t)$ denotes the state delay of the system itself, which can be considered as the summation of computation time and execution time. $d_2(t)$ denotes the input delay, which is caused by communication and measuring. $w_i(t) \in \mathbb{R}^{m_2}$ denotes the external disturbances belonging to $\mathscr{L}_2[0, \infty)$. $\Delta A(t)$ and $\Delta A_d(t)$ are matrix valued function representing time-varying parameter uncertainties. The parameter uncertainties are assumed to be norm bounded and $\Delta A(t) = GF(t)E$ and $\Delta A_d(t) = G_d F(t) E_d$, where $G$, $G_d$, $E$, and $E_d$

are known constant matrices with appropriate dimensions, which represent the structure of uncertainties, and $F(t)$ is an unknown matrix function with Lebesgue measurable elements satisfying $F^T(t)F(t) \leqslant I$ for all $t \geqslant 0$.

*Remark 1.* The system model (1) is a more generalized form. On one hand, it can cover the first-order dynamics in [10], second-order dynamics in [4, 11], and high-order dynamics in [19]. On the other hand, compared with [12, 13, 15], it includes the parameter uncertainties and nonidentical time-varying delay, and compared with [8, 9, 16], it includes parameter uncertainties and external disturbance.

*Definition 2.* The multiagent system (1) reaches consensus, if and only if the states of agents satisfy

$$\lim_{t\to+\infty} \left(x_i(t) - x_j(t)\right) = 0 \tag{4}$$

$\forall i, j \in \mathcal{I}$.

Define a vector function $z(t) = [z_1^T(t), \dots, z_n^T(t)]^T \in \mathbb{R}^{mn}$, where $z_i^T(t) = [z_{i1}, \dots, z_{im}]^T \in \mathbb{R}^m$ measures the disagreement of the state of the $i$th agent to the average state of all agents for $i \in \mathcal{I}$ and its value is as follows:

$$z_i(t) = x_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t). \tag{5}$$

Note that if $z(t) = 0$ can be satisfied, then $x_i(t) = x_j(t)$ holds $\forall i, j \in \mathcal{I}$. That is to say, the multiagent system (1) reaches consensus. This implies that $z(t)$ can quantitatively reflect the disagreement degree of all agents on their states and the norm of $z(t)$ indicates consensus performance. Therefore, $z(t)$ is defined as the controlled output of the multiagent system (1) for analysing its consensus behaviour.

Let

$$\begin{aligned} x(t) &= \left[x_1^T(t), \dots, x_n^T(t)\right]^T \in \mathbb{R}^{mn}, \\ u(t) &= \left[u_1^T(t), \dots, u_n^T(t)\right]^T \in \mathbb{R}^{m_1 n}, \\ w(t) &= \left[w_1^T(t), \dots, w_n^T(t)\right]^T \in \mathbb{R}^{m_2 n}. \end{aligned} \tag{6}$$

Then combining the dynamic equation (1) with controlled output defined in (5) yields the following system in matrix form:

$$\begin{aligned} \dot{x}(t) &= \left[I_n \otimes (A + \Delta A(t))\right] x(t) \\ &\quad + \left[I_n \otimes \left(A_d + \Delta A_d(t)\right)\right] x\left(t - d_1(t)\right) \\ &\quad + \left(I_n \otimes B_1\right) u\left(t - d_2(t)\right) + \left(I_n \otimes B_2\right) w(t), \\ z(t) &= \left(C \otimes I_m\right) x(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} C &= \left[C_{ij}\right]_{i,j=1}^{n}, \\ C_{ij} &= \begin{cases} \dfrac{n-1}{n}, & i = j, \\ -\dfrac{1}{n}, & i \neq j. \end{cases} \end{aligned} \tag{8}$$

According to robust control theory, the attenuating ability of consensus performance for the multiagent system (1) against external disturbances can be quantitatively measured by the $H_\infty$ norm of the closed-loop transfer function matrix $T_{wz}(s)$ from the external disturbance $w(t)$ to the controlled output $z(t)$. In order to obtain prescribed performance against external disturbances, we need to design a distributed state feedback protocol $u_i(t)$ such that

$$\left\|T_{wz}(s)\right\|_\infty < \gamma \tag{9}$$

holds for a prescribed $H_\infty$ disturbance attenuation index $\gamma > 0$, or equivalently, the closed-loop system satisfies the following dissipation inequality:

$$\begin{aligned} &\int_0^\infty \left[z^T(t) z(t) - \gamma^2 w^T(t) w(t)\right] dt < 0, \\ &\qquad \forall w(t) \in \mathcal{L}_2[0, \infty). \end{aligned} \tag{10}$$

## 4. Main Results

In this section, we will solve $H_\infty$ consensus control problem for multiagent systems (7). Firstly, we will design a distributed control protocol with an undetermined feedback matrix $K$ and build the closed-loop dynamics of the system. By applying matrix theory tools, multiagent system is decoupled and $H_\infty$ consensus control problem for multiagent systems is transformed into $H_\infty$ control problem for the decoupling subsystems. Then employing Lyapunov-Krasovskii functional and free-weighting matrices, a BRL will be derived in terms of LMIs such that a class of time-delay system is guaranteed to be globally asymptotically stable with the desired $H_\infty$ performance index. Extending BRL, a sufficient delay-dependent condition in terms of the matrix inequalities is obtained to make all agents asymptotically reach consensus with the desired $H_\infty$ performance index. Furthermore, an algorithm is elaborately designed to get feasible solution to this condition. Extending this algorithm, an optimization algorithm for control protocol parameter is proposed to improve the disturbance attenuation capacity or allowable delay bounds.

*4.1. Protocol Design.* To solve $H_\infty$ consensus control problem for the multiagent systems (1), the distributed control protocol is given as

$$u_i(t) = K \sum_{s_j \in N_i} a_{ij}\left(x_j(t) - x_i(t)\right) \tag{11}$$

for $i \in \mathcal{I}$, where $K \in \mathbb{R}^{m_1 \times m}$ is an undetermined feedback matrix and $a_{ij}$ are the adjacency elements of interaction graph $\mathcal{G}$.

By substituting the protocol (11) into system (7), the closed-loop dynamics of system (1) can be written as

$$\begin{aligned} \dot{x}(t) &= \left[I_n \otimes (A + \Delta A(t))\right] x(t) \\ &\quad + \left[I_n \otimes \left(A_d + \Delta A_d(t)\right)\right] x\left(t - d_1(t)\right) \end{aligned}$$

$$\begin{aligned} &- \left[L \otimes \left(B_1 K\right)\right] x\left(t - d_2(t)\right) \\ &+ \left(I_n \otimes B_2\right) w(t) \\ z(t) &= \left(C \otimes I_m\right) x(t), \end{aligned} \tag{12}$$

where $L$ is the Laplacian matrix of the graph $\mathcal{G}$.

On the basis of the above analysis, it is clear that the multiagent system (1) reaches consensus while satisfying the desired $H_\infty$ disturbance attenuation index $\gamma$ by control protocol (11), if and only if the closed-loop system (12) reaches consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$. In the next section, we will find design rules for $K$ such that system (1) reaches consensus with the $H_\infty$ disturbance attenuation index $\gamma$.

*4.2. Some Necessary Lemma*

**Lemma 3** (see [20]). *Let L be the Laplacian of an undirected graph $\mathcal{G}$. Then L has at least one zero eigenvalue and all of the nonzero eigenvalues are positive. Furthermore, matrix L has exactly one zero eigenvalue if and only if the graph $\mathcal{G}$ is connected, and the eigenvector associated with zero is* $\mathbf{1}$.

**Lemma 4** (see [10]). *Consider the matrix C. The following statements hold.*

(a) *The eigenvalues of C are 1 with multiplicity $n-1$ and 0 with multiplicity 1. The vectors $\mathbf{1}_n^T$ and $\mathbf{1}_n$ are the left and right eigenvectors of C associated with the zero eigenvalue, respectively.*

(b) *There exists an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $U^T C U = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ and the column is $\mathbf{1}_n/\sqrt{n}$. Let $\Xi_1 \in \mathbb{R}^{n\times n}$ be the Laplacian of any directed graph; then $U^T \Xi_1 U = \begin{bmatrix} v_1 & 0 \end{bmatrix}, v_1 \in \mathbb{R}^{n\times(n-1)}$.*

*For convenience, denote $U = \begin{bmatrix} U_1 & \overline{U}_1 \end{bmatrix}$, where $\overline{U}_1 = \mathbf{1}_n/\sqrt{n}$ is the last column of U and $U_1 \in \mathbb{R}^{n\times(n-1)}$ is the remaining part.*

**Lemma 5** (Schur complement formula [21]). *For a given symmetric matrix S with the form $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$, $S_{11} \in \mathbb{R}^{r\times r}$, $S_{12} \in \mathbb{R}^{r\times(n-r)}$, and $S_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$; then $S < 0$ if and only if $S_{11} < 0$ and $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$ or $S_{22} < 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.*

*4.3. System Decoupling*

**Theorem 6** (system decoupling). *Assume that the interaction graph $\mathcal{G}$ is connected. For a given $\gamma > 0$, the closed-loop system (12) can reach consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$, if the following $n-1$ subsystems are simultaneously asymptotically stable with $\|T_{\widehat{w}_i \widehat{z}_i}(s)\|_\infty < \gamma$ for $i = 1, 2, \ldots, n-1$.*

$$\begin{aligned} \dot{\widehat{\delta}}_i(t) &= \left(A + \Delta A(t)\right) \widehat{\delta}_i(t) \\ &\quad + \left(A_d + \Delta A_d(t)\right) \widehat{\delta}_i\left(t - d_1(t)\right) \\ &\quad - \lambda_i B_1 K \widehat{\delta}_i\left(t - d_2(t)\right) + B_2 \widehat{w}_i(t), \\ \widehat{z}_i(t) &= \widehat{\delta}_i(t), \end{aligned} \tag{13}$$

*where $\lambda_i$ $(i = 1, 2, \ldots, n-1)$ are the positive eigenvalues of the Laplacian matrix L and $\widehat{\delta}_i(t)$, $\widehat{w}_i(t)$, $\widehat{z}_i(t) \in \mathbb{R}^m$.*

*Proof.* See Appendix. □

*4.4. Delay-Dependent $H_\infty$ Control.* In order to investigate the $H_\infty$ stability, let us consider a class of time-delay system as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x\left(t - d(t)\right) + A_h x\left(t - h(t)\right) \\ &\quad + B_w w(t), \\ z(t) &= Cx(t), \\ 0 &< d(t) \leqslant \tau_1, \\ \left|\dot{d}(t)\right| &\leqslant \mu_1, \\ 0 &< h(t) \leqslant \tau_2, \\ \left|\dot{h}(t)\right| &\leqslant \mu_2. \end{aligned} \tag{14}$$

A bounded real lemma (BRL) will be derived in the following part.

