Research Article

Resonance Spectra of Caged Stringy Black Hole and Its Spectroscopy

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Maggiore's method (MM), which evaluates the transition frequency that appears in the adiabatic invariant from the highly damped quasinormal mode (QNM) frequencies, is used to investigate the entropy/area spectra of the Garfinkle–Horowitz–Strominger black hole (GHSBH). Instead of the ordinary QNMs, we compute the boxed QNMs (BQNMs) that are the characteristic resonance spectra of the confined scalar fields in the GHSBH geometry. For this purpose, we assume that the GHSBH has a confining cavity (mirror) placed in the vicinity of the event horizon. We then show how the complex resonant frequencies of the caged GHSBH are computed using the Bessel differential equation that arises when the scalar perturbations around the event horizon are considered. Although the entropy/area is characterized by the GHSBH parameters, their quantization is shown to be independent of those parameters. However, both spectra are equally spaced.

1. Introduction

Currently, one of the greatest projects in theoretical physics is to unify general relativity (GR) with quantum mechanics (QM). Such a new unified theory is known as the quantum gravity theory (QGT) [1]. Recent developments in physics show that our universe has a more complex structure than that predicted by the standard model [2]. The QGT is considered to be an important tool that can tackle this problem. However, current QGT still requires further extensive development to reach completion. The development of the QGT began in the seventies when Hawking [3, 4] and Bekenstein [5–9] considered the black hole (BH) as a quantum mechanical system rather than a classical one. In particular, Bekenstein showed that the area of the BH should have a discrete and equally spaced spectrum

$$\mathscr{A}_n = \epsilon n \hbar = 8\pi \xi n \hbar, \quad (n = 0, 1, 2, \ldots), \tag{1}$$

where ϵ is the undetermined dimensionless constant and ξ is of the order of unity. The above expression also shows that the minimum increase in the horizon area is $\Delta \mathscr{A}_{\min} = \epsilon \hbar$. Bekenstein [7, 8] also conjectured that for the Schwarzschild BH (also for the Kerr-Newman BH) the value of ϵ is 8π (or

$\xi = 1$). Following the seminal work of Bekenstein, various methods have been suggested to compute the area spectrum of the BHs. Some methods used for obtaining the spectrum can admit that the value of ϵ is different than that obtained by Bekenstein; this has led to the discussion of this subject in the literature (for a review of this topic, see [10] and references therein). Among those methods, the MM's results [11] show a perfect agreement with Bekenstein's result by modifying the Kunstatter's [12] formula as

$$I_{\mathrm{adb}} = \int \frac{dM}{\Delta \omega}, \tag{2}$$

where I_{adb} denotes the adiabatic invariant quantity and $\Delta \omega = \omega_{n-1} - \omega_n$ represents the transition frequency between the subsequent levels of an uncharged and static BH with the total energy (or mass) M. However, the researchers [13–15] working on this issue later realized that the above definition is not suitable for the charged rotating (hairy) BHs that the generalized form of the definition should be given by

$$I_{\mathrm{adb}} = \int \frac{TdS}{\Delta \omega}, \tag{3}$$

where T and S denote the temperature and the entropy of the BH, respectively. Thus, using the first law of BH thermodynamics, (3) can be modified for the considered BH. According to the Bohr-Sommerfeld quantization rule, I_{adb} behaves as a quantized quantity ($I_{\text{adb}} \simeq n\hbar$) while the quantum number n tends to infinity. To determine $\Delta\omega$, Maggiore considered the BH as a damped harmonic oscillator that has a proper physical frequency in the form of $\omega = (\omega_R^2 + \omega_I^2)^{1/2}$, where ω_R and ω_I are the real and imaginary parts of the frequency, respectively. For the highly excited modes ($n \to \infty$), $\omega_I \gg \omega_R$ and therefore $\Delta\omega \simeq \Delta\omega_I$. Hod [16, 17] was the first to argue that the QNMs can be used in the identification of the quantum transitions for the I_{adb}. Subsequently, there have been other published papers that use the MM to achieve similar results (see, for instance, [18–25]).

