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TOTALLY REAL SURFACES IN CP^2 WITH PARALLEL MEAN CURVATURE VECTOR

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Abstract. It has been shown that a totally real surface in CP^2 with parallel mean curvature vector and constant Gaussian curvature is either flat or totally geodesic.

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1. INTRODUCTION.

Let J be the almost complex structure on CP^2 and g be the Hermitian metric on CP^2 of constant holomorphic sectional curvature 4. If $\overline{\nabla}$ is the Riemannian connection with respect to g and $\overline{R}$ is the curvature tensor of $\overline{\nabla}$, then

$$(\overline{\nabla}_X J)(Y) = 0, \tag{1.1}$$

$$\overline{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ, \tag{1.2}$$

where X, Y, Z are vector fields on CP^2.

Let M be a 2-dimensional totally real submanifold of CP^2 and ν be the normal bundle of M. If $\chi(M)$ is the lie-algebra of vector fields on M, then for each $X \in \chi(M)$, $JX \in \nu$. The Riemannian connection $\overline{\nabla}$ induces the Riemannian connection ∇ on M and the connection $\nabla^\perp$ in the normal bundle ν. We then have the following Gauss and Weingarten formulae

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \nabla_X N = -A_N X + \nabla_X^\perp N,\ X,\ Y \in \chi(M),\ N \in \nu, \tag{1.3}$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms and are related by $g(h(X,Y),N) = g(A_N X,Y)$. The mean curvature vector H of M is given by

$$H = (1/2)\sum h(e_i,e_i),$$

where $\{e_1,e_2)$ is a local orthonormal frame on M. If $H = 0$, then M is said to be a minimal submanifold of CP^2. It is known that if M is a minimal totally real surface of constant Gaussian curvature in CP^2, then either M is flat or totally geodesic (cf. [2]). The mean curvature vector H is said to be parallel if $\nabla_X^\perp H = 0$, $X \in \chi(M)$. In this paper we consider the totally real surfaces of constant Gaussian curvature with parallel mean curvature vector in CP^2.

The Gaussian curvature K of M is given by

$$K = 1 + g(h(X,X), 2h(Y,Y)) - g(h(X,Y), h(X,Y)), \tag{1.4}$$

where $\{X,Y\}$ is an orthonormal frame on M. The Codazzi equation gives

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \; X,Y,Z \in \chi(M). \tag{1.5}$$

For a totally real surface M, using (1.1) and (1.3), we get

$$h(X,Y) = JA_{JY}X, \; \nabla_X^{\perp} JY = J\nabla_X Y, \; X,Y \in \chi(M). \tag{1.6}$$

Using (1.6) and the symmetry of $h(X,Y)$, we have

$$g(h(Y,Z),JX) = g(h(X,Y),JZ) = g(h(X,Z),JY), \; X,Y,Z \in \chi(M). \tag{1.7}$$

2. MAIN RESULTS

THEOREM 2.1. Let M be a connected totally real surface in CP^2 of constant Gaussian curvature c with parallel mean curvature vector. Then either M is flat or totally geodesic.

PROOF. Let $UM = \{X \in TM : \|X\| = 1\}$ be the unit tangent bundle of M. Define the function $f: UM \to R$ by $F(X) = g(h(X,X),JX)$, which is clearly a smooth function. First suppose that f is constant. Then $f(-X) = -f(X)$ gives $f(X) = 0$ and therefore $g(h(X,X),JX) = 0$, $X \in UM$. Now consider a local orthonormal frame $\{X,Y\}$ on M. Then we have $g(h(X,X),JX) = 0$, $g(h(Y,Y),JY) = 0$,

$$g\left(h\left(\frac{X+Y}{\sqrt{2}}, \frac{X+Y}{\sqrt{2}}\right), J\left(\frac{X+Y}{\sqrt{2}}\right)\right) = 0, \; g\left(h\left(\frac{X-Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}}\right), J\left(\frac{X-Y}{\sqrt{2}}\right)\right) = 0$$

These equations, in view of (1.7), imply that $g(h(X,X),JY) = 0$, $g(h(Y,Y),JX) = 0$, $g(h(X,Y),JX) = 0$, and $g(h(X,Y),JY) = 0$. Since $\{JX,JY\}$ is a local orthonormal frame in the normal bundle ν, we conclude that $h(X,X) = 0$, $h(X,Y) = 0$ and $h(Y,Y) = 0$, which means that M is totally geodesic.

We therefore assume that f is not a constant. Since the unit tangent bundle UM is compact, f attains a maximum at some $e_1 \in UM$. It is known that $g(h(e_1,e_1),JY) = 0$ for any vector in TM which is orthogonal to e_1 (cf. [1]). Choose e_2 such that $\{e_1,e_2\}$ is an orthonormal frame on M. Then we can set

$$h(e_1,e_1) = \alpha Je_1, \; h(e_2,e_2) = \beta Je_1 + \gamma Je_2 \text{ and } h(e_1,e_2) = \beta Je_2, \tag{2.1}$$

where α, β and γ are smooth functions. Using the structure equations of M we have locally

$$\nabla_{e_1} e_1 = ae_2, \; \nabla_{e_2} e_2 = be_1, \; \nabla_{e_1} e_2 = -ae_1, \; \nabla_{e_2} e_1 = -be_2, \tag{2.2}$$

where a, b are smooth functions. Inserting different combinations of the frame vectors e_1, e_2 in (1.5) and using (2.1) and (2.2) we get, upon equating components,

$$e_1 \cdot \beta = a\gamma + 2b\beta - b\alpha, \; e_2 \cdot \alpha = a(\alpha - 2\beta), \; e_2 \cdot \beta - e_1 \cdot \gamma = 3a\beta - b\gamma. \tag{2.3}$$