**Lemma 7** (BRL). *For a given $\gamma > 0$, the time-delay system (14) is asymptotically stable with $\|T_{wz}(s)\|_\infty < \gamma$, if there exist positive definite matrices $P$, $Q_1$, $Q_2$, $R_1$, $R_2$, $S_1$, and $S_2$ and matrices $M_1$, $M_2$, $N_1$, $N_2$, $U_1$, $U_2$, $V_1$, and $V_2$ with appropriate dimensions such that*

$$\Gamma = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 \\ * & -S & 0 & 0 & 0 & 0 \\ * & * & -\tau_1 S_1 & 0 & 0 & 0 \\ * & * & * & -\tau_1 S_1 & 0 & 0 \\ * & * & * & * & -\tau_2 S_2 & 0 \\ * & * & * & * & * & -\tau_2 S_2 \end{bmatrix} < 0, \tag{15}$$

*where $S = \tau_1 S_1 + \tau_2 S_2$,*

$$\Gamma_0 = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & -M_1 & -U_1 & PB_w \\ * & \Gamma_{22} & 0 & -M_2 & 0 & 0 \\ * & * & \Gamma_{33} & 0 & -U_2 & 0 \\ * & * & * & -R_1 & 0 & 0 \\ * & * & * & * & -R_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\Gamma_1 = \left[SA, SA_d, SA_h, 0, 0, SB_w\right]^T,$$
$$\Gamma_2 = \left[\tau_1 M_1^T, \tau_1 M_2^T, 0, 0, 0, 0\right]^T,$$
$$\Gamma_3 = \left[\tau_1 N_1^T, \tau_1 N_2^T, 0, 0, 0, 0\right]^T,$$
$$\Gamma_4 = \left[\tau_2 U_1^T, 0, \tau_2 U_2^T, 0, 0, 0\right]^T,$$
$$\Gamma_5 = \left[\tau_2 V_1^T, 0, \tau_2 V_2^T, 0, 0, 0\right]^T,$$
$$\Gamma_{11} = PA + A^T P + Q_1 + Q_2 + R_1 + R_2 + N_1 + N_1^T + V_1 + V_1^T + C^T C,$$
$$\Gamma_{12} = PA_d + M_1 - N_1 + N_2^T,$$
$$\Gamma_{13} = PA_h + U_1 - V_1 + V_2^T,$$
$$\Gamma_{22} = (\mu_1 - 1) Q_1 + M_2 + M_2^T - N_2 - N_2^T,$$
$$\Gamma_{33} = (\mu_2 - 1) Q_2 + U_2 + U_2^T - V_2 - V_2^T. \quad (16)$$

*Proof.* See Appendix. □

**Theorem 8.** *Assume that the interaction graph $\mathcal{G}$ is connected. For given positive scalar constants $\gamma$, by distributed protocol (11), the multiagent system (1) can reach consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$, if there exist positive definite matrices $Q_1, Q_2, R_1, R_2, S_1, S_2, T, X \in \mathbb{R}^{m\times m}$, matrices $M_1, M_2, N_1, N_2, U_1, U_2, V_1, V_2 \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{m_1\times m}$, and positive scalars $\varepsilon_1$, $\varepsilon_2$ such that*

$$\Phi^{(i)} = \begin{bmatrix} \Phi_0^{(i)} & \Pi_1^{(i)T} & \Pi_2^T & H^T & H_d^T \\ * & -T + \varepsilon_1 GG^T + \varepsilon_2 G_d G_d^T & 0 & 0 & 0 \\ * & * & -XT^{-1}X & 0 & 0 \\ * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad (17)$$

*where*

$$\Phi_0^{(i)} = \begin{bmatrix} \Phi_{0x}^{(i)} & 0 & \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 \\ * & -S & 0 & 0 & 0 & 0 & 0 \\ * & * & -\tau_1 S_1 & 0 & 0 & 0 & 0 \\ * & * & * & -\tau_1 S_1 & 0 & 0 & 0 \\ * & * & * & * & -\tau_2 S_2 & 0 & 0 \\ * & * & * & * & * & -\tau_2 S_2 & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

$$\Phi_{0x}^{(i)} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13}^{(i)} & -M_1 & -U_1 & B_2 \\ * & \Phi_{22} & 0 & -M_2 & 0 & 0 \\ * & * & \Phi_{33} & 0 & -U_2 & 0 \\ * & * & * & -R_1 & 0 & 0 \\ * & * & * & * & -R_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$
$$S = \tau_1 S_1 + \tau_2 S_2,$$
$$\Phi_{11} = AX + XA^T + Q_1 + Q_2 + R_1 + R_2 + N_1 + N_1^T + V_1 + V_1^T + \varepsilon_1 GG^T + \varepsilon_2 G_d G_d^T,$$
$$\Phi_{12} = A_d X + M_1 - N_1 + N_2^T,$$
$$\Phi_{13}^{(i)} = -\lambda_i B_1 Y + U_1 - V_1 + V_2^T,$$
$$\Phi_{22} = (\mu_1 - 1) Q_1 + M_2 + M_2^T - N_2 - N_2^T,$$
$$\Phi_{33} = (\mu_2 - 1) Q_2 + U_2 + U_2^T - V_2 - V_2^T,$$
$$\Phi_1 = \left[\tau_1 M_1^T, \tau_1 M_2^T, 0, 0, 0, 0\right]^T,$$
$$\Phi_2 = \left[\tau_1 N_1^T, \tau_1 N_2^T, 0, 0, 0, 0\right]^T,$$
$$\Phi_3 = \left[\tau_2 U_1^T, 0, \tau_2 U_2^T, 0, 0, 0\right]^T,$$
$$\Phi_4 = \left[\tau_2 V_1^T, 0, \tau_2 V_2^T, 0, 0, 0\right]^T,$$
$$\Phi_5 = [X, 0, 0, 0, 0, 0]^T,$$
$$\Pi_1^{(i)} = \left[AX + \varepsilon_1 GG^T + \varepsilon_2 G_d G_d^T, A_d X, -\lambda_i B_1 Y, 0, 0, B_2, 0, 0, 0, 0, 0, 0\right],$$
$$\Pi_2 = [0, 0, 0, 0, 0, 0, \tau_1 S_1 + \tau_2 S_2, 0, 0, 0, 0, 0],$$
$$H = [EX, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$H_d = \left[0, E_d X, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]. \quad (18)$$

*Furthermore, the feedback matrix $K$ in the proposed control protocol (11) can be designed by $K = YX^{-1}$.*

*Proof.* See Appendix. □

*4.5. Algorithm of Solving Delay-Dependent Condition.* In the following section, an algorithm will be designed to get the feasible solution to the matrix inequality condition (17).

Due to the existence of nonlinear entry $XT^{-1}X$, the matrix inequality condition (17) is not yet in the form of an LMI. Therefore, we cannot directly use LMI method to solve the matrix inequality (17). But we can turn this problem into the LMI optimization problem by the cone-complementary linearization algorithm [22].

Define a new positive definite matrix variable $W$ such that $W \leqslant XT^{-1}X$. It is easy to derive that $W^{-1} - X^{-1}TX^{-1} \geqslant 0$. Furthermore, by defining $\overline{X} \triangleq X^{-1}$, $\overline{T} \triangleq T^{-1}$, and $\overline{W} \triangleq W^{-1}$ and using Lemma 5 (Schur complement formula), we can turn condition (17) into

$$\begin{bmatrix} \Phi_0^{(i)} & \Pi_1^{(i)T} & \Pi_2^T & H^T & H_d^T \\ * & -T + \varepsilon_1 GG^T + \varepsilon_2 G_d G_d^T & 0 & 0 & 0 \\ * & * & -W & 0 & 0 \\ * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{19}$$

$$\begin{gathered} \begin{bmatrix} \overline{W} & \overline{X} \\ \overline{X} & \overline{T} \end{bmatrix} \geqslant 0, \\ \begin{bmatrix} \overline{X} & I \\ I & X \end{bmatrix} \geqslant 0, \\ \begin{bmatrix} \overline{T} & I \\ I & T \end{bmatrix} \geqslant 0, \\ \begin{bmatrix} \overline{W} & I \\ I & W \end{bmatrix} \geqslant 0. \end{gathered} \tag{20}$$

Then, we may solve the following optimization problem and find a feasible solution satisfying $W \leqslant XT^{-1}X$:

$$\begin{aligned} \min \quad & \operatorname{Trace}\left(\overline{X}X + \overline{T}T + \overline{W}W\right) \\ \text{s.t.} \quad & (19), (20). \end{aligned} \tag{21}$$

In a word, in order to solve the matrix inequality condition (17), an algorithm is designed as follows.

*Algorithm 9.* (1) Solve the LMI (19) and (20) for given positive scalar constants $\tau_1$, $\tau_2$, $\mu_1$, $\mu_2$, and $\gamma$. There exists a feasible solution set $\{\overline{X}_0, X_0, \overline{T}_0, T_0, \overline{W}_0, W_0\}$ and set $k = 0$.