In this paper, we focus on the investigation of the GHSBH [26, 27] spectra. The GHSBH is a member of a family of solutions for the low-energy limit of the string theory. This spacetime is obtained when the field content of the Einstein–Maxwell theory is enlarged to cover a dilaton field, which couples to the metric and the gauge field nontrivially. This causes the charged stringy BHs to differ significantly from the Reissner–Nordström (RN) BH. To employ the MM, the QNMs (a set of complex frequencies arising from the perturbed BH) of the GHSBH should be computed. To achieve this, we first consider the KGE for a massless scalar field in the background of the GHSBH. After separating the angular and the radial equations, we obtain a Schrödinger-like wave equation, which is the so-called Zerilli equation [28].

In fact, the spectroscopy of the GHSBH was previously studied by Wei et al. [29]. They used the QNMs of Chen and Jing [30] who studied the monodromy method [31] and obtained an equal spacing of GHSBH spectra at the high frequency modes. There are several methods to calculate the QNMs, such as the WKB method, the phase integral method, continued fractions, and direct integrations of the wave equation in the frequency domain [32]. Our main goal in this study is to consider a recent analytical method, which is invented by Hod [33] for obtaining the GHSBH's resonance spectra or the BQNMs [34–37] and is different from the monodromy method. Thus, we seek to support the study of Wei et al. [29] because we believe that the studies that obtain the same conclusion using different methods are more reliable. For this purpose, we consider the GHSBH as a caged BH, which describes a BH confined within a finite-volume cavity. Hod's idea is indeed based on the recent study [38], which provides compelling evidence that spherically symmetric confined scalar fields in a cavity of Einstein-Klein-Gordon system generically collapse to form caged BHs. To this end, we consider a mirror (confining cavity) surrounding the GHSBH that is placed at a constant radial coordinate with a radius r_m. The scalar field Φ is imposed to terminate at the mirror's location, which requires to use the Dirichlet boundary condition (DBC) ($\Phi(r)|_{r=r_m} = 0$) and Neumann boundary condition (NBC) (($d\Phi(r)/dr)|_{r=r_m} = 0$). In the framework of this scenario, we focus our analysis of the radial wave equation on the near-horizon (NH) region [33]. We then derive the BQNMs of the GHSBH based on the fact that, for the QNMs to exist, the outgoing waves must be terminated

at the event horizon. The NH form of the Zerilli equation is reduced to a Bessel differential equation [39]. After choosing the expedient solution, we impose the Dirichlet and Neumann boundary conditions. Then, we consider some of the transformed features of the Bessel functions for finding the resonance conditions. Next, we use an iteration scheme to define the BQNMs of the GHSBH. Once the BQNMs are obtained, we use the transition frequency $\Delta\omega_I$ in (3) and obtain the GHSBH area/entropy spectra.

The remainder of this paper is arranged as follows. Section 2 introduces the GHSBH metric and analyzes the KGE for a massless scalar field in this geometry. Using the separation of variables technique, we then reduce the physical problem to the Zerilli equation. In Section 3, we show that the Zerilli equation reduces to a Bessel differential equation in the vicinity of the event horizon. The Dirichlet and Neumann boundary conditions at the surface $r = r_m$ of the confining cavity single out two discrete sets of complex BQNMs of the caged GHSBH. Finally, we apply the MM to obtain the quantum spectra of the entropy/area of the GHSBH. Conclusions are presented in Section 4. (Throughout the paper, we set $c = G = k_B = 1$).

2. GHSBH and the Separation of the Massless KGE on It

In this section, we represent the geometry and some of the thermodynamical properties of the GHSBH [26, 27]. We also derive the Zerilli equation and its corresponding effective potential for a massless scalar field propagating in the GHSBH background.

In the low-energy limit of the string field theory, the four-dimensional Einstein-Maxwell-dilaton low-energy action (in Einstein frame) describing the dilaton field ϕ coupled to a $U(1)$ gauge field is given by

$$S = \int d^4x \sqrt{-g} \left(-R + 2\left(\nabla\phi\right)^2 + e^{-2\phi}F^2 \right), \qquad (4)$$

with $F^2 = F_{\mu\nu}F^{\mu\nu}$ in which $F_{\mu\nu}$ is the Maxwell field associated with the $U(1)$ subgroup of $E_8 \times E_8$ or $Spin(32)/Z_2$ [26, 27]. After applying the variational principle to the above action, we obtain the following field equations:

$$\nabla_\mu \left(e^{-2\phi}F^{\mu\nu} \right) = 0,$$

$$\nabla^2\phi + \frac{1}{2}e^{-2\phi}F^2 = 0,$$

$$R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\nabla\phi\right)^2 + 2e^{-2\phi}F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2.$$

