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Research Article

Strong Convergence Theorems for an Implicit Iterative Algorithm for the Split Common Fixed Point Problem

Huimin He,[1] Sanyang Liu,[1] and Rudong Chen[2]

[1] *School of Mathematics and Statistics, Xidian University, Xi'an 710071, China*
[2] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*

Correspondence should be addressed to Huimin He; huiminhe@126.com

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The aim of this paper is to construct a novel implicit iterative algorithm for the split common fixed point problem for the demicontractive operators U, T, and $x_n = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda(x_n - \rho_n A^*(I-T)Ax_n)$, $n \geq 0$, where $U_\lambda = (1-\lambda)I + \lambda U$, and we obtain the sequence which strongly converges to a solution $\widehat{x}$ of this problem, and the solution $\widehat{x}$ satisfies the variational inequality. $\langle \widehat{x} - f(\widehat{x}), \widehat{x} - z\rangle \leq 0$, $\forall z \in S$, where S denotes the set of all solutions of the split common fixed point problem.

1. Introduction

The split feasibility problem (SFP) is to find a point

$$x \in C \quad \text{such that } Ax \in Q, \tag{1}$$

where C is a nonempty closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

This problem was proposed by Censor and Elfving [1] in 1994.

Since the SFP can extensively be applied in fields such as intensity-modulated radiation therapy, signal processing, and image reconstruction, then the SFP has received so much attention by so many scholars; see [2–23].

In 1994, Censor and Elfving [1] proposed the original algorithm in R^n,

$$x_{n+1} = A^{-1} P_Q P_{A(C)} A x_n, \tag{2}$$

where C and Q are nonempty closed convex subsets of R^n, A in the finite-dimensional R^n is a $n \times n$ matrix, and P_Q is the projection operator from H_2 onto Q.

As we know, the computation of the inverse A^{-1} is not easy if the inverse of A existed. So, the algorithm (2) does not become popular.

In 2002 and 2004, Byrne [2, 3] gave the so-called CQ algorithm as follows:

$$x_{n+1} = P_C\left(x_n - \gamma A^*\left(I - P_Q\right) A x_n\right), \quad n \geq 0, \tag{3}$$

where $0 < \gamma < 2/\rho$ with ρ taken as the largest eigenvalue of the operator A^*A and P_C and P_Q denote the projection operators from H_1 and H_2 onto the sets C, Q, respectively.

For the stepsize of algorithm (3) is fixed and closely related to spectral radius of A^*A, then the projection operators P_C and P_Q are not easily calculated usually.

The split common fixed point problem (SCFP) is to find a point

$$x \in \operatorname{Fix}(U) \quad \text{such that } Ax \in \operatorname{Fix}(T), \tag{4}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$, and $\operatorname{Fix}(U)$ and $\operatorname{Fix}(T)$ denote the fixed point sets of U and T.

This problem was proposed by Censor and Segal [12] in 2009. Note that the SCFP is closely related to SFP and it is a particular case of SFP.

In 2009, Censor and Segal [12] introduced the original algorithm for directed operators as follows:

$$x_{n+1} = U\left(x_n - \rho A^*(I - T) A x_n\right), \quad n \geq 0, \tag{5}$$

where the step size ρ satisfies $0 < \rho < 2/\|A\|^2$, and they obtained that $\{x_n\}$ weakly converges to a solution of the SCFP

(4) if the solution of SCFP exists. But it is obvious that the choice of the step size ρ depends on the norm of operator, A, which is the disadvantage of this algorithm.

The next two years, some extension results on the operators are obtained, such as Moudafi (2010) [24], Moudafi (2011) [25], and Wang and Xu (2011) [14].