(2) Solve the following optimization problem for the variables

$$\begin{aligned} & \left\{\overline{X}, X, \overline{T}, T, \overline{W}, W\right\} \\ \min \quad & \operatorname{Trace}\left(\overline{X}_k X + \overline{T}_k T + \overline{W}_k W + \overline{X} X_k + \overline{T} T_k + \overline{W} W_k\right) \\ \text{s.t.} \quad & (19), (20) \end{aligned} \tag{22}$$

and set $\overline{X}_{k+1} = \overline{X}$, $X_{k+1} = X$, $\overline{T}_{k+1} = \overline{T}$, $T_{k+1} = T$, $\overline{W}_{k+1} = \overline{W}$, and $W_{k+1} = W$.

(3) If $W \leqslant XT^{-1}X$ for the above solution set, then save the current $X$, $Y$ and exit. Otherwise, set $k = k + 1$, go to Step (2), and repeat the optimization for a prescribed maximum iterative number $k_{\max}$ until finding a feasible solution satisfying $W \leqslant XT^{-1}X$. If such a solution does not exist, then exit.

If a feasible solution set is found by Algorithm 9, given the disturbance attenuation index $\gamma$ and delay parameters $\tau_1$, $\tau_2$, $\mu_1$, and $\mu_2$, by distributed protocol (11), the multiagent system (1) can reach consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$ and the feedback matrix can be constructed by $K = YX^{-1}$.

*4.6. Optimization for $H_\infty$ Control Parameters.* According to Theorem 8, assume that delay parameters $\tau_1$, $\tau_2$, $\mu_1$, and $\mu_2$ have been given; in order to achieve a minimum $H_\infty$ disturbance attenuation index $\gamma$, we can solve the following optimization problem for the positive definite decision variables $Q_1, Q_2, R_1, R_2, S_1, S_2, T, X \in \mathbb{R}^{m\times m}$, matrices $M_1, M_2, N_1, N_2, U_1, U_2, V_1, V_2 \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{m_1\times m}$, and positive scalars $\varepsilon_1, \varepsilon_2$:

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & (17). \end{aligned} \tag{23}$$

Because the matrix inequality condition (17) is not yet in the form of an LMI, we cannot find global minima for the optimization problem (23) using convex optimization algorithms. However, by the similar methods to Algorithm 9, we may still obtain a suboptimal controller for the optimization problem (23) using an iterative algorithm presented in the following sequel.

*Algorithm 10.* (1) Assume that the delay parameters $\tau_1$, $\tau_2$, $\mu_1$, and $\mu_2$ have been given. Choose a sufficiently large initial $\gamma$ such that there exists a feasible solution set $\{\overline{X}_0, X_0, \overline{T}_0, T_0, \overline{W}_0, W_0\}$ to the LMIs (19) and (20). Set $k = 0$.

(2) Solve the optimization problem (22) and set $\overline{X}_{k+1} = \overline{X}$, $X_{k+1} = X$, $\overline{T}_{k+1} = \overline{T}$, $T_{k+1} = T$, $\overline{W}_{k+1} = \overline{W}$, and $W_{k+1} = W$.

(3) If $W \leqslant XT^{-1}X$ for the above solution set, then set $\gamma_{\min} = \gamma$ and return to Step (1) after decreasing $\gamma = \gamma - \Delta\gamma$, where $\Delta\gamma$ is predefined step-sizes. Otherwise, set $k = k + 1$, go to Step (2), and repeat the optimization for a prescribed maximum iterative number $k_{\max}$ until finding a feasible solution satisfying $W \leqslant XT^{-1}X$. If such a solution does not exist, then exit.

If a feasible solution set is found by Algorithm 10, for the given delay parameters $\tau_1$, $\tau_2$, $\mu_1$, and $\mu_2$, by distributed protocol (11), the multiagent system (1) can reach consensus with the suboptimal $H\infty$ disturbance attenuation index $\gamma_{\min}$ and the suboptimal feedback matrix can be constructed by $K = YX^{-1}$.

*Remark 11.* Assuming that the disturbance attenuation index $\gamma$ and delay parameters $\mu_1$ (or the proportion coefficient of $\mu_1$ and $\tau_1$), $\tau_2$, and $\mu_2$ have been given, we can raise the maximum allowable delay bound to $\tau_{1\max}$ and the corresponding feedback matrix $K = YX^{-1}$ by the similar methods to Algorithm 10. Similarly, we also can raise the maximum allowable delay bound to $\tau_{2\max}$ and the corresponding feedback matrix $K = YX^{-1}$.

## 5. Simulation Results

To illustrate the correctness of the theoretical results and the effectiveness of the algorithms, numerical simulations will be given in this section. Consider a multiagent system with four agents and the dynamics of each agent is described by (1), with

$$A_d = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0.2 & 0.5 \\ 0 & 0.2 \end{bmatrix}, \quad (24)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix},$$

$$G = G_d = \begin{bmatrix} 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.02 & 0 \\ 0 & 0 & 0 & 0.03 \\ 0 & 0 & 0.01 & 0 \end{bmatrix},$$

$$F(t) = \operatorname{diag}\{\sin 10t, \sin 20t, \cos 10t, \cos 20t\},$$

$$E = E_d = I_4.$$

The connected interaction graph $\mathcal{G}$ is shown in Figure 1, whose nonzero weighting factors $a_{ij}$ are all 1. So the corresponding Laplacian matrix is

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \quad (25)$$

So its smallest and largest nonzero eigenvalues are $\lambda_1 = 2$ and $\lambda_3 = 4$, respectively. We assume that the external disturbance

$$w(t) = [1, 1.2, 0.8, 1.5, 1, 1.4, 2, 1.2]^T \varepsilon(t),$$

$$\varepsilon(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1 \\ 0, & \text{otherwise} \end{cases} \quad (26)$$

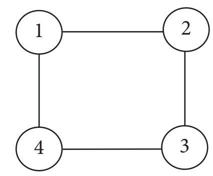


Figure 1: The communication topology.

and the initial state of the multiagent system is taken as $x(t = 0) = \mathbf{0}_{16}$.

*Example 12.* Suppose that the $H_\infty$ performance index $\gamma = 1$, the delay $d_1(t) = 0.1\sin t$, and $d_2(t) = 0.15\cos t$ (so $\tau_1 = \mu_1 = 0.1$ and $\tau_2 = \mu_2 = 0.15$). By Theorem 8 and Algorithm 9, we can figure out the feedback matrix $K$ in the control protocol (11) as follows:

$$K = YX^{-1} = \begin{bmatrix} 0.5103 & 1.0701 & 0.8234 & -0.1057 \\ -0.3108 & -0.6331 & -0.3141 & 0.8557 \end{bmatrix}. \quad (27)$$

On the one hand, Figure 2 shows the trajectories of $z_i(t) = [z_{i1}, \ldots, z_{i4}]^T$ when $\gamma = 1$, $d_1(t) = 0.1\sin t$, and $d_2(t) = 0.15\cos t$ for all $i = 1, 2, 3, 4$, where $z_{ik} = x_{ik}(t) - (1/n)\sum_{j=1}^{n} x_{jk}(t)$, $k = 1, 2, 3, 4$. It is clear that consensus is asymptotically achieved for the multiagent systems. On the other hand, Figure 3 shows the energy relation of the controlled output and the external disturbance. Obviously, the closed-loop system satisfies $\int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt < 0$; that is, the $H_\infty$ disturbance attenuation index is achieved. Therefore, applying the distributed protocol (11) and calculating the feedback matrix $K$ by Theorem controller synthesis theorem and Algorithm 9, the multiagent system can reach consensus while satisfying the desired $H_\infty$ disturbance attenuation index $\gamma = 1$. So we validate the correctness of Theorem 8 and the effectiveness of Algorithm 9.

*Example 13.* Suppose that the delay $d_1(t) = 0.1\sin t$ and $d_2(t) = 0.15\cos t$ (so $\tau_1 = \mu_1 = 0.1$ and $\tau_2 = \mu_2 = 0.15$). Using the distributed protocol (11) and optimizing the feedback matrix $K$ by Algorithm 10, we can figure out the suboptimal $H_\infty$ disturbance attenuation index $\gamma_{\min} = 0.7$ and the corresponding feedback matrix as follows:

$$K = YX^{-1} = \begin{bmatrix} 0.6677 & 1.2483 & 0.7142 & -0.1105 \\ -0.3815 & -0.6790 & -0.1872 & 0.8701 \end{bmatrix}. \quad (28)$$

Figure 3 shows the energy relation of the controlled output and the external disturbance. Obviously, the $H_\infty$ disturbance attenuation index is smaller than its value in Example 12. That is to say, by Algorithm 10, the disturbance attenuation capacity of the multiagent system (1) is obviously improved. So the effectiveness of Algorithm 10 is validated.