$$(5)$$

Their solutions are expressed by the following static and spherically symmetric metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + A(r)d\Omega^2, \qquad (6)$$

where $d\Omega^2$ is the standard metric on the 2-sphere. The metric functions are given by

$$f(r) = 1 - \frac{r_+}{r},$$
$$A(r) = r(r - 2a). \qquad (7)$$

The physical parameter a is defined by

$$a = \frac{Q^2 e^{-2\phi_0}}{2M}, \qquad (8)$$

where Q, M, and ϕ_0 describe the magnetic charge, mass, and the asymptotic constant value of the dilaton, respectively. Besides, $r_+ = 2M$ represents the event horizon of the GHSBH. In this spacetime, the dilaton field is governed by

$$e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{2a}{r}\right), \qquad (9)$$

and the Maxwell field reads

$$F = Q\sin\theta d\theta \wedge d\varphi. \qquad (10)$$

For the electric charge case, one can simply apply the following duality transformations:

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}e^{-2\phi}\epsilon_{\mu\nu}^{\lambda\rho}F_{\lambda\rho}, \quad \phi \longrightarrow -\phi. \qquad (11)$$

Since R^2 part of the GHSBH metric (6) is identical to the Schwarzschild BH, the surface gravity [40] naturally coincides with Schwarzschild's one:

$$\kappa = \lim_{r \to r_+}\sqrt{-\frac{1}{2}\nabla^\mu\chi^\nu\nabla_\mu\chi_\nu} = \left.\frac{f'(r)}{2}\right|_{r=r_+} = \frac{1}{4M}, \qquad (12)$$

where the timelike killing vector is $\chi^\nu = [1, 0, 0, 0]$. Therefore, the Hawking temperature T_H of the GHSBH reads

$$T_H = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi M}. \qquad (13)$$

Thus, the Hawking temperature of the GHSBH is independent of the amount of the charge. But, the similarity between the GHSBH and the Schwarzschild BH is apparent since the radial coordinate does not belong to the areal radius. So, the entropy of the GHSBH is different than the Schwarzschild BH's entropy:

$$S^{\text{BH}} = \frac{\mathscr{A}}{4\hbar} = \frac{\pi r_+(r_+ - 2a)}{\hbar}. \qquad (14)$$

In fact, at extremal charge $Q = \sqrt{2}Me^{\phi_0}$, that is, $a = M$, the BH has a vanishing area and hence its entropy is zero. The extremal GHSBH is not a BH in the ordinary sense since its area has become degenerate and singular: it is indeed a naked singularity. Unlike the singularity of RN, which is timelike, this singularity is null and whence outward-directed radial null geodesics cannot hit it. For a detailed study of the null geodesics of the GHSBH, one may refer to [41]. On the other hand, one can easily prove that the first law of thermodynamics,

$$T_H dS^{\text{BH}} = dM - V_H dQ, \qquad (15)$$

holds for the GHSBH. In (15), the electric potential on the horizon is $V_H = a/Q$. To obtain the GHSBH spectra via the MM, we shall first consider the massless scalar field Ψ satisfying the KGE:

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\Psi\right) = 0. \qquad (16)$$

The chosen ansatz for the scalar field Ψ has the following form:

$$\Psi = A(r)^{-1/2}F(r)e^{-i\omega t}Y_l^m(\theta, \varphi), \quad \text{Re}(\omega) > 0, \qquad (17)$$

in which ω and $Y_l^m(\theta, \varphi)$ represent the frequency of the propagating scalar wave and the spheroidal harmonics with the eigenvalue $L = -l(l + 1)$, respectively. After some algebra, the radial equation can be reduced to the following form:

$$\left[-\frac{d^2}{dr^{*2}} + V(r)\right]F(r) = \omega^2 F(r), \qquad (18)$$

which is nothing but the Zerilli equation [28]. Employing the tortoise coordinate r^* defined as

$$r^* = \int\frac{dr}{f(r)}, \qquad (19)$$

we get

$$r^* = r + r_+\ln\left(\frac{r}{r_+} - 1\right). \qquad (20)$$

Inversely, one obtains

$$r = r_+[1 + W(u)], \qquad (21)$$

where $u = e^{(r^*/r_+ - 1)}$ and $W(u)$ represents the Lambert-W or the omega function [42]. It can be checked that

$$\lim_{r \to r_+}r^* = -\infty, \qquad \lim_{r \to \infty}r^* = \infty. \qquad (22)$$