In order to overcome this disadvantage, Cui and Wang [26] proposed the following algorithm in 2014:

$$x_{n+1} = U_\lambda \left(x_n - \rho_n A^* (I - T) A x_n\right), \quad n \geq 0, \tag{6}$$

where $U_\lambda = (1-\lambda)I + \lambda U$ and the step size ρ_n is chosen by the following way:

$$\rho_n = \begin{cases} \dfrac{(1-\tau)\left\|(I-T)Ax_n\right\|^2}{2\left\|A^*(I-T)Ax_n\right\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

and they proved that the sequence $\{x_n\}$ converges weakly to a solution of the SCFP (4). Note that the advantage of this algorithm is that the step size ρ_n searches automatically and does not depend on the norm of operator A.

Recently, Byrne et al. [27] introduced the split common null point problem (SCNPP) for set-valued maximal monotone mappings in Hilbert spaces. Given set-valued mappings $B_i : H_1 \to 2^{H_1}$, $1 \leq i \leq p$, and $F_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$, respectively, and the bounded linear operators $A_j : H_1 \to H_2$, $1 \leq j \leq r$, the SCNPP is formulated as follows:

$$\begin{aligned} &\text{find} && x \in H_1 \\ &\text{such that} && 0 \in \bigcap_{i=1}^{p} B_i(x) \\ &\text{such that} && y_j = A_j(x) \in H_2 \\ & && \text{solve } 0 \in \bigcap_{j=1}^{r} F_J\left(y_j\right). \end{aligned} \tag{8}$$

As we know, the SCNPP generalizes the split common fixed point problem and the split variational inequality problem [28, 29].

Motivated by the viscosity idea of [30], in this paper, we construct a novel algorithm for demicontractive operators to approximate the solution of the SCFP (4), that is, the following implicit iterative algorithm:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n) U_\lambda \left(x_n - \rho_n A^* (I-T) A x_n\right), \quad n \geq 0, \tag{9}$$

where $U_\lambda = (1-\lambda)I + \lambda U$ and the step size ρ_n is also chosen as (7).

The research highlight of this paper is that the strong convergence of the SCFP (4) is constructed; that is to say the sequence $\{x_n\}$ generated by (9) converges strongly to a solution of the SCFP.

2. Preliminaries

Throughout this paper, we denote the set of all solutions of the SCFP (4) by S. We use $x_n \rightharpoonup x$ to indicate that $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ symbolizes the sequence $\{x_n\}$ which converges strongly to x.

Let H, H_1, and H_2 be Hilbert spaces endowed with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and C and Q are nonempty closed convex subsets of H_1 and H_2, respectively.

Some concepts and lemmas are given in the following and they are useful in proving our main results.

Definition 1. A operator $T : H \to H$ is said to be

(i) nonexpansive if

$$\left\|Tx - Ty\right\| \leq \|x - z\|, \quad \forall x, y \in H \tag{10}$$

(ii) quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \quad \forall x \in H, \ \forall z \in \operatorname{Fix}(T) \tag{11}$$

(iii) directed if

$$\langle z - Tx, x - Tx \rangle \leq 0, \quad \forall x \in H, \ \forall z \in \operatorname{Fix}(T) \tag{12}$$

(iv) τ-demicontractive with $\tau < 1$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \tau \|x - Tx\|^2, \quad \forall x \in H, \ \forall z \in \operatorname{Fix}(T) \tag{13}$$

Note that (12) is equivalent to

$$\|z - Tx\|^2 + \|x - Tx\|^2 - \|x - z\|^2 \leq 0, \quad \forall x \in H, \ \forall z \in \operatorname{Fix}(T) \tag{14}$$

Definition 2. Let $T : H \to H$ be an operator, then $I - T$ is said to be demiclosed at zero, if for any $\{x_n\}$ in H, the following implication holds

$$\begin{gathered} x_n \rightharpoonup x \\ (I - T)x_n \longrightarrow 0 \\ \Downarrow \\ x = Tx \end{gathered} \tag{15}$$

As we know, the nonexpansive mappings are demiclosed at zero [31].