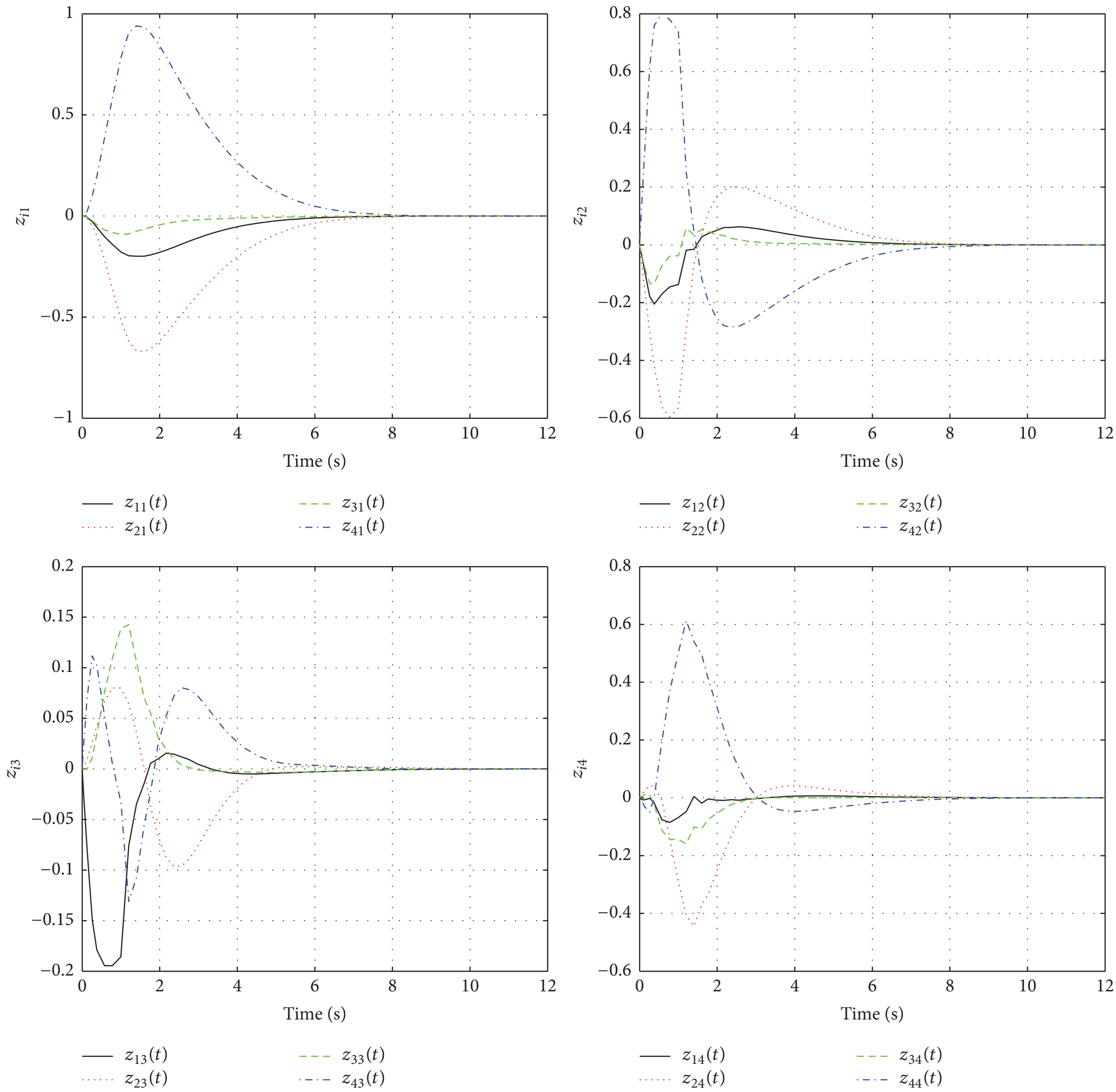


FIGURE 2: The trajectories of $z_i(t) = [z_{i1}, \ldots, z_{i4}]^T$ in Example 12.

## 6. Conclusion

This paper studies robust $H_\infty$ consensus control problem for multiagent systems with parameter uncertainties, external disturbances, nonidentical time-varying state, and input delays. The main contributions of this paper are as follows: first, considering a more generalized form, which includes parameter uncertainties, external disturbances, nonidentical time-varying state, and input delays; second, a lower conservative BRL; third, a lower complexity delay-dependent control condition; fourth, an algorithm for calculating parameters of control protocol; fifth, an optimization algorithm for improving the disturbance attenuation capacity or allowable delay bounds. In further research, we will research robust $H_\infty$ consensus control problem for multiagent systems with heterogeneous uncertainties and nonuniform time-varying delays.

## Appendix

*Proof of Theorem 6.* Let

$$\delta(t) = x(t) - \frac{\mathbf{1}_n}{n} \otimes \left( \sum_{i=1}^{n} x_i(t) \right)$$
$$= (C \otimes I_m)\, x(t),$$

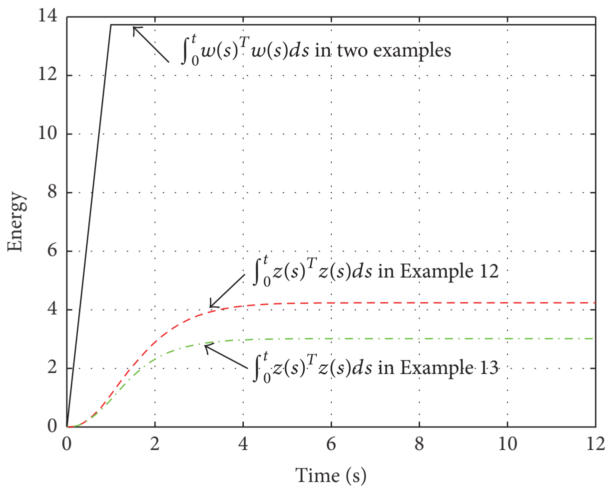


Figure 3: The energy trajectories of the controlled output $z(t)$ and the external disturbance $w(t)$ in four examples.

$$\begin{bmatrix} \bar{\delta}(t) \\ \bar{\delta}_n(t) \end{bmatrix} = \left(U^T \otimes I_m\right) \delta(t),$$
$$\begin{bmatrix} \overline{w}(t) \\ \overline{w}_n(t) \end{bmatrix} = \left(U^T \otimes I_m\right) w(t),$$
$$\begin{bmatrix} \bar{z}(t) \\ \bar{z}_n(t) \end{bmatrix} = \left(U^T \otimes I_m\right) z(t). \tag{A.1}$$

By Lemmas 3 and 4, we know that 0 is a common eigenvalue of $L$ and $C$ when the graph $\mathcal{G}$ is connected, and $\overline{U}_1 = \mathbf{1}_n/\sqrt{n}$ is a common eigenvector associated with 0. Therefore, it is easy to derive that $L\mathbf{1}_n = C\mathbf{1}_n = 0$ and $U^T CLU = U^T CUU^T LU = \left[\begin{smallmatrix} U_1^T LU_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Denote $\overline{L} = U_1^T LU_1$, $A_1 = A + \Delta A(t)$, and $A_2 = A_d + \Delta A_d(t)$. Then combining with system (12), we have

$$\begin{aligned}
\begin{bmatrix} \dot{\bar{\delta}}(t) \\ \dot{\bar{\delta}}_n(t) \end{bmatrix} &= \left(U^T \otimes I_m\right)\left(C \otimes I_m\right)\dot{x}(t) = \left(U^T C \otimes I_m\right) \\
&\cdot \left[\left(I_n \otimes A_1\right) x(t) + \left(I_n \otimes A_2\right) x\left(t - d_1(t)\right) - L\right. \\
&\left.\otimes \left(B_1 K\right) x\left(t - d_2(t)\right) + \left(I_n \otimes B_2\right) w(t)\right] = \left(U^T C\right. \\
&\left.\otimes A_1\right)\left[\delta(t) + \frac{\mathbf{1}_n}{n} \otimes \left(\sum_{i=1}^n x_i(t)\right)\right] + \left(U^T C \otimes A_2\right) \\
&\cdot \left[\delta\left(t - d_1(t)\right) + \frac{\mathbf{1}_n}{n} \otimes \left(\sum_{i=1}^n x_i\left(t - d_1(t)\right)\right)\right] \\
&- \left[U^T CL \otimes \left(B_1 K\right)\right]\left[\delta\left(t - d_2(t)\right) + \frac{\mathbf{1}_n}{n}\right. \\
&\left.\otimes \left(\sum_{i=1}^n x_i\left(t - d_2(t)\right)\right)\right] + \left(U^T C \otimes B_2\right) w(t) \\
&= \left(U^T C \otimes A_1\right)\left(U \otimes I_m\right)\begin{bmatrix} \bar{\delta}(t) \\ \bar{\delta}_n(t) \end{bmatrix} + \left(U^T C \otimes A_2\right) \\
&\cdot \left(U \otimes I_m\right)\begin{bmatrix} \bar{\delta}\left(t - d_1(t)\right) \\ \bar{\delta}_n\left(t - d_1(t)\right) \end{bmatrix} - \left[U^T CL \otimes \left(B_1 K\right)\right] \\
&\cdot \left(U \otimes I_m\right)\begin{bmatrix} \bar{\delta}\left(t - d_2(t)\right) \\ \bar{\delta}_n\left(t - d_2(t)\right) \end{bmatrix} + \left(U^T C \otimes B_2\right)\left(U\right. \\
&\left.\otimes I_m\right)\begin{bmatrix} \overline{w}(t) \\ \overline{w}_n(t) \end{bmatrix} = \left(\begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \otimes A_1\right)\begin{bmatrix} \bar{\delta}(t) \\ \bar{\delta}_n(t) \end{bmatrix} \\
&+ \left(\begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \otimes A_2\right)\begin{bmatrix} \bar{\delta}\left(t - d_1(t)\right) \\ \bar{\delta}_n\left(t - d_1(t)\right) \end{bmatrix} \\
&- \left(\begin{bmatrix} \overline{L} & 0 \\ 0 & 0 \end{bmatrix} \otimes B_1 K\right)\begin{bmatrix} \bar{\delta}\left(t - d_2(t)\right) \\ \bar{\delta}_n\left(t - d_2(t)\right) \end{bmatrix} \\
&+ \left(\begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \otimes B_2\right)\begin{bmatrix} \overline{w}(t) \\ \overline{w}_n(t) \end{bmatrix},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\begin{bmatrix} \bar{z}(t) \\ \bar{z}_n(t) \end{bmatrix} &= \left(U^T \otimes I_m\right) z(t) = \left(U^T \otimes I_m\right)\left(C \otimes I_m\right) \\
&\cdot x(t) = \left(U^T C \otimes I_m\right)\left[\delta(t) + \frac{\mathbf{1}_n}{n} \otimes \left(\sum_{i=1}^n x_i(t)\right)\right] \\
&= \left(U^T \otimes I_m\right)\left(U \otimes I_m\right)\begin{bmatrix} \bar{\delta}(t) \\ \bar{\delta}_n(t) \end{bmatrix} = \left(\begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}\right. \\
&\left.\otimes I_m\right)\begin{bmatrix} \bar{\delta}(t) \\ \bar{\delta}_n(t) \end{bmatrix}.
\end{aligned} \tag{A.3}$$

Due to the fact that the rows of $\dot{\bar{\delta}}_n(t)$ and $\bar{z}_n(t)$ are all 0 in (A.2) and (A.3), we can obtain the reduced-order system (A.4) that is equivalent to a system, which is made up of (A.2) and (A.3), regarding the $H_\infty$ performance. Considering that $\bar{\delta}(t) = (U_1^T \otimes I_m)\delta(t)$, $\delta(t) = x(t) - (\mathbf{1}_n/n) \otimes (\sum_{i=1}^n x_i(t))$, and $U_1^T U_1 = I_{n-1}$, it is easy to derive that $\bar{\delta}(t) = \mathbf{0}$ leads to $x(t) = (\mathbf{1}_n/n) \otimes (\sum_{i=1}^n x_i(t))$. So system (12) reaches consensus if system (A.4) is asymptotically stable. In addition, by $\overline{w}(t) = (U_1^T \otimes I_m)w(t)$, $\bar{z}(t) = (U_1^T \otimes I_m)z(t)$, and (A.2) and (A.3), it can be easily proved that $T_{\overline{w}\bar{z}}(s) = (U_1^T \otimes I_m)T_{wz}(U_1^T \otimes I_m)$. So $\|T_{wz}(s)\|_\infty = \|T_{\overline{w}\bar{z}}(s)\|_\infty$.

