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Research Article

δ-Primary Hyperideals on Commutative Hyperrings

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The purpose of this paper is to define the hyperideal expansion. Hyperideal expansion is associated with prime hyperideals and primary hyperideals. Then, we define some of their properties. Prime and primary hyperideals' numerous results can be extended into expansions.

1. Introduction

The hyperstructure theory was introduced by Marty (1934). Hyperstructures have many applications to several sectors of both pure and applied mathematics. A hypergroup in the sense of Marty is a nonempty set H endowed by a hyperoperation $* : H \times H \rightarrow P^*(H)$ [1], the set of the entire nonempty set H, which satisfies the associative law and reproduction axiom. Canonical hypergroups are a special class of the hypergroup of Marty. The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. There are different notions of hyperrings [1]. If only the addition $+$ is hyperoperation and the multiplication $\cdot$ is usual operation, then we say that R is an additive hyperring. A special case of this type is the hyperring introduced by Krasner (1957) [2]. Also, Krasner (1983) introduced a class of hyperring and hyperfields and the quotient hyperrings and hyperfields. If only $\cdot$ is a hyperoperation, we shall say that R is a multiplicative hyperring [2]. Rota (1982) introduced the multiplicative hyperrings; subsequently, many authors worked on this field (Nakassis, 1988; Olson and Ward, 1997; Procesi and Rota, 1999; Rota, 1996) [2]. Algebraic hyperstructures have been studied in the following decades and nowadays by many mathematicians.

Although the δ-primary ideals have been investigated by Dongsheng [3], the concept of δ-primary hyperideals which unify prime hyperideals and primary hyperideals has not been studied yet. So, this work shows some elementary properties of the hyperideal expansion; then we show some new results of δ-primary hyperideals. After this introductory section, Section 2 is devoted to some definitions and properties related to δ primary ideals and hyperideals that will be needed later. In Section 3, the definitions of hyperideal expansion and δ primary hyperideals will be given and some basic properties of these concepts will be studied.

2. Preliminaries

Throughout this paper $(R, +, \cdot)$ denotes the Krasner hyperring.

Definition 1 (see [4]). A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup; that is,

(i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$,

(ii) for every $x, y \in R$, $x + y = y + x$,

(iii) there exists $0 \in R$ such that $0 + x = \{x\}$ for every $x \in R$,

(iv) for every $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$,

(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$,

(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element; that is, $x \cdot 0 = 0 \cdot x = 0$.

(3) The multiplication is distributive with respect to the hyperoperation $+$.

Definition 2 (see [2]). Let $(R, +, \cdot)$ be a hyperring and A be a nonempty subset of R. Then A is said to be a subhyperring of R if $(A, +, \cdot)$ is itself a hyperring.

Definition 3 (see [1]). A subhyperring A of a hyperring R is a left (right) hyperideal of R if $ra \in A$ ($ar \in A$) for all $r \in R$ and $a \in A$. A is called a hyperideal if A is both a left and a right hyperideal.

Lemma 4 (see [2]). *A nonempty subset A of a hyperring R is a hyperideal if and only if*

(1) *$a, b \in A$ implies $a - b \subseteq A$,*

(2) *$a \in A$ and $r \in R$ imply $ra \in A$.*

Definition 5 (see [2]). Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a good (strong) homomorphism if, for all $a, b \in R_1$,

$$\begin{gathered} \varphi(a+b)=\varphi(a)+\varphi(b), \\ \varphi(a\cdot b)=\varphi(a)\cdot\varphi(b), \\ \varphi(0)=0. \end{gathered} \tag{1}$$

Definition 6 (see [1]). Let $f : R \rightarrow S$ be a hyperring homomorphism. The kernel of f, denoted $\ker(f)$, is the set of elements of R that map to 0 in S; that is, $\ker(f) = \{x \in R \mid f(x) = 0\}$.

Definition 7 (see [2]). A hyperideal P of a hyperring R is called a prime hyperideal if whenever $a \cdot b \in P$, either $a \in P$ or $b \in P$.

Definition 8 (see [2]). Let I be a hyperideal of the hyperring R. Then the radical of I, denoted by $\sqrt{I}$, is defined as $\sqrt{I} = \{x \mid x^n \in I \text{ for some } n \in N\}$.

Definition 9 (see [2]). A hyperideal I of a hyperring R is called a primary hyperideal if whenever $a \cdot b \in P$, either $a \in P$ or $b \in \sqrt{P}$.