Definition 3. Let C be a nonempty closed convex subset of a Hilbert space H, the metric (nearest point) projection P_C from H to C is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\left\|x - P_C x\right\| = \inf\left\{\left\|x - y\right\| : y \in C\right\}. \tag{16}$$

Lemma 4 (see [32]). *Let C be a nonempty closed convex subset of a Hilbert space H, P_C is a nonexpansive mapping from H onto C and is characterized as: Given $x \in H$, there holds the inequality*

$$\left\langle x - P_C x, y - P_C x \right\rangle \leq 0, \quad \forall y \in C. \tag{17}$$

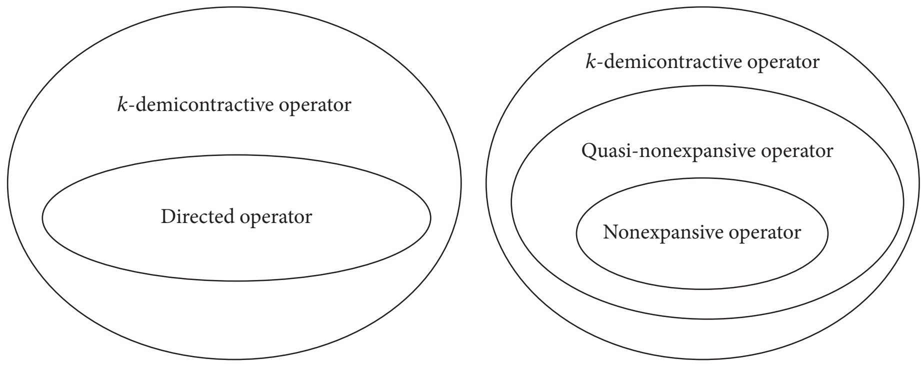

Figure 1: The relations of k-demicontractive operator, directed operator, quasi-nonexpansive operator, and nonexpansive operator.

Lemma 5 (see [32]). *Let H be a Hilbert space, then the following inequality holds,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{18}$$

Lemma 6 (Cui and Wang [26]). *Let $A : H_1 \to H_2$ be a bounded linear operator and $T : H_2 \to H_2$ a τ-demicontractive operator with $\tau < 1$. If $A^{-1}Fix(T) \neq \emptyset$, then*

(a) $(I-T)Ax = 0 \Leftrightarrow A^*(I-T)Ax = 0$, $\forall x \in H_1$.

(b) *In addition, for $z \in A^{-1}Fix(T)$*

$$\begin{aligned} &\left\|x - \rho A^*(I-T)Ax - z\right\|^2 \\ &\quad \leq \|x-z\|^2 - \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax\|^4}{\|A^*(I-T)Ax\|^2} \end{aligned} \tag{19}$$

where $x \in H_1$, $Ax \neq T(Ax)$ and

$$\rho := \frac{1-\tau}{2}\frac{\|(I-T)Ax\|^2}{\|A^*(I-T)Ax\|^2}. \tag{20}$$

Lemma 7 (Cui and Wang [26]). *Let $U : H_1 \to H_1$ be a k-demicontractive operator with $k < 1$. Denote $U_\lambda := (1-\lambda)I + \lambda U$ for $\lambda \in (0, 1-k)$. Then for any $x \in H_1$ and $z \in Fix(U)$,*

$$\left\|U_\lambda x - z\right\|^2 \leq \|x-z\|^2 - \lambda(1-k-\lambda)\|x-Ux\|^2. \tag{21}$$

3. Main Results

Proposition 8. *Based on the definitions in preliminaries, the classes of k-demicontractive operators, directed operators, quasi-nonexpansive operators, and nonexpansive operators have close relations. We can visually use the following Venn diagram (Figure 1) to denote their relations.*

Proof. From Definition 1, the following conclusion is obtained easily.

(i) The nonexpansive operator is quasi-nonexpansive operator.

(ii) The quasi-nonexpansive operator is 0-demicontractive operator.

(iii) The directed operator is -1-demicontractive operator. □

Next, we give the novel implicit algorithm to solve the SCFP (4) for demicontractive operators. In the sequel, the assumptions are given as follows:

(i) $U : H_1 \to H_1$ is a k-demicontractive operator with $k < 1$.