Therefore, the closed-loop system (12) reaches consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$

($\|T_{wz}(s)\|_\infty < \gamma$), if the following system is asymptotically stable with $\|T_{\overline{w}\,\overline{z}}(s)\|_\infty < \gamma$.

$$\begin{aligned}\dot{\overline{\delta}}(t) &= \left[I_{n-1} \otimes (A + \Delta A(t))\right] \overline{\delta}(t) \\ &\quad + \left[I_{n-1} \otimes \left(A_d + \Delta A_d(t)\right)\right] \overline{\delta}\left(t - d_1(t)\right) \\ &\quad - \left[\overline{L} \otimes \left(B_1 K\right)\right] \overline{\delta}\left(t - d_2(t)\right) \\ &\quad + \left(I_{n-1} \otimes B_2\right) \overline{w}(t), \\ \overline{z}(t) &= \overline{\delta}(t),\end{aligned} \tag{A.4}$$

where $\overline{L} = U_1^T L U_1$ and $\overline{\delta}(t), \overline{w}(t), \overline{z}(t) \in \mathbb{R}^{m(n-1)}$.

When the interaction graph $\mathscr{G}$ is connected, the matrix $\overline{L} = U_1^T L U_1$ is positive definite and its eigenvalues are the positive eigenvalues of the Laplacian matrix $L$ by Lemmas 3 and 4. Then there exists an orthogonal matrix $U_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$U_2^T \overline{L} U_2 = \operatorname{diag}\{\lambda_1, \ldots, \lambda_{n-1}\} \triangleq \Lambda, \tag{A.5}$$

where $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_{n-1}$.

Let

$$\begin{aligned}\widehat{\delta}(t) &= \left(U_2^T \otimes I_m\right) \overline{\delta}(t) \triangleq \left[\widehat{\delta}_1^T(t), \ldots, \widehat{\delta}_{n-1}^T(t)\right]^T, \\ \widehat{w}(t) &= \left(U_2^T \otimes I_m\right) \overline{w}(t) \triangleq \left[\widehat{w}_1^T(t), \ldots, \widehat{w}_{n-1}^T(t)\right]^T, \\ \widehat{z}(t) &= \left(U_2^T \otimes I_m\right) \overline{z}(t) \triangleq \left[\widehat{z}_1^T(t), \ldots, \widehat{z}_{n-1}^T(t)\right]^T.\end{aligned} \tag{A.6}$$

Then system (A.4) can be rewritten in terms of $\widehat{\delta}(t)$, $\widehat{w}(t)$, and $\widehat{z}(t)$ as follows:

$$\begin{aligned}\dot{\widehat{\delta}}(t) &= \left[I_{n-1} \otimes (A + \Delta A(t))\right] \widehat{\delta}(t) \\ &\quad + \left[I_{n-1} \otimes \left(A_d + \Delta A_d(t)\right)\right] \widehat{\delta}\left(t - d_1(t)\right) \\ &\quad - \left[\Lambda \otimes \left(B_1 K\right)\right] \widehat{\delta}\left(t - d_2(t)\right) \\ &\quad + \left(I_{n-1} \otimes B_2\right) \widehat{w}(t), \\ \overline{z}(t) &= \widehat{\delta}(t).\end{aligned} \tag{A.7}$$

By the variable substitution in (A.6), we know that $\overline{\delta}(t)$ is asymptotically stable if $\widehat{\delta}(t)$ is asymptotically stable. In addition, it is easy to derive that $T_{\widehat{w}\widehat{z}} = (U_2^T \otimes I_m) T_{\overline{w}\,\overline{z}} (U_2 \otimes I_m)$. So $\|T_{\widehat{w}\,\widehat{z}}(s)\|_\infty = \|T_{\overline{w}\,\overline{z}}(s)\|_\infty$. Therefore, system (A.4) is asymptotically stable with $\|T_{\overline{w}\,\overline{z}}(s)\|_\infty < \gamma$, if system (A.7) is asymptotically stable with $\|T_{\widehat{w}\,\widehat{z}}(s)\|_\infty < \gamma$.

Due to the diagonal property of matrix $\Lambda$, system (A.7) realises the complete decoupling of state variables $\widehat{\delta}_i(t)$. So system (A.7) is asymptotically stable if its $n-1$ diagonal subsystems (13) are simultaneously asymptotically stable. Besides that, it follows from (13), (A.6), and (A.7) that $T_{\widehat{w}\widehat{z}} = \operatorname{diag}\{T_{\widehat{w}_1\widehat{z}_1}, \ldots, T_{\widehat{w}_{n-1}\widehat{z}_{n-1}}\}$, which implies that $\|T_{\widehat{w}\widehat{z}}(s)\|_\infty = \max_{i=1,\ldots,n-1} \|T_{\widehat{w}_i\widehat{z}_i}(s)\|_\infty$. Hence system (A.7) is asymptotically stable with $\|T_{\widehat{w}\,\widehat{z}}(s)\|_\infty < \gamma$, if its $n-1$ diagonal subsystems (13) are simultaneously asymptotically stable with $\|T_{\widehat{w}_i\widehat{z}_i}(s)\|_\infty < \gamma$ for $i = 1, 2, \ldots, n-1$.

In conclusion, assume that the interaction graph $\mathscr{G}$ is connected; the closed-loop system (12) can reach consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$, if the $n-1$ systems (13) are simultaneously asymptotically stable with $\|T_{\widehat{w}_i\widehat{z}_i}(s)\|_\infty < \gamma$ for $i = 1, 2, \ldots, n-1$. This completes the proof. □

*Proof of Lemma 7.* Let us define a Lyapunov-Krasovskii function for the multiagent system (14) as $V(x(t), t) = \sum_{i=1}^{7} V_i(t)$ with positive definite matrices $P$, $Q_1$, $Q_2$, $R_1$, $R_2$, $S_1$, and $S_2$, where

$$\begin{aligned}V_1(t) &= x^T(t) P x(t), \\ V_2(t) &= \int_{t-d_1(t)}^{t} x^T(s) Q_1 x(s)\, ds, \\ V_3(t) &= \int_{t-\tau_1}^{t} x^T(s) R_1 x(s)\, ds, \\ V_4(t) &= \int_{-\tau_1}^{0} \int_{t+\beta}^{t} \dot{x}^T(s) S_1 \dot{x}(s)\, ds\, d\beta, \\ V_5(t) &= \int_{t-d_2(t)}^{t} x^T(s) Q_2 x(s)\, ds, \\ V_6(t) &= \int_{t-\tau_2}^{t} x^T(s) R_2 x(s)\, ds, \\ V_7(t) &= \int_{-\tau_2}^{0} \int_{t+\beta}^{t} \dot{x}^T(s) S_2 \dot{x}(s)\, ds\, d\beta.\end{aligned} \tag{A.8}$$

The time derivative of $V(x(t), t)$ along the solution of system (14) is $\dot{V}(x(t), t) = \sum_{i=1}^{7} \dot{V}_i(t)$, where $\dot{V}_i(t)$ is as follows:

$$\begin{aligned}\dot{V}_1(t) &= 2x^T(t) P \left(Ax(t) + A_d x(t - d(t))\right. \\ &\quad \left. + A_h x(t - h(t)) + B_w w(t)\right),\end{aligned} \tag{A.9}$$

$$\begin{aligned}\dot{V}_2(t) &= x^T(t) Q_1 x(t) + \left(\dot{d}(t) - 1\right) x^T(t - d(t)) \\ &\quad \cdot Q_1 x(t - d(t)) \leqslant x^T(t) Q_1 x(t) + \left(\mu_1 - 1\right) x^T(t \\ &\quad - d(t)) Q_1 x(t - d(t)),\end{aligned} \tag{A.10}$$

$$\dot{V}_3(t) = x^T(t) R_1 x(t) - x^T\left(t - \tau_1\right) R_1 x\left(t - \tau_1\right), \tag{A.11}$$

$$\begin{aligned}\dot{V}_4(t) &= \tau_1 \left(Ax(t) + A_d x(t - d(t))\right. \\ &\quad \left. + A_h x(t - h(t)) + B_w w(t)\right)^T S_1 \left(Ax(t)\right. \\ &\quad \left. + A_d x(t - d(t)) + A_h x(t - h(t)) + B_w w(t)\right) \\ &\quad - \int_{t-\tau_1}^{t} \dot{x}^T(s) S_1 \dot{x}(s)\, ds = \tau_1 \left(Ax(t)\right. \\ &\quad \left. + A_d x(t - d(t)) + A_h x(t - h(t)) + B_w w(t)\right)^T \\ &\quad \cdot S_1 \left(Ax(t) + A_d x(t - d(t)) + A_h x(t - h(t))\right.\end{aligned}$$