3. Hyperideal Expansion and δ Primary Hyperideals

Definition 10. An expansion of hyperideals, or briefly hyperideal expansion, is a function δ which assigns to each hyperideal I of a hyperring R another hyperideal $\delta(I)$ of the same ring such that the following conditions are satisfied:

(i) $I \subseteq \delta(I)$.

(ii) $P \subseteq Q$ implies $\delta(P) \subseteq \delta(Q)$ for P, Q hyperideals of R.

Example 11. Let $\mathrm{Id}(R)$ denote the set of all hyperideals of the hyperring R. The identity function δ_0, where $\delta(I) = I$ for every $I \in \mathrm{Id}(R)$, is an expansion of hyperideals.

For each I hyperideal define $\delta_1(I) = \sqrt{I}$, the radical of I. Then δ_1 is an expansion of hyperideals.

Definition 12. Given an expansion δ of hyperideals, a hyperideal I of R is called δ-primary if $ab \in I$ and $a \notin I$ imply $b \in \delta(I)$ for all $a, b \in R$.

Obviously the definition of δ-primary hyperideals can be also stated as $ab \in I$ and $a \notin \delta(I)$ implies $b \in I$ for all $a, b \in R$.

Example 13.

(1) A Hyperideal I Is δ_0-Primary If and Only If It Is Prime. Let I be δ_0-primary hyperideal. We show that I is prime. Assume that $ab \in I$ and that I is δ_0-primary $a \in I$ or $b \in \delta_0(I) = I$ so I is a prime hyperideal.

Conversely, let I be a prime hyperideal. Assume that $ab \in I$. Since I is prime $a \in I$ or $b \in I = \delta_0(I)$ and I is δ_0-primary.

(2) A Hyperideal I is δ_1-Primary If and Only If It Is Primary. Let I be δ_1-primary. We show that I is primary. Assume that $ab \in I$. Since I is δ_1-primary then we can say that $a \in I$ or $b \in \delta_1(I)$. That is $a \in I$ or $b \in \sqrt{I}$. So I is primary.

Conversely let I be a primary hyperideal. Assume that $ab \in I$. Since I primary hyperideal $a \in I$ or $b \in \sqrt{I}$, thus $a \in I$ or $b \in \sqrt{I} = \delta_1(I)$.

Remark 14. (1) If δ and γ are two hyperideal expansions and $\delta(I) \subseteq \gamma(I)$ for each hyperideal I, then every δ-primary hyperideal is also γ-primary. Thus, in particular, a prime hyperideal is δ-primary for every δ hyperideal expansion. Let I be δ-primary. Assume that, for all $a, b \in R$, $ab \in I$ and $a \notin I$. Since I is δ-primary and $\delta(I) \subseteq \gamma(I)$ and $b \in \delta(I) \subseteq \gamma(I)$, thus $b \in \gamma(I)$.

(2) Given two hyperideal expansions δ_1 and δ_2, define $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then δ is also a hyperideal expansion. Since δ_1 and δ_2 are hyperideal expansions $I \subseteq \delta_1(I)$ and $I \subseteq \delta_2(I)$, then $I \subseteq \delta_1 \cap \delta_2(I) = \delta(I)$ and $I \subseteq \delta(I)$.

Let P and Q be any hyperideals of R and $P \subseteq Q$. Thus $\delta_1(P) \subseteq \delta_1(Q)$ and $\delta_2(P) \subseteq \delta_2(Q)$. Finally we find $\delta_1(P) \cap \delta_2(P) \subseteq \delta_1(Q) \cap \delta_2(Q)$.

(3) Let δ be a hyperideal expansion. Define $E_\delta(P) = \bigcap\{J \in \mathrm{Id}(R) \mid P \subseteq J,\ J \text{ is } \delta\text{-primary}\}$. Then E_δ is still a hyperideal expansion.

For all $P \in \mathrm{Id}(R)$, we show that $P \subseteq E_\delta(P)$ for any $K, L \in \mathrm{Id}(R)$, if $K \subseteq L$; then $E_\delta(K) \subseteq E_\delta(L)$. By the definition of $E_\delta(P)$, we conclude that $P \subseteq E_\delta(P)$. For any $K, L \in \mathrm{Id}(R)$, if $K \subseteq L$, then the δ-primary hyperideals which contain L contain also K. In addition, there may be δ-primary hyperideals which contained L but did not contain K. Hence, we conclude that $E_\delta(K) \subseteq E_\delta(L)$.