Since the mean curvature vector $H = (1/2)(h(e_1,e_1) + h(e_2,e_2))$ is parallel, we have

$$\nabla_{e_1}^{\perp}(h(e_1,e_1) + h(e_2,e_2)) = 0 \text{ and } \nabla_{e_2}^{\perp}(h(e_1,e_1) + h(e_2,e_2)) = 0.$$

Using (1.6), (2.1) and (2.2) in the above equations we conclude, upon equating components, that

$$e_1 \cdot (\alpha + \beta) = a\gamma, \quad e_1 \cdot \gamma = -a(\alpha + \beta) \tag{2.4}$$

$$e_2 \cdot (\alpha + \beta) = -b\gamma, \quad e_2 \cdot \gamma = b(\alpha + \beta). \tag{2.5}$$

From (2.3), (2.4) and (2.5), we have

$$
\begin{aligned}
&e_1 \cdot \alpha = b(\alpha - 2\beta), \quad e_1 \cdot \beta = a\nu + 2b\beta - b\alpha, \quad e_1 \cdot \gamma = -a(\alpha + \beta), \\
&e_2 \cdot \alpha = a(\alpha - 2\beta), \quad e_2 \cdot \beta = -b\gamma + 2a\beta - a\alpha, \quad e_2 \cdot \gamma = b(\alpha + \beta).
\end{aligned} \tag{2.6}
$$

In view of (2.1) and (1.4), the Gaussian curvature c is given by $c = 1 + \alpha\beta - \beta^2$. If we operate on this equation by e_1 and e_2 with c constant, and use (2.6), we obtain

$$(\alpha - 2\beta)(a\gamma + b(3\beta - \alpha)) = 0 \text{ and } (\alpha - 2\beta)(-b\gamma + a(3\beta - \alpha)) = 0. \tag{2.7}$$

We have two cases:

Case (i). Suppose $\alpha \neq 2\beta$, then the two equations in (2.7) give $(a^2 + b^2)\gamma = 0$ and $(a^2 + b^2)(3\beta - \alpha) = 0$. If $a^2 + b^2 = 0$, then from (2.2) it follows that M is flat (as c is constant). If $a^2 + b^2 = 0$, then we have $\gamma = 0$ and $3\beta - \alpha = 0$. Since a and b cannot both be zero and $\gamma = 0$ it follows from equations (2.4) and (2.5) that $\alpha + \beta = 0$. Thus we have $\gamma = 0$ and $\alpha + \beta = 0$, which implies that $H = 0$, that is, M is minimal.

Case (ii). Suppose $\alpha = 2\beta$. Then from (2.6) we get that α is constant, and consequently β is also constant. With $\alpha = 2\beta$ and β constant equations (2.6) give $a\gamma = 0$ and $b\gamma = 0$. Thus either $a = b = 0$ or $\gamma = 0$, which results in either M being flat or $\gamma = 0$. If M is not flat, that is, not both a and b are zero, and $\gamma = 0$, then from (2.4) and (2.5) we get $\alpha + \beta = 0$. This shows that $H = 0$. Hence either M is flat or minimal. But since a minimal totally real surface is constant curvature in CP^2 is either flat or totally geodesic [2], the theorem is proved.

In the following we first prove that in any submanifold of a Riemannian manifold if the second fundamental form is parallel, then the mean curvature vector is parallel. Though this is a simple observation, it does not seem to appear in the literature and is worth mentioning. As a corollary then we obtain the same result as in Section 2 for the totally real surfaces of CP^2 with parallel second fundamental form.

THEOREM 2.2. Let M be a submanifold of a Riemannian manifold $\overline{M}$ with parallel second fundamental form. Then the mean curvature vector of M is parallel.

PROOF. Suppose $dim M = n$. Then for a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ of M, the mean curvature H is given by

$$H = (1/n)\sum_{i=1}^{n} h(e_i, e_i).$$

Since the second fundamental form is parallel we have

$$(\overline{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0 \text{ for } X, Y, Z \in \chi(M).$$

Thus for each frame vector e_i we can write

$$\nabla_X^{\perp} h(e_i, e_i) = 2h(\nabla_X e_i, e_i).$$

Adding these equations we get

$$n\nabla_X^{\perp} H = 2\sum_{i=1}^{n} h(\nabla_X e_i, e_i).$$

Let ω_j^i be the connections forms on M. Then we have

$$\nabla_X e_i = \sum_{j=1}^{n} \omega_i^j(x) e_j.$$

Substituting this into the above equation we get

$$n\nabla_X{}^{\perp} H = 2 \sum_{i,j=1}^{n} \omega_i^j(X)\, h(e_i, e_j).$$

Since $\omega_i^j(X) = -\omega_j^i(X)$ and $h(e_i, e_j) = h(e_j, e_i)$, we conclude that $\nabla_X{}^{\perp} H = 0$, $X \in \chi(M)$.

As a direct consequence of this theorem and the theorem in the previous section we have

COROLLARY 2.1. Let M be a connected totally real surface in CP^2 with parallel second fundamental form and constant Gaussian curvature. Then M is either flat or totally geodesic.

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