The effective or the so-called Zerilli potential $V(r)$ is given by

$$V(r) = \frac{f(r)}{r(r - 2a)}\left[L - \frac{a^2}{r(r - 2a)}f(r) + \frac{2M(r - a)}{r^2}\right]. \qquad (23)$$

3. BQNM Frequencies and Entropy/Area Spectra

In this section, we are interested in solutions of the Zerilli equation (17) around the NH. In computing the BQNM frequencies, we first impose the condition that QNMs should be ingoing plane waves at the event horizon. Secondly, we

borrow ideas from recent study [33] and impose the DBC and NBC to have the resonance conditions. In computing the BQNMs, we use an iteration scheme to resolve the resonance conditions.

The metric function $f(r)$ can be rewritten as follows:

$$f(r) \longrightarrow f(x) = \frac{x}{x+1}, \qquad (24)$$

where

$$x = \frac{r - r_+}{r_+}. \qquad (25)$$

Thus, one finds

$$f(x) \cong x + O\left(x^2\right), \qquad (26)$$

$$r^* = \int \frac{r_+ dx}{f(x)} \cong r_+ \ln(x) = \frac{1}{2\kappa} \ln x, \qquad (27)$$

in the NH region ($x \rightarrow 0$). From (27), one reads

$$x = e^{2y}, \qquad (28)$$

where

$$y = \kappa r^*. \qquad (29)$$

After substituting (25) into (23), the NH form of the Zerilli potential can be approximated by

$$V_{NH}(x) = \frac{L + \beta}{r_+ \gamma} x + O\left(x^2\right), \qquad (30)$$

where the parameters are given by

$$\beta = \frac{r_+ - a}{r_+}, \qquad \gamma = r_+ - 2a. \qquad (31)$$

Substituting (28)–(30) into (18), one obtains the following NH form of the Zerilli equation:

$$\left[-\frac{d^2}{dy^2} + \frac{4r_+(L+\beta)}{\gamma} e^{2y} \right] F(y) = \tilde{\omega}^2 F(y). \qquad (32)$$

The above differential equation has two linearly independent solutions:

$$F(y) = C_1 J_{-i\tilde{\omega}} \left(2i\sqrt{\Delta} e^y \right) + C_2 Y_{-i\tilde{\omega}} \left(2i\sqrt{\Delta} e^y \right), \qquad (33)$$

and correspondingly

$$F(x) = C_1 J_{-i\tilde{\omega}} \left(2i\sqrt{\Delta x} \right) + C_2 Y_{-i\tilde{\omega}} \left(2i\sqrt{\Delta x} \right), \qquad (34)$$

where $J_v(z)$ and $Y_v(z)$ are called Bessel functions [39] of the first and second kinds, respectively. C_1 and C_2 are constants. The parameters of the special functions are given by

$$\tilde{\omega} = \frac{\omega}{\kappa}, \qquad (35)$$

$$\Delta = \frac{r_+(L+\beta)}{\gamma}. \qquad (36)$$

The following limiting forms (when v is fixed and $z \rightarrow 0$) of the Bessel functions are needed for our analysis [39, 43]:

$$J_v(z) \sim \frac{[(1/2)z]^v}{\Gamma(1+v)}, \qquad (v \neq -1, -2, -3, \ldots),$$

$$Y_v(z) \sim -\frac{1}{\pi} \Gamma(v) \left(\frac{1}{2} z \right)^{-v}, \qquad (\Re v > 0). \qquad (37)$$

By using them, we obtain the NH ($e^y \ll 1$) behavior of the solution (33) as

$$F \sim C_1 \frac{\left(i\sqrt{\Delta} \right)^{-i\tilde{\omega}}}{\Gamma(1 - i\tilde{\omega})} e^{-i\tilde{\omega}y} - C_2 \frac{1}{\pi} \Gamma(-i\tilde{\omega}) \left(i\sqrt{\Delta} \right)^{i\tilde{\omega}} e^{i\tilde{\omega}y}$$

$$= C_1 \frac{\left(i\sqrt{\Delta} \right)^{-i\tilde{\omega}}}{\Gamma(1 - i\tilde{\omega})} e^{-i\omega r^*} - C_2 \frac{1}{\pi} \Gamma(-i\tilde{\omega}) \left(i\sqrt{\Delta} \right)^{i\tilde{\omega}} e^{i\omega r^*}, \qquad (38)$$