Lemma 15. *A hyperideal P is δ-primary if and only if for any two hyperideals I and J, if $IJ \subseteq P$ and $I \nsubseteq P$ then $J \subseteq \delta(P)$.*

Proof. Let P be δ-primary. Suppose $IJ \subseteq P$ and $I \nsubseteq P$, but $J \nsubseteq \delta(P)$, and then we can choose $a \in I - P$ and $b \in J - \delta(P)$. Then $ab \in IJ \subseteq P$ but $a \notin P$ and $b \notin \delta(P)$. This contradicts the assumption that P is δ-primary.

Conversely, if the condition is satisfied, for any two elements a and b, suppose $ab \in P$ and $a \notin P$. Then $(a)(b) \subseteq P$ and $(a) \nsubseteq P$. So $(b) \subseteq \delta(P)$. Hence $b \in (b) \subseteq \delta(P)$ implies $b \in \delta(P)$. Thus P is δ-primary. □

Recall that if I and J are ideals of a commutative ring R, then their ideal quotient denotes $(I : J)$ defined by $(I : J) = \{r \in R \mid rJ \subset I\}$. We recall also ideal quotient $(I : J)$ is itself an ideal in R.

Theorem 16. *Let δ be a hyperideal expansion. Then*

(1) *if P is a δ-primary hyperideal and I is a hyperideal with $I \nsubseteq \delta(P)$, then $(P : I) = P$,*

(2) *for any δ-primary hyperideal P and any subset N of the R, $(P : N)$ is also δ-primary.*

Proof. (1) From the definition of $(P : I)$, for all $x \in I \cdot (P : I)$, $x \in \sum_{i=1}^{n} a_i p_i$, $a_i \in I$ and $p_i \in (P : I)$. Since P is a hyperideal $x \in \sum_{i=1}^{n} a_i p_i \subseteq P$, then we get $x \in P$. In other words $I \cdot (P : I) \subseteq P$. Since P is δ-primary, if $I \nsubseteq \delta(P)$ then $(P : I) \subseteq P$.

Conversely, since $P \subseteq (P : I)$ then $(P : I) = P$.

(2) For all $a, b \in R$, assume that $ab \in (P : N)$ and $a \notin (P : N)$. Then there exists a $n \in N$ such that $an \notin P$. But $anb = abn \in P$. Thus $b \in \delta(P)$. Since $\delta(P) \subseteq \delta(P : N)$, $b \in \delta(P : N)$. By this way we get that $(P : N)$ is δ-primary. □

Theorem 17. *If δ is a hyperideal expansion such that $\delta(I) \subseteq \delta_1(I)$ for every hyperideal I, then, for any δ-primary hyperideal P, $\delta(P) = \delta_1(P)$.*

Proof. For all I hyperideals, since $\delta(I) \subseteq \delta_1(I)$, $\delta(P) \subseteq \delta_1(P)$.

Conversely, let $a \in \delta_1(P)$. We show that $a \in \delta(P)$.

Then there exists k which is the least positive integer k with $a^k \in P$. If $k = 1$ then $a \in P \subseteq \delta(P)$. If $k > 1$ then $a^{k-1}a \in P$. But $a^{k-1} \notin P$, so $a \in \delta(P)$. Hence $\delta_1(P) \subseteq \delta(P)$ and $\delta(P) = \delta_1(P)$. □

4. Expansions with Extra Properties

In this section we investigate δ-primary hyperideals where δ satisfy additional conditions and prove more results with respect to such expansions.

Definition 18. A hyperideal expansion δ is intersection preserving if it satisfies

$$\delta(I \cap J) = \delta(I) \cap \delta(J) \quad \text{for any } I, J \in \mathrm{Id}(R). \tag{2}$$

An expansion is said to be global if for any hyperring homomorphism $f : R \to S$

$$\delta\left(f^{-1}(I)\right) = f^{-1}(\delta(I)) \quad \forall I \in \mathrm{Id}(S). \tag{3}$$

The expansions δ_0 and δ_1 are both intersection preserving and global.

For any $I, J \in \mathrm{Id}(R)$, $\delta_0(I \cap J) = I \cap J = \delta_0(I) \cap \delta_0(J) = I \cap J$:

$$\delta_1(I \cap J) = \sqrt{I \cap J} = \delta_1(I) \cap \delta_1(J) = \sqrt{I} \cap \sqrt{J}. \tag{4}$$

And $\delta_0(f^{-1}(I)) = \delta(I) = I = f^{-1}(I)$.