(ii) $T : H_2 \to H_2$ is a τ-demicontractive operator with $\tau < 1$.

(iii) Both $I-U$ and $I-T$ are demiclosed at zero.

Algorithm 9. Choose an initial guess $x_0 \in H_1$ arbitrarily. Let f be a fixed contraction on Fix(U) with coefficient α $(0 < \alpha < 1)$, $\lambda \in (0, 1-\tau)$. Assume that the nth iteration x_n has been constructed. Then the $(n+1)$th iteration is via the following formula:

$$\begin{aligned} x_n &= \alpha_n f(x_n) \\ &\quad + (1-\alpha_n) U_\lambda\left(x_n - \rho_n A^*(I-T)Ax_n\right), \\ &\qquad n \geq 0, \end{aligned} \tag{22}$$

where A^* is the adjoint of bounded linear operator A and the step size ρ_n is chosen in the following way:

$$\rho_n = \begin{cases} \dfrac{(1-\tau)\left\|(I-T)Ax_n\right\|^2}{2\left\|A^*(I-T)Ax_n\right\|^2}, & Ax_n \neq T(Ax_n) \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Theorem 10. *Assume the solution set of the SCFP (4) $S \neq \emptyset$. If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then the sequence $\{x_n\}$ generated by implicit algorithm (22) converges strongly to a point $\widehat{x} \in S$, and $\widehat{x} = P_S f(\widehat{x})$; that is, $\widehat{x}$ satisfies the following variational inequality:*

$$\left\langle \widehat{x} - f(\widehat{x}), \widehat{x} - z\right\rangle \leq 0, \quad \forall z \in S. \tag{24}$$

Proof. The proof is divided into three steps.

Step 1. We show that $\{x_n\}$ is bounded.

Denote $y_n := x_n - \rho_n A^*(I-T)Ax_n$ and take $z \in S$; it follows from (22) that

$$\begin{aligned}\|x_n - z\| &= \|\alpha_n (f(x_n) - z) + (1-\alpha_n)(U_\lambda y_n - z)\| \\ &\le \alpha_n \|f(x_n) - f(z)\| + (1-\alpha_n)\|U_\lambda y_n - z\| \\ &\quad + \alpha_n \|f(z) - z\| \\ &\le \alpha\alpha_n \|x_n - z\| + (1-\alpha_n)\|U_\lambda y_n - z\| \\ &\quad + \alpha_n \|f(z) - z\|.\end{aligned} \tag{25}$$

(i) If $\rho_n = 0$. Then $y_n = x_n$; from (21), we get

$$\|U_\lambda x_n - z\| \le \|x_n - z\|. \tag{26}$$

Thus

$$\begin{aligned}\|x_n - z\| &\le \alpha\alpha_n \|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &\quad + \alpha_n \|f(z) - z\|.\end{aligned} \tag{27}$$

Hence

$$\|x_n - z\| \le \frac{1}{1-\alpha}\|f(z) - z\|. \tag{28}$$

So, $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.

(ii) If $\rho_n \neq 0$. It follows from (19) and (21) that we get

$$\begin{aligned}\|U_\lambda y_n - z\|^2 &\le \|y_n - z\|^2 - \lambda(1-\lambda-k)\|y_n - Uy_n\|^2 \\ &= \|x_n - \rho_n A^*(I-T)Ax_n - z\|^2 \\ &\quad - \lambda(1-\lambda-k)\|y_n - Uy_n\|^2 \\ &\le \|x_n - z\|^2 \\ &\quad - \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_n\|^4}{\|A^*(I-T)Ax_n\|^2} \\ &\quad - \lambda(1-\lambda-k)\|y_n - Uy_n\|^2.\end{aligned} \tag{29}$$

Thus,

$$\|U_\lambda y_n - z\| \le \|x_n - z\|. \tag{30}$$

Combining with (30) and (25), we get (28). So, $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.