$$+ B_w w(t)) - \int_{t-\tau_1}^{t-d(t)} \dot{x}^T(s) S_1 \dot{x}(s)\, ds - \int_{t-d(t)}^{t} \dot{x}^T(s) S_1 \dot{x}(s)\, ds, \tag{A.12}$$

and similarly,

$$\dot{V}_5(t) \leqslant x^T(t) Q_2 x(t) + (\mu_2 - 1) x^T(t-h(t)) \cdot Q_2 x(t-h(t)), \tag{A.13}$$

$$\dot{V}_6(t) = x^T(t) R_2 x(t) - x^T(t-\tau_2) R_2 x(t-\tau_2), \tag{A.14}$$

$$\begin{aligned}\dot{V}_7(t) &= \tau_2 (Ax(t) + A_d x(t-d(t)) \\ &\quad + A_h x(t-h(t)) + B_w w(t))^T S_2 (Ax(t) \\ &\quad + A_d x(t-d(t)) + A_h x(t-h(t)) + B_w w(t)) \\ &\quad - \int_{t-\tau_2}^{t-h(t)} \dot{x}^T(s) S_2 \dot{x}(s)\, ds \\ &\quad - \int_{t-h(t)}^{t} \dot{x}^T(s) S_2 \dot{x}(s)\, ds.\end{aligned} \tag{A.15}$$

For any matrices $M_1$, $M_2$, $N_1$, and $N_2$ with appropriate dimensions, using Newton-Leibnitz formula with free-weighting matrices, we can construct the following null equations:

$$\begin{aligned}&2\left(x^T(t) M_1 + x^T(t-d(t)) M_2\right) \\ &\quad \cdot \left(x(t-d(t)) - x(t-\tau_1) - \int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds\right) \\ &\quad = 0, \\ &2\left(x^T(t) N_1 + x^T(t-d(t)) N_2\right) \\ &\quad \cdot \left(x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)\, ds\right) = 0.\end{aligned} \tag{A.16}$$

According to square inequality

$$\pm 2x^T y \leqslant x^T \Upsilon^{-1} x + y^T \Upsilon y \tag{A.17}$$

for any $x, y \in \mathbb{R}^n$ and any positive definite matrix $\Upsilon \in \mathbb{R}^{n\times n}$, it is easy to construct the following inequality:

$$\begin{aligned}&-2\left(x^T(t) M_1 + x^T(t-d(t)) M_2\right) \int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds \\ &\quad \leqslant \tau_1 \left(x^T(t) M_1 + x^T(t-d(t)) M_2\right) \\ &\quad \cdot S_1^{-1} \left(x^T(t) M_1 + x^T(t-d(t)) M_2\right)^T \\ &\quad + \tau_1^{-1} \left(\int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds\right)^T S_1 \left(\int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds\right).\end{aligned} \tag{A.18}$$

From Jensen integral inequality and $0 < d(t) \leqslant \tau_1$, we have

$$\begin{aligned}&\left(\int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds\right)^T S_1 \left(\int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds\right) \\ &\quad \leqslant \tau_1 \int_{t-\tau_1}^{t-d(t)} \dot{x}^T(s) S_1 \dot{x}(s)\, ds.\end{aligned} \tag{A.19}$$

Therefore, from inequalities (A.18) and (A.19), we have

$$\begin{aligned}&-2\left(x^T(t) M_1 + x^T(t-d(t)) M_2\right) \int_{t-\tau_1}^{t-d(t)} \dot{x}(s)\, ds \\ &\quad \leqslant \tau_1 x^T(t) M_1 S_1^{-1} M_1^T x(t) \\ &\qquad + \tau_1 x^T(t) M_1 S_1^{-1} M_2^T x(t-d(t)) \\ &\qquad + \tau_1 x^T(t-d(t)) M_2 S_1^{-1} M_1^T x(t) \\ &\qquad + \tau_1 x^T(t-d(t)) M_2 S_1^{-1} M_2^T x(t-d(t)) \\ &\qquad + \int_{t-\tau_1}^{t-d(t)} \dot{x}^T(s) S_1 \dot{x}(s)\, ds,\end{aligned} \tag{A.20}$$

and similarly,

$$\begin{aligned}&-2\left(x^T(t) N_1 + x^T(t-d(t)) N_2\right) \int_{t-d(t)}^{t} \dot{x}(s)\, ds \\ &\quad \leqslant \tau_1 x^T(t) N_1 S_1^{-1} N_1^T x(t) \\ &\qquad + \tau_1 x^T(t) N_1 S_1^{-1} N_2^T x(t-d(t)) \\ &\qquad + \tau_1 x^T(t-d(t)) N_2 S_1^{-1} N_1^T x(t) \\ &\qquad + \tau_1 x^T(t-d(t)) N_2 S_1^{-1} N_2^T x(t-d(t)) \\ &\qquad + \int_{t-d(t)}^{t} \dot{x}^T(s) S_1 \dot{x}(s)\, ds,\end{aligned} \tag{A.21}$$

Then adding the null equations (A.16) to the right-hand side of (A.12) and replacing the corresponding terms with inequalities (A.20) and (A.21), we can obtain

$$\begin{aligned}\dot{V}_4(t) &\leqslant \tau_1 (Ax(t) + A_d x(t-d(t)) \\ &\quad + A_h x(t-h(t)) + B_w w(t))^T S_1 (Ax(t) \\ &\quad + A_d x(t-d(t)) + A_h x(t-h(t)) + B_w w(t)) \\ &\quad + 2x^T(t) M_1 x(t-d(t)) - 2x^T(t) M_1 x(t-\tau_1) \\ &\quad + 2x^T(t-\mathrm{d}(t)) M_2 x(t-d(t)) - 2x^T(t-d(t)) \\ &\quad \cdot M_2 x(t-\tau_1) + \tau_1 x^T(t) M_1 S_1^{-1} M_1^T x(t) \\ &\quad + \tau_1 x^T(t) M_1 S_1^{-1} M_2^T x(t-d(t)) + \tau_1 x^T(t \\ &\quad - d(t)) M_2 S_1^{-1} M_1^T x(t) + \tau_1 x^T(t-d(t))\end{aligned}$$

$$\cdot M_2S_1^{-1}M_2^T x(t-d(t)) + 2x^T(t) N_1 x(t) - 2x^T(t) N_1 x(t-d(t)) + 2x^T(t-d(t)) N_2 x(t) - 2x^T(t-d(t)) N_2 x(t-d(t)) + \tau_1 x^T(t) \cdot N_1S_1^{-1}N_1^T x(t) + \tau_1 x^T(t) N_1S_1^{-1}N_2^T x(t-d(t)) + \tau_1 x^T(t-d(t)) N_2S_1^{-1}N_1^T x(t) + \tau_1 x^T(t - d(t)) N_2S_1^{-1}N_2^T x(t-d(t)). \tag{A.22}$$

For any matrices $U_1$, $U_2$, $V_1$, and $V_2$ with appropriate dimensions, by using the method similar to $\dot{V}_4(t)$, we can obtain

$$\begin{aligned}\dot{V}_7(t) \leqslant{} & \tau_2 \left(Ax(t) + A_d x(t-d(t)) + A_h x(t-h(t)) + B_w w(t)\right)^T S_2 \left(Ax(t) + A_d x(t-d(t)) + A_h x(t-h(t)) + B_w w(t)\right) \\ & + 2x^T(t) U_1 x(t-h(t)) - 2x^T(t) U_1 x(t-\tau_2) + 2x^T(t-h(t)) U_2 x(t-h(t)) - 2x^T(t-h(t)) \cdot U_2 x(t-\tau_2) + \tau_2 x^T(t) U_1S_2^{-1}U_1^T x(t) \\ & + \tau_2 x^T(t) U_1S_2^{-1}U_2^T x(t-h(t)) + \tau_2 x^T(t-h(t)) \cdot U_2S_2^{-1}U_1^T x(t) + \tau_2 x^T(t-h(t)) U_2S_2^{-1}U_2^T x(t - h(t)) \\ & + 2x^T(t) V_1 x(t) - 2x^T(t) V_1 x(t-h(t)) + 2x^T(t-h(t)) V_2 x(t) - 2x^T(t-h(t)) V_2 x(t - h(t)) + \tau_2 x^T(t) V_1S_2^{-1}V_1^T x(t) + \tau_2 x^T(t) \\ & \cdot V_1S_2^{-1}V_2^T x(t-h(t)) + \tau_2 x^T(t-h(t)) \cdot V_2S_2^{-1}V_1^T x(t) + \tau_2 x^T(t-h(t)) V_2S_2^{-1}V_2^T x(t - h(t)).\end{aligned} \tag{A.23}$$