in which C_1 and C_2 correspond to the amplitudes of the NH ingoing and outgoing waves, respectively. Since the QNMs impose that the outgoing waves must spontaneously terminate at the horizon, we deduce that $C_2 = 0$. Thus, the acceptable solution of the radial equation (34) is given by

$$F(x) = C_1 J_{-i\tilde{\omega}} \left(2i\sqrt{\Delta x} \right). \qquad (39)$$

Following [33, 38], we consider the DBC at the surface $x = x_m$ of the confining cage:

$$F(x)|_{x=x_m} = 0. \qquad (40)$$

Thus, we have

$$J_{-i\tilde{\omega}} \left(2i\sqrt{\Delta x_m} \right) = 0. \qquad (41)$$

Using the following relation [39]

$$Y_v(z) = J_v(z) \cot(v\pi) - J_{-v}(z) \csc(v\pi), \qquad (42)$$

we can express the condition (41) as

$$\tan(i\tilde{\omega}\pi) = \frac{J_{i\tilde{\omega}} \left(2i\sqrt{\Delta x_m} \right)}{Y_{i\tilde{\omega}} \left(2i\sqrt{\Delta x_m} \right)}, \qquad (43)$$

which is called the resonance condition. According to the definition of the caged BHs, the boundary of the confining cavity is located at the vicinity of the event horizon [33]. Namely, we have

$$z_m \equiv \Delta x_m \ll 1 \longrightarrow r_m \approx r_+, \qquad (44)$$

in the NH region. Using (37), we can rewrite the resonance condition (43) as

$$\tan(i\tilde{\omega}\pi) \sim -\frac{\pi \left(i\sqrt{z_m} \right)^{i\tilde{\omega}}}{\Gamma(i\tilde{\omega}) \Gamma(i\tilde{\omega}+1) \left(i\sqrt{z_m} \right)^{-i\tilde{\omega}}}$$

$$= i \frac{\pi e^{-\pi\tilde{\omega}}}{\tilde{\omega}\Gamma^2(i\tilde{\omega})} z_m^{i\tilde{\omega}}. \qquad (45)$$

The NBC is given by [33, 38]

$$\left.\frac{d_F(x)}{dx}\right|_{x=x_m} = 0. \tag{46}$$

Using the derivative features of the Bessel functions given in [39], we obtain

$$J_{-i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right) - J_{-i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right) = 0. \tag{47}$$

Using (42), we derive the following relation:

$$Y_{\nu+1}(z) - Y_{\nu-1}(z) = \cot(\nu\pi)\left[J_{\nu+1}(z) - J_{\nu-1}(z)\right]$$
$$- \csc(\nu\pi)\left[J_{-\nu-1}(z) - J_{-\nu+1}(z)\right]. \tag{48}$$

Combining (47) and (48), we further get NBC's resonance condition:

$$\tan(i\widetilde{\omega}\pi)$$
$$= \frac{J_{i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right)}{Y_{i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right)}\left[\frac{-1 + J_{i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right)/J_{i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right)}{1 - Y_{i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right)/Y_{i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right)}\right]. \tag{49}$$

From (37), we find

$$\frac{J_{i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right)}{J_{i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right)} \equiv \frac{Y_{i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right)}{Y_{i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right)} \sim O(z_m), \tag{50}$$

in the NH region. Thus, the resonance condition (49) becomes

$$\tan(i\widetilde{\omega}\pi) \sim -\frac{J_{i\widetilde{\omega}-1}\left(2i\sqrt{z_m}\right)}{Y_{i\widetilde{\omega}+1}\left(2i\sqrt{z_m}\right)} = -i\frac{\pi e^{-\pi\widetilde{\omega}}}{\widetilde{\omega}\Gamma^2(i\widetilde{\omega})}z_m^{i\widetilde{\omega}}. \tag{51}$$