$\delta_1(f^{-1}(I)) = \delta_1(I) = \sqrt{I} = f^{-1}(I)$. Thus δ_0 and δ_1 are both intersection preserving and global.

Theorem 19. *Let δ be an intersection preserving hyperideal expansion. If $Q_1, Q_2, \ldots, Q_n$ are δ-primary hyperideals of R and $P = \delta(Q_i)$ for all i, then $Q = \bigcap_{i=1}^{n} Q_i$ is δ-primary.*

Proof. Let δ be an intersection preserving hyperideal expansion. If $x, y \in Q$ and $x \notin Q$, then $x \notin Q_k$ for some k. But $xy \in Q \subseteq Q_k$ and Q_k is δ-primary, so $y \in \delta(Q_i)$. But $\delta(Q) = \delta(\bigcap_{i=1}^{n} Q_i) = \bigcap_{i=1}^{n} \delta(Q_i) = P = \delta(Q_k)$. Thus $y \in \delta(Q)$. So Q is δ-primary. □

Definition 20. Let R be a hyperring and δ be a hyperideal expansion. If for an $a \in R$, $a \in \delta\{0\}$ then a is called δ nilpotent.

Note that δ_0 nilpotent element of a ring is the zero element of the ring. Also δ_1 nilpotent elements are exactly the ordinary nilpotent elements.

Theorem 21. *Let δ be a global expansion. Let a hyperideal I of R be δ-primary and then every zero divisor of the quotient hyperring R/I is δ nilpotent.*

Proof. Let I be δ-primary. If $\tilde{r} = r + I$ is a zero divisor of R/I, then there is a $\tilde{s} = s + I \neq I$ with $\tilde{r}\tilde{s} = rs + I = I$. This means that $rs \in I$ and $s \notin I$. Since I is δ-primary so $r \in \delta(I)$; that is, $\tilde{r} \in \delta(I)/I$.

Let $q : R \to R/I$ be the natural quotient hyperring homomorphism. As δ is global, we have $\delta(I) = \delta(q^{-1}(0_{R/I})) = q^{-1}(\delta(0_{R/I}))$.

Since q is onto, so $\delta(I)/I = q(\delta)(I) = \delta(0_{R/I})$. Hence we get $\tilde{r} \in \delta(0_{R/I})$, so $\tilde{r}$ is δ nilpotent. □

Theorem 22. *If δ is global and $f : R \to S$ is a hyperring homomorphism, then, for any δ-primary hyperideal I of S, f^{-1} is a δ-primary hyperideal of R.*

Proof. Let $a, b \in R$ with $ab \in f^{-1}(I)$. If $a \notin f^{-1}(I)$ then $f(a)f(b) \in I$ but $f(a) \notin I$. So, as I is δ-primary, $f(b) \in \delta(I)$. So $b \in f^{-1}(\delta(I)) = \delta(f^{-1}(I))$. Hence $f^{-1}(I)$ is δ-primary. □

Theorem 23. *Let $f : R \to S$ be a surjective hyperring homomorphism. Then a hyperideal I of R that contains* $\ker(f)$ *is δ-primary hyperideal of S.*

Proof. If $f(I)$ is δ-primary, then by $I = f^{-1}(f(I))$ and Theorem 22, I is δ-primary. Now suppose $f(I)$ is δ-primary. If $a, b \in S$ and $ab \in f(\mathrm{I})$ and $a \notin f(I)$, then there are $x, y \in R$ with $f(x) = a$, $f(y) = b$. Then $f(xy) = f(x)f(y) = ab \in f(I)$ implies $xy \in f^{-1}(f(I)) = I$ and $f(x) = a \notin f(I)$ implies $xy \in f^{-1}(f(I)) = I$ and $f(x) = a \notin f(I)$ implies $x \notin I$.

So $y \in \delta(I)$, and hence $b = f(y) \in f(\delta(I))$. Now one only needs to prove $f(\delta(I)) = \delta(f(I))$. But this follows directly from $\delta(I) = \delta(f^{-1}(f(I))) = f^{-1}(\delta(f(I)))$ and that f is surjective. □

The following theorem does not need a proof because it is a consequence of Theorems 22 and 23.