As a result, let us define an extended state vector as $\chi(t) \triangleq [x^T(t), x^T(t-d(t)), x^T(t-h(t)), x^T(t-\tau_1), x^T(t-\tau_2), w^T(t)]^T$ and substitute $\dot{V}_i(t)$ $(i = 1, \ldots, 7)$ computed in (A.9), (A.10), (A.11), (A.22),(A.13), (A.14), and (A.23) into $\dot{V}(x(t), t) = \sum_{i=1}^{7} \dot{V}_i(t)$. Then, we can calculate

$$\dot{V}(x(t), t) + z^T(t) z(t) - \gamma^2 w^T(t) w(t) \leqslant \chi^T(t) \Psi \chi(t), \tag{A.24}$$

where

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & -M_1 & U_1 & PB_w + A^T(\tau_1 S_1 + \tau_2 S_2) B_w \\ * & \Psi_{22} & \Psi_{23} & -M_2 & 0 & A_d^T(\tau_1 S_1 + \tau_2 S_2) B_w \\ * & * & \Psi_{33} & 0 & U_2 & A_h^T(\tau_1 S_1 + \tau_2 S_2) B_w \\ * & * & * & -R_1 & 0 & 0 \\ * & * & * & * & -R_2 & 0 \\ * & * & * & * & * & -\gamma^2 I + B_w^T(\tau_1 S_1 + \tau_2 S_2) B_w \end{bmatrix} \tag{A.25}$$

with

$$\begin{aligned}\Psi_{11} &= \Gamma_{11} + \tau_1 M_1S_1^{-1}M_1^T + \tau_2 U_1S_2^{-1}U_1^T + \tau_1 N_1S_1^{-1}N_1^T + \tau_2 V_1S_2^{-1}V_1^T, \\ \Psi_{12} &= \Gamma_{12} + A^T(\tau_1 S_1 + \tau_2 S_2) A_d + \tau_1 M_1S_1^{-1}M_2^T + \tau_1 N_1S_1^{-1}N_2^T, \\ \Psi_{13} &= \Gamma_{12} + A^T(\tau_1 S_1 + \tau_2 S_2) A_h + \tau_2 U_1S_2^{-1}U_2^T + \tau_2 V_1S_2^{-1}V_2^T, \\ \Psi_{22} &= \Gamma_{22} + A_d^T(\tau_1 S_1 + \tau_2 S_2) A_d + \tau_1 M_2S_1^{-1}M_2^T + \tau_1 N_2S_1^{-1}N_2^T, \\ \Psi_{23} &= A_d^T(\tau_1 S_1 + \tau_2 S_2) A_h, \\ \Psi_{33} &= \Gamma_{33} + A_h^T(\tau_1 S_1 + \tau_2 S_2) A_h \tau_2 U_2S_2^{-1}U_2^T + \tau_2 V_2S_2^{-1}V_2^T.\end{aligned} \tag{A.26}$$

If $\Psi < 0$ is satisfied, then $\dot{V}(x(t), t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq \chi^T(t)\Psi\chi(t) < 0$. It is apparently seen that when $w(t) \equiv 0$, $\forall t \geqslant 0$, $\dot{V}(x(t), t) < 0$ is ensured guaranteeing that system (14) without the disturbances is globally asymptotically stable. Moreover, integrating both sides of $\dot{V}(x(t), t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0$ from 0 to infinity allows getting $\lim_{t\to+\infty} V(x(t), t) - V(x(0), 0) + \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)] < 0$. Since $\lim_{t\to+\infty} V(x(t), t) > 0$ and $V(x(0), 0) = 0$, we have $\int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)] < 0$. Finally, applying Lemma 5 (Schur complement formula) to $\Psi$ we can get (15). This completes the proof. □

*Proof of Theorem 8.* On the basis of Lemma 7, subsystem (13) is asymptotically stable with $\|T_{\widehat{w}_i \widehat{z}_i}(s)\|_\infty < \gamma$, if there exist positive definite matrices $P$, $Q_1$, $Q_2$, $R_1$, $R_2$, $S_1$, and $S_2$ and matrices $M_1, M_2, N_1, N_2, U_1, U_2, V_1$, and $V_2$ with appropriate dimensions such that

$$\Gamma^{(i)} = \begin{bmatrix} \Gamma_0^{(i)} & \Gamma_1^{(i)} & \Gamma_2^{(i)} & \Gamma_3^{(i)} & \Gamma_4^{(i)} & \Gamma_5^{(i)} \\ * & -S & 0 & 0 & 0 & 0 \\ * & * & -\tau_1 S_1 & 0 & 0 & 0 \\ * & * & * & -\tau_1 S_1 & 0 & 0 \\ * & * & * & * & -\tau_2 S_2 & 0 \\ * & * & * & * & * & -\tau_2 S_2 \end{bmatrix} < 0, \quad \text{(A.27)}$$

where $S = \tau_1 S_1 + \tau_2 S_2$, $A_1 = A + \Delta A(t)$, $A_2 = A_d + \Delta A_d(t)$, $B_{ki} = -\lambda_i B_1 K$,

$$\Gamma_0^{(i)} = \begin{bmatrix} \Gamma_{11}^{(i)} & \Gamma_{12}^{(i)} & \Gamma_{13}^{(i)} & -M_1 & -U_1 & PB_2 \\ * & \Gamma_{22}^{(i)} & 0 & -M_2 & 0 & 0 \\ * & * & \Gamma_{33}^{(i)} & 0 & -U_2 & 0 \\ * & * & * & -R_1 & 0 & 0 \\ * & * & * & * & -R_2 & 0 \\ * & * & * & * & * & \gamma^2 I \end{bmatrix},$$

$$\begin{aligned} \Gamma_1^{(i)} &= \left[SA_1, SA_2, SB_{ki}, 0, 0, SB_2\right]^T, \\ \Gamma_2^{(i)} &= \left[\tau_1 M_1^T, \tau_1 M_2^T, 0, 0, 0, 0\right]^T, \\ \Gamma_3^{(i)} &= \left[\tau_1 N_1^T, \tau_1 N_2^T, 0, 0, 0, 0\right]^T, \\ \Gamma_4^{(i)} &= \left[\tau_2 U_1^T, 0, \tau_2 U_2^T, 0, 0, 0\right]^T, \\ \Gamma_5^{(i)} &= \left[\tau_2 V_1^T, 0, \tau_2 V_2^T, 0, 0, 0\right]^T, \\ \Gamma_{11}^{(i)} &= PA_1 + A_1^T P + Q_1 + Q_2 + R_1 + R_2 + N_1 + N_1^T \\ &\quad + V_1 + V_1^T + I, \\ \Gamma_{12}^{(i)} &= PA_2 + M_1 - N_1 + N_2^T, \\ \Gamma_{13}^{(i)} &= PB_{ki} + U_1 - V_1 + V_2^T, \\ \Gamma_{22}^{(i)} &= (\mu_1 - 1) Q_1 + M_2 + M_2^T - N_2 - N_2^T, \\ \Gamma_{33}^{(i)} &= (\mu_2 - 1) Q_2 + U_2 + U_2^T - V_2 - V_2^T. \end{aligned} \quad \text{(A.28)}$$

Due to the convex property of LMIs, $\Gamma^{(i)} < 0$ for all $i = 1, 2, \ldots, n-1$, if and only if $\Gamma^{(1)} < 0$ and $\Gamma^{(n-1)} < 0$, which are associated with the smallest eigenvalues $\lambda_1$ and the largest eigenvalues $\lambda_{n-1}$, respectively.

Pre- and postmultiplying inequality (A.27) with $\operatorname{diag}\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, I, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}\}$ and applying the variable changes $X = P^{-1}$, $\widehat{*} = P^{-1} * P^{-1}$, where $*$ denote $Q_1$, $Q_2$, $R_1$, $R_2$, $S_1$, $S_2$, $M_1$, $M_2$, $N_1$, $N_2$, $U_1$, $U_2$, $V_1$, and $V_2$, inequality (A.27) can be decomposed as $\widehat{\Gamma}^{(i)} = \widehat{\Gamma}_x^{(i)} + \widehat{\Gamma}_{xx}^{(i)} + \widehat{\Gamma}_{xx}^{(i)T}$, where

$$\widehat{\Gamma}_x^{(i)} = \begin{bmatrix} \widehat{\Gamma}_0^{(i)} & 0 & \widehat{\Gamma}_2^{(i)} & \widehat{\Gamma}_3^{(i)} & \widehat{\Gamma}_4^{(i)} & \widehat{\Gamma}_5^{(i)} \\ * & -\widehat{S} & 0 & 0 & 0 & 0 \\ * & * & -\tau_1 \widehat{S}_1 & 0 & 0 & 0 \\ * & * & * & -\tau_1 \widehat{S}_1 & 0 & 0 \\ * & * & * & * & -\tau_2 \widehat{S}_2 & 0 \\ * & * & * & * & * & -\tau_2 \widehat{S}_2 \end{bmatrix},$$

$$\widehat{\Gamma}_{xx}^{(i)} = \begin{bmatrix} 0 & \widehat{\Gamma}_1^{(i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{S} = \tau_1 \widehat{S}_1 + \tau_2 \widehat{S}_2,$$

$$\widehat{\Gamma}_0^{(i)} = \begin{bmatrix} \widehat{\Gamma}_{11}^{(i)} & \widehat{\Gamma}_{12}^{(i)} & \widehat{\Gamma}_{13}^{(i)} & -\widehat{M}_1 & -\widehat{U}_1 & B_2 \\ * & \widehat{\Gamma}_{22}^{(i)} & 0 & -\widehat{M}_2 & 0 & 0 \\ * & * & \widehat{\Gamma}_{33}^{(i)} & 0 & -\widehat{U}_2 & 0 \\ * & * & * & -\widehat{R}_1 & 0 & 0 \\ * & * & * & * & -\widehat{R}_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\widehat{\Gamma}_1^{(i)} = \begin{bmatrix} XA_1^T X^{-1} \widehat{S} \\ XA_2^T X^{-1} \widehat{S} \\ XB_{ki}^T X^{-1} \widehat{S} \\ 0 \\ 0 \\ B_2^T X^{-1} \widehat{S} \end{bmatrix},$$

$$\begin{aligned} \widehat{\Gamma}_2^{(i)} &= \left[\tau_1 \widehat{M}_1^T, \tau_1 \widehat{M}_2^T, 0, 0, 0, 0\right]^T, \\ \widehat{\Gamma}_3^{(i)} &= \left[\tau_1 \widehat{N}_1^T, \tau_1 \widehat{N}_2^T 0, 0, 0, 0\right]^T, \\ \widehat{\Gamma}_4^{(i)} &= \left[\tau_2 \widehat{U}_1^T, 0, \tau_2 \widehat{U}_2^T, 0, 0, 0\right]^T, \\ \widehat{\Gamma}_5^{(i)} &= \left[\tau_2 \widehat{V}_1^T, 0, \tau_2 \widehat{V}_2^T, 0, 0, 0\right]^T, \\ \widehat{\Gamma}_{11}^{(i)} &= A_1 X + XA_1^T + \widehat{Q}_1 + \widehat{Q}_2 + \widehat{R}_1 + \widehat{R}_2 + \widehat{N}_1 \\ &\quad + \widehat{N}_1^T + \widehat{V}_1 + \widehat{V}_1^T + XX, \\ \widehat{\Gamma}_{12}^{(i)} &= A_2 X + \widehat{M}_1 - \widehat{N}_1 + \widehat{N}_2^T, \\ \widehat{\Gamma}_{13}^{(i)} &= B_{ki} X + \widehat{U}_1 - \widehat{V}_1 + \widehat{V}_2^T, \end{aligned}$$

$$\widehat{\Gamma}_{22}^{(i)} = (\mu_1 - 1)\widehat{Q}_1 + \widehat{M}_2 + \widehat{M}_2^T - \widehat{N}_2 - \widehat{N}_2^T,$$
$$\widehat{\Gamma}_{33}^{(i)} = (\mu_2 - 1)\widehat{Q}_2 + \widehat{U}_2 + \widehat{U}_2^T - \widehat{V}_2 - \widehat{V}_2^T. \tag{A.29}$$