One can immediately realize from (44) that the resonance conditions (49) and (51) are small quantities. We can therefore use an iteration scheme to resolve the resonance conditions. The 0th order resonance equation is given by [33]

$$\tan\left(i\widetilde{\omega}_n^{(0)}\pi\right) = 0, \tag{52}$$

which implies that

$$\widetilde{\omega}_n^{(0)} = -in, \quad (n = 0, 1, 2, \ldots). \tag{53}$$

The 1st order resonance condition is obtained after substituting (53) into r.h.s of (49) and (51). Hence, we have

$$\tan\left(i\widetilde{\omega}_n^{(1)}\pi\right) = \pm i\frac{\pi e^{i\pi n}}{(-in)\Gamma^2(n)}z_m^n, \tag{54}$$

which reduces to

$$\tan\left(i\widetilde{\omega}_n^{(1)}\pi\right) = \mp n\frac{\pi(-z_m)^n}{(n!)^2}, \tag{55}$$

where minus (plus) stands for the DBC (NBC). For having the general characteristic resonance spectra of the caged GHSBH, we use the fact that

$$\tan(x + n\pi) = \tan(x) \approx x, \tag{56}$$

in the $x \ll$ regime. Namely, we obtain

$$i\widetilde{\omega}_n\pi = n\pi \mp n\frac{\pi(-z_m)^n}{(n!)^2}. \tag{57}$$

Therefore, one finds

$$\widetilde{\omega}_n = -in\left[1 \mp \frac{(-z_m)^n}{(n!)^2}\right]. \tag{58}$$

From (35), we read the BQNMs as

$$\omega_n = -i\kappa n\left[1 \mp \frac{(-z_m)^n}{(n!)^2}\right], \quad (n = 0, 1, 2, \ldots). \tag{59}$$

Here, n is called the overtone quantum number or the so-called resonance parameter [44]. For the highly excited states $(n \to \infty)$, (59) behaves as

$$\omega_n \approx -i\kappa n, \quad (n \longrightarrow \infty). \tag{60}$$

The above result is in accordance with the results of [33, 45–48]. Hence, the transition frequency becomes

$$\Delta\omega_I = \kappa = \frac{2\pi T_H}{\hbar}. \tag{61}$$

Substituting (61) into (3), we obtain

$$I_{\text{adb}} = \frac{S^{\text{BH}}}{2\pi}\hbar. \tag{62}$$

Acting upon the Bohr-Sommerfeld quantization rule $(I_{\text{adb}} = \hbar n)$, we find the entropy spectrum as

$$S_n^{\text{BH}} = 2\pi n. \tag{63}$$

Furthermore, since $S^{\text{BH}} = \mathscr{A}/4\hbar$, we can also read the area spectrum:

$$\mathscr{A}_n = 8\pi\hbar n. \tag{64}$$

Thus, the minimum area spacing becomes

$$\Delta\mathscr{A}_{\text{min}} = 8\pi\hbar, \tag{65}$$

which represents that the entropy/area spectra of the GHSBH are evenly spaced. It is obvious that the spectra of the GHSBH are clearly independent of the dilaton parameter a. Furthermore, the spectral-spacing coefficient becomes $\epsilon = 8\pi$, which is in agreement with the Bekenstein's original result [7–9]. In short, our result (65) supports the study of Wei et al. [29].

4. Conclusion

In this paper, the quantum entropy/area spectra of the GHSBH are investigated via the MM that is based on the adiabatic invariant formulation (3) of the BHs. For this purpose, we have considered caged GHSBH whose confining cage (mirror) is placed in the NH region $r_m \approx r_+$ [33]. We have

therefore attempted to find the BQNMs (resonance spectra) of the GHSBH. The massless KGE for the GHSBH geometry has been separated into the angular and the radial parts. In particular, the Zerilli equation (18) with its effective potential (23) of the associated radial equation has been obtained. The NH form of the Zerilli equation is well approximated by a Bessel differential equation. After imposing the boundary conditions appropriate for purely ingoing waves at the event horizon with the DBC and NBC, we have obtained the resonant frequencies of the caged GHSBH. We have then applied the MM to the highly damped BQNMs to derive the entropy/area spectra of the GHSBH. The obtained spectra are equally spaced and are independent of the physical parameters of the GHSBH as concluded in the study of Wei et al. [29]. Moreover, our results support the Kothawala et al.'s conjecture [18], which states that the BHs in Einstein's gravity theory have equispaced area spectrum.