Correspondence Theorem for δ-Primary Hyperideals. *Let f be a hyperring homomorphism of a hyperring R onto a* *hyperring S and let δ be global hyperring expansion. Then f induces a one-one inclusion preserving correspondence between δ-primary hyperideals of R containing $\ker f$ and the δ-primary hyperideals of S in such a way that if I is a δ-primary hyperideal of R that I contains $\ker f$, then $f(I)$ is the corresponding δ-primary hyperideal of S, and if J is a δ-primary hyperideal of S, then $f^{-1}(J)$ is the corresponding δ-primary hyperideal of R.*

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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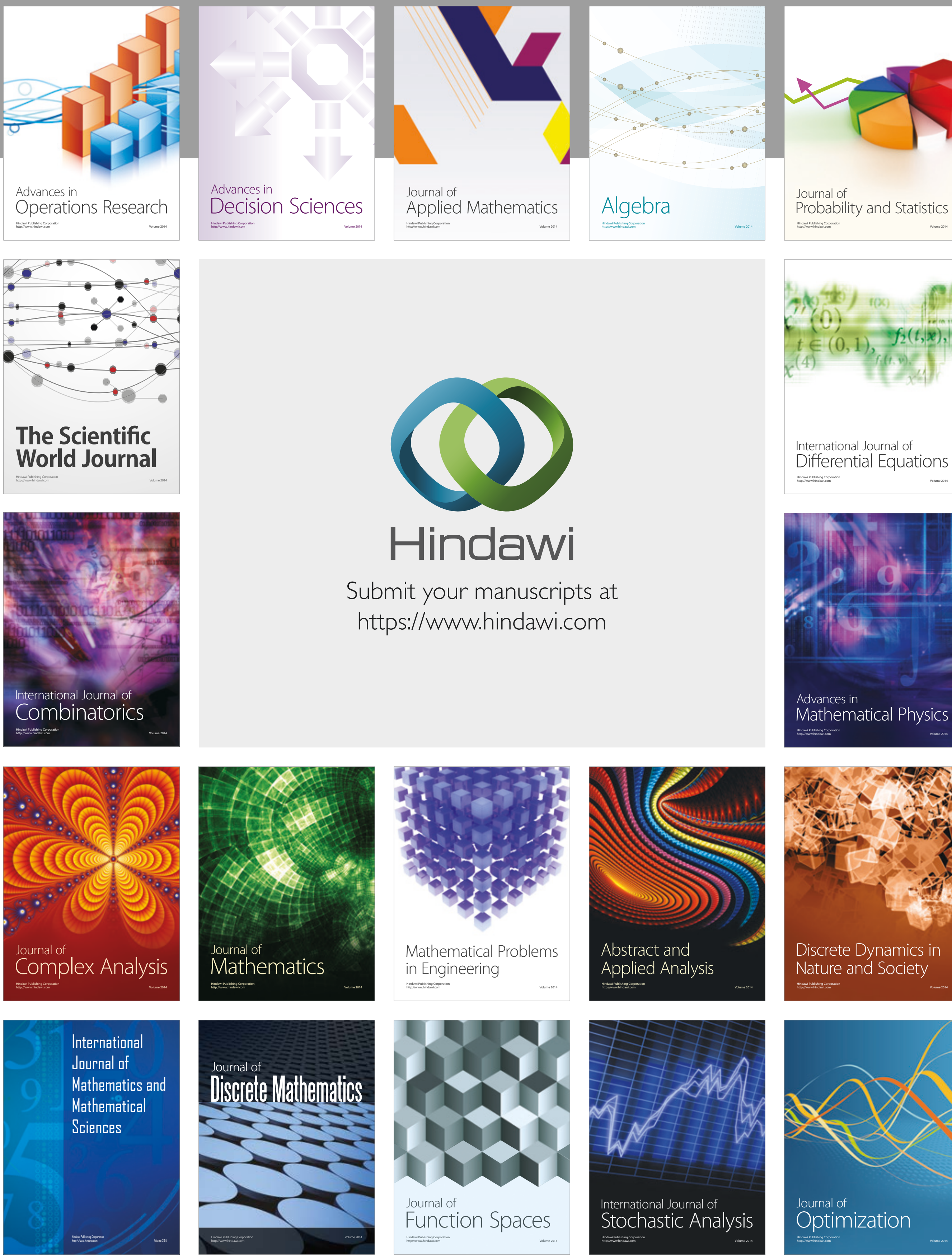
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