On one hand, $\widehat{\Gamma}_{xx}^{(i)}$ can be rewritten as $\widehat{\Gamma}_{xx}^{(i)} = \widehat{\Pi}_1^{(i)T} X^{-1} \widehat{\Pi}_2$, where

$$\begin{aligned}\widehat{\Pi}_1^{(i)} &= [A_1 X, A_2 X, B_{ki} X, 0, 0, B_2, 0, 0, 0, 0, 0], \\ \widehat{\Pi}_2 &= [0, 0, 0, 0, 0, 0, \widehat{S}, 0, 0, 0, 0].\end{aligned} \tag{A.30}$$

On the other hand, according to square inequality, it is easy to construct the following inequality:

$$\begin{aligned}&\widehat{\Pi}_1^{(i)T} X^{-1} \widehat{\Pi}_2 + \left(\widehat{\Pi}_1^{(i)T} X^{-1} \widehat{\Pi}_2\right)^T \\ &\quad \leqslant \widehat{\Pi}_1^{(i)T} T^{-1} \widehat{\Pi}_1^{(i)} + \widehat{\Pi}_2^T X^{-1} T X^{-1} \widehat{\Pi}_2,\end{aligned} \tag{A.31}$$

where $T$ is any positive definite matrix with appropriate dimensions. Therefore, $\widehat{\Gamma}_x^{(i)} + \widehat{\Pi}_1^{(i)T} T^{-1} \widehat{\Pi}_1^{(i)} + \widehat{\Pi}_2^T X^{-1} T X^{-1} \widehat{\Pi}_2 < 0$ implies that $\widehat{\Gamma}^{(i)} < 0$. Then, defining $Y = KX$ and applying Lemma 5 (Schur complement formula) on $\widehat{\Gamma}_x^{(i)} + \widehat{\Pi}_1^{(i)T} T^{-1} \widehat{\Pi}_1^{(i)} + \widehat{\Pi}_2^T X^{-1} T X^{-1} \widehat{\Pi}_2 < 0$, we can obtain the matrix inequality condition:

$$\overline{\Phi}^{(i)} = \begin{bmatrix} \overline{\Phi}_0^{(i)} & \overline{\Pi}_1^{(i)T} & \overline{\Pi}_2^T \\ * & -T & 0 \\ * & * & -XT^{-1}X \end{bmatrix} < 0, \tag{A.32}$$

where

$$\overline{\Phi}_0^{(i)} = \begin{bmatrix} \overline{\Phi}_{0x}^{(i)} & 0 & \widehat{\Gamma}_2^{(i)} & \widehat{\Gamma}_3^{(i)} & \widehat{\Gamma}_4^{(i)} & \widehat{\Gamma}_5^{(i)} & \overline{\Phi}_6 \\ * & -\widehat{S} & 0 & 0 & 0 & 0 & 0 \\ * & * & -\tau_1 \widehat{S}_1 & 0 & 0 & 0 & 0 \\ * & * & * & -\tau_1 \widehat{S}_1 & 0 & 0 & 0 \\ * & * & * & * & -\tau_2 \widehat{S}_2 & 0 & 0 \\ * & * & * & * & * & -\tau_2 \widehat{S}_2 & 0 \\ * & * & * & * & * & * & -I \end{bmatrix},$$

$$\overline{\Phi}_{0x}^{(i)} = \begin{bmatrix} \overline{\Phi}_{11} & \overline{\Phi}_{12} & \overline{\Phi}_{13}^{(i)} & -\widehat{M}_1 & -\widehat{U}_1 & B_2 \\ * & \overline{\Phi}_{22} & 0 & -\widehat{M}_2 & 0 & 0 \\ * & * & \overline{\Phi}_{33} & 0 & -\widehat{U}_2 & 0 \\ * & * & * & -\widehat{R}_1 & 0 & 0 \\ * & * & * & * & -\widehat{R}_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\begin{aligned}\overline{\Phi}_{11} &= A_1 X + X A_1^T + \widehat{Q}_1 + \widehat{Q}_2 + \widehat{R}_1 + \widehat{R}_2 + \widehat{N}_1 + \widehat{N}_1^T \\ &\quad + \widehat{V}_1 + \widehat{V}_1^T, \\ \overline{\Phi}_{12} &= A_2 X + \widehat{M}_1 - \widehat{N}_1 + \widehat{N}_2^T, \\ \overline{\Phi}_{13}^{(i)} &= -\lambda_i B_1 Y + \widehat{U}_1 - \widehat{V}_1 + \widehat{V}_2^T, \\ \overline{\Phi}_{22} &= (\mu_1 - 1)\widehat{Q}_1 + \widehat{M}_2 + \widehat{M}_2^T - \widehat{N}_2 - \widehat{N}_2^T, \\ \overline{\Phi}_{33} &= (\mu_2 - 1)\widehat{Q}_2 + \widehat{U}_2 + \widehat{U}_2^T - \widehat{V}_2 - \widehat{V}_2^T, \\ \overline{\Phi}_6 &= [X, 0, 0, 0, 0, 0]^T, \\ \overline{\Pi}_1^{(i)} &= [A_1 X, A_2 X, -\lambda_i B_1 Y, 0, 0, B_2, 0, 0, 0, 0, 0, 0], \\ \overline{\Pi}_2 &= [0, 0, 0, 0, 0, 0, \widehat{S}, 0, 0, 0, 0, 0].\end{aligned} \tag{A.33}$$

Note that $A_1 = A + \Delta A(t) = A + GF(t)E$ and $A_2 = A_d + \Delta A_d(t) = A_d + G_d F(t) E_d$. Define that

$$\begin{aligned}\widehat{J} &= \left[G^T, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, G^T, 0\right]^T, \\ \widehat{H} &= [EX, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], \\ \widehat{J}_d &= \left[G_d^T, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, G_d^T, 0\right]^T, \\ \widehat{H}_d &= [0, E_d X, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].\end{aligned} \tag{A.34}$$

It can be obtained that $\overline{\Phi}^{(i)} = \widehat{\Phi}^{(i)} + \widehat{J} F(t) \widehat{H} + \widehat{H}^T F^T(t) \widehat{J}^T + \widehat{J}_d F(t) \widehat{H}_d + \widehat{H}_d^T F^T(t) \widehat{J}_d^T$, where $\widehat{\Phi}^{(i)}$ is constructed through replacing $A_1$ and $A_2$ term in $\overline{\Phi}^{(i)}$ with $A$ and $A_d$, respectively. According to square inequality (A.17) and $F^T(t)F(t) \leqslant I$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $\widehat{\Phi}^{(i)} + \varepsilon_1 \widehat{J}\widehat{J}^T + \varepsilon_1^{-1} \widehat{H}^T \widehat{H} + \varepsilon_2 \widehat{J}_d \widehat{J}_d^T + \varepsilon_2^{-1} \widehat{H}_d^T \widehat{H}_d < 0$ guaranteeing $\overline{\Phi}^{(i)} < 0$. Then, applying Lemma 5 (Schur complement formula) on $\widehat{\Phi}^{(i)} + \varepsilon_1 \widehat{J}\widehat{J}^T + \varepsilon_1^{-1} \widehat{H}^T \widehat{H} + \varepsilon_2 \widehat{J}_d \widehat{J}_d^T + \varepsilon_2^{-1} \widehat{H}_d^T \widehat{H}_d < 0$ and using the definitions $Q_1 \triangleq \widehat{Q}_1$, $Q_2 \triangleq \widehat{Q}_2$, $R_1 \triangleq \widehat{R}_1$, $R_2 \triangleq \widehat{R}_2$, $S_1 \triangleq \widehat{S}_1$, $S_2 \triangleq \widehat{S}_2$, $M_1 \triangleq \widehat{M}_1$, $M_2 \triangleq \widehat{M}_2$, $N_1 \triangleq \widehat{N}_1$, $N_2 \triangleq \widehat{N}_2$, $U_1 \triangleq \widehat{U}_1$, $U_2 \triangleq \widehat{U}_2$, $V_1 \triangleq \widehat{V}_1$, and $V_2 \triangleq \widehat{V}_2$, we can obtain that the matrix inequality condition $\Phi^{(i)} < 0$ (see (17)) and $K = YX^{-1}$.

On the basis of the above analysis, if $\Phi^{(1)} < 0$, $\Phi^{(n-1)} < 0$ and $K = YX^{-1}$, the subsystems (13) are simultaneously asymptotically stable with $\|T_{\widehat{w}_i \widehat{z}_i}(s)\|_\infty < \gamma$. Further, by Theorem 6, the closed-loop system (12) reaches consensus with the desired $H_\infty$ disturbance attenuation index $\gamma$; that is, applying the distributed protocol (11) with $K = YX^{-1}$, the multiagent system (1) can reach consensus while satisfying the desired $H_\infty$ disturbance attenuation index $\gamma$. This completes the proof. □

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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