Conflict of Interests

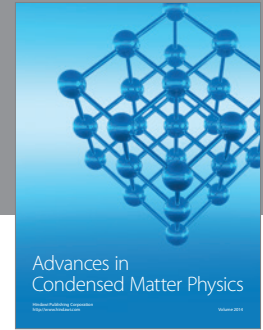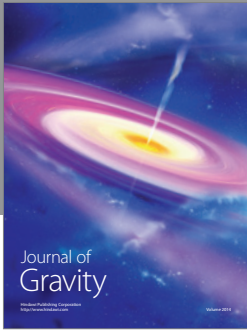The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. Robson, *Keeping It Real (Quantum Gravity, Book 1)*, Pyr Press, London, UK, 2006.

[2] R. Oerter, *The Theory of Almost Everything: The Standard Model, The Unsung Triumph of Modern Physics*, Plume, New York, NY, USA, 2006.

[3] S. W. Hawking, "Black hole explosions?" *Nature*, vol. 248, no. 5443, pp. 30–31, 1974.

[4] S. W. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, no. 3, pp. 199–220, 1975.

[5] J. D. Bekenstein, "Black holes and the second law," *Lettere Al Nuovo Cimento Series 2*, vol. 4, no. 15, pp. 737–740, 1972.

[6] J. D. Bekenstein, "Black holes and entropy," *Physical Review D*, vol. 7, pp. 2333–2346, 1973.

[7] J. D. Bekenstein, "The quantum mass spectrum of the Kerr black hole," *Lettere Al Nuovo Cimento Series 2*, vol. 11, no. 9, pp. 467–470, 1974.

[8] J. D. Bekenstein, "Quantum black holes as atoms," http://arxiv.org/abs/gr-qc/9710076.

[9] J. D. Bekenstein, "Black holes: classical properties, thermodynamics and heuristic quantization," http://arxiv.org/abs/gr-qc/9808028.

[10] G. Gua, "Area spectra of extreme Kerr and nearly extreme Kerr-Newman black holes from quasinormal modes," *General Relativity and Gravitation*, vol. 45, no. 9, pp. 1711–1721, 2013.

[11] M. Maggiore, "Physical interpretation of the spectrum of black hole quasinormal modes," *Physical Review Letters*, vol. 100, Article ID 141301, 2008.

[12] G. Kunstatter, "d-dimensional black hole entropy spectrum from quasinormal modes," *Physical Review Letters*, vol. 90, Article ID 161301, 2003.

[13] E. C. Vagenas, "Area spectrum of rotating black holes via the new interpretation of quasinormal modes," *Journal of High Energy Physics*, vol. 11, article 073, 2008.

[14] A. J. M. Medved, "On the Kerr quantum area spectrum," *Classical and Quantum Gravity*, vol. 25, no. 20, Article ID 205014, 2008.

[15] I. Sakalli, "Quantization of rotating linear dilaton black holes," http://arxiv.org/abs/1406.5130.

[16] S. Hod, "Bohr's correspondence principle and the area spectrum of quantum black holes," *Physical Review Letters*, vol. 81, no. 20, pp. 4293–4296, 1998.

[17] S. Hod, "Best approximation to a reversible process in black-hole physics and the area spectrum of spherical black holes," *Physical Review D*, vol. 59, no. 2, Article ID 024014, 1998.

[18] D. Kothawala, T. Padmanabhan, and S. Sarkar, "Is gravitational entropy quantized?" *Physical Review D: Particles, Fields, Gravitation, and Cosmology*, vol. 78, no. 10, Article ID 104018, 5 pages, 2008.

[19] W. Li, L. Xu, and J. Lu, "Area spectrum of near-extremal SdS Black holes via the new interpretation of quasinormal modes," *Physics Letters B*, vol. 676, no. 4-5, pp. 177–179, 2009.

[20] A. López-Ortega, "Area spectrum of the D-dimensional de Sitter spacetime," *Physics Letters B*, vol. 682, no. 1, pp. 85–88, 2009.

[21] M. R. Setare and D. Momeni, "Spacing of the entropy spectrum for KS black hole in Hořava-Lifshitz gravity," *Modern Physics Letters A*, vol. 26, no. 2, pp. 151–159, 2011.

[22] C. Z. Liu, "Black hole spectroscopy via adiabatic invariant in a quantum corrected spacetime," *The European Physical Journal C*, vol. 72, article 2009, 2012.

[23] S. Sebastian and V. C. Kuriakose, "Spectroscopy and thermodynamics of MSW black hole," *Modern Physics Letters A*, vol. 28, no. 33, Article ID 1350149, 2013.

[24] I. Sakalli, "Quasinormal modes of charged dilaton black holes and their entropy spectra," *Modern Physics Letters A: Particles and Fields, Gravitation, Cosmology, Nuclear Physics*, vol. 28, no. 27, Article ID 1350109, 10 pages, 2013.

[25] I. Sakalli and S. F. Mirekhtiary, "Spectroscopy of Rindler modified Schwarzschild black hole," *Astrophysics and Space Science*, vol. 350, no. 2, pp. 727–731, 2014.

[26] D. Garfinkle, G. T. Horowitz, and A. Strominger, "Charged black holes in string theory," *Physical Review D*, vol. 43, no. 10, 3140 pages, 1991.

[27] D. Garfinkle, G. T. Horowitz, and A. Strominger, "Erratum: charged black holes in string theory," *Physical Review D*, vol. 43, p. 3140, 1992.

[28] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press, New York, NY, USA, 1998.

[29] S.-W. Wei, Y.-X. Liu, K. Yang, and Y. Zhong, "Entropy/area spectra of the charged black hole from quasinormal modes," *Physical Review D*, vol. 81, Article ID 104042, 2010.

[30] S. Chen and J. Jing, "Asymptotic quasinormal modes of a coupled scalar field in the Garfinkle-Horowitz-Strominger dilaton spacetime," *Classical and Quantum Gravity*, vol. 22, no. 3, p. 533, 2005.

[31] L. Motl and A. Neitzke, "Asymptotic black hole quasinormal frequencies," *Advances in Theoretical and Mathematical Physics*, vol. 7, no. 2, pp. 307–330, 2003.

[32] E. Berti, "Black hole quasinormal modes: hints of quantum gravity?" in *Proceedings of the International Workshop on Dynamics and Thermodynamics of Black Holes and Naked Singularities*, Milan, Italy, May 2004.

[33] S. Hod, "Resonance spectra of caged black holes," *The European Physical Journal C*, vol. 74, no. 11, article 3137, 2014.

[34] V. Cardoso, O. J. C. Dias, J. P. S. Lemos, and S. Yoshida, "Black-hole bomb and superradiant instabilities," *Physical Review D*, vol. 70, no. 4, Article ID 044039, 9 pages, 2004.

[35] V. Cardoso, O. J. C. Dias, J. P. S. Lemos, and S. Yoshida, "Erratum: Black-hole bomb and superradiant instabilities," *Physical Review D*, vol. 70, no. 4, Article ID 049903, p. 1, 2004.

[36] R. Li and J. Zhao, "Superradiant instability of the charged scalar field in stringy black hole mirror system," *The European Physical Journal C*, vol. 74, article 3051, 2014.

[37] R. Li and J. Zhao, "Numerical study of superradiant instability for charged stringy black hole-mirror system," *Physics Letters B*, vol. 740, pp. 317–321, 2015.

[38] H. Okawa, V. Cardoso, and P. Pani, "Study of the nonlinear instability of confined geometries," *Physical Review D*, vol. 90, Article ID 104032, 2014.

[39] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, NY, USA, 1965.

[40] R. M. Wald, *General Relativity*, University of Chicago Press, Chicago, Ill, USA, 1984.

[41] S. Fernando, "Null geodesics of charged black holes in string theory," *Physical Review D*, vol. 85, Article ID 024033, 2012.

[42] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," *Advances in Computational Mathematics*, vol. 5, no. 4, pp. 329–359, 1996.

[43] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, USA, 2010.

[44] S. Hod, "Kerr-Newman black holes with stationary charged scalar clouds," *Physical Review D*, vol. 90, Article ID 024051, 2014.

[45] H.-P. Nollert, "Quasinormal modes of Schwarzschild black holes: the determination of quasinormal frequencies with very large imaginary parts," *Physical Review D*, vol. 47, no. 12, pp. 5253–5258, 1993.

[46] S. Hod, "Slow relaxation of rapidly rotating black holes," *Physical Review D: Particles, Fields, Gravitation, and Cosmology*, vol. 78, no. 8, Article ID 084035, 5 pages, 2008.

[47] S. Hod, "Black-hole quasinormal resonances: wave analysis versus a geometric-optics approximation," *Physical Review D*, vol. 80, no. 6, Article ID 064004, 4 pages, 2009.

[48] S. Hod, "Relaxation dynamics of charged gravitational collapse," *Physics Letters A*, vol. 374, no. 29, pp. 2901–2903, 2010.