Step 2. We show that there exists a subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $x_{n_j} \to \hat{x}$ as $j \to \infty$, and $\hat{x} \in S$ solves the variational inequality (24).

By the reflexivity of Hilbert space H_1 and the boundedness of $\{x_n\}$, there exists a weakly convergence subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $x_{n_j} \rightharpoonup \hat{x}$, as $j \to \infty$.

First, we show that $x_{n_j} \to \hat{x}$, as $j \to \infty$.

Next, we denote x_{n_j} by x_j.

(i) If $\rho_{n_j} = 0$. From (18) and (21), we have

$$\begin{aligned}&\|x_j - \hat{x}\|^2 \\ &\quad \le (1-\alpha_j)\|U_\lambda x_j - \hat{x}\|^2 \\ &\qquad + 2\alpha_j \langle f(x_j) - \hat{x}, x_j - \hat{x}\rangle \\ &\quad \le (1-\alpha_j)\left[\|x_j - \hat{x}\|^2 - \lambda(1-k-\lambda)\|x_j - Ux_j\|^2\right] \\ &\qquad + 2\alpha_j \langle f(x_j) - \hat{x}, x_j - \hat{x}\rangle \\ &\quad \le (1-\alpha_j)\|x_j - \hat{x}\|^2 + 2\alpha_j \langle f(x_j) - \hat{x}, x_j - \hat{x}\rangle.\end{aligned} \tag{31}$$

Hence

$$\begin{aligned}\|x_j - \hat{x}\|^2 &\le 2\langle f(x_j) - \hat{x}, x_j - \hat{x}\rangle \\ &\le 2\langle f(x_j) - f(\hat{x}), x_j - \hat{x}\rangle \\ &\quad + 2\langle f(\hat{x}) - \hat{x}, x_j - \hat{x}\rangle \\ &\le 2\alpha\|x_j - \hat{x}\|^2 + 2\langle f(\hat{x}) - \hat{x}, x_j - \hat{x}\rangle.\end{aligned} \tag{32}$$

So

$$\|x_j - \hat{x}\|^2 \le \frac{2}{1-2\alpha}\langle f(\hat{x}) - \hat{x}, x_j - \hat{x}\rangle. \tag{33}$$

For $\{x_j\} \rightharpoonup \hat{x}$ as $j \to \infty$, the above inequality implies that

$$x_j \longrightarrow \hat{x} \quad \text{as } j \longrightarrow \infty. \tag{34}$$

(ii) If $\rho_{n_j} \neq 0$. From (18), (19) and (21), we have

$$\begin{aligned}\|x_j - \hat{x}\|^2 &\le (1-\alpha_j)\|U_\lambda y_j - \hat{x}\|^2 \\ &\quad + 2\alpha_j \langle f(x_j) - \hat{x}, x_{n+1} - \hat{x}\rangle \le (1-\alpha_j) \\ &\quad \cdot \left(\|y_j - \hat{x}\|^2 - \lambda(1-k-\lambda)\|y_j - Uy_j\|^2\right) \\ &\quad + 2\alpha_j \langle f(x_j) - \hat{x}, x_{n+1} - \hat{x}\rangle \le (1-\alpha_j) \\ &\quad \cdot \left(\|x_j - \hat{x}\|^2 - \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_j\|^4}{\|A^*(I-T)Ax_j\|^2}\right) \\ &\quad - (1-\alpha_j)\lambda(1-k-\lambda)\|y_j - Uy_j\|^2 \\ &\quad + 2\alpha_j \langle f(x_j) - \hat{x}, x_j - \hat{x}\rangle \le (1-\alpha_j)\|x_j - \hat{x}\|^2 \\ &\quad + 2\alpha_j \langle f(x_j) - \hat{x}, x_j - \hat{x}\rangle.\end{aligned} \tag{35}$$

Then, (33) is obtained. By the similar proofs of the case of $\rho_{n_j} = 0$, we conclude that

$$x_j \longrightarrow \hat{x} \quad \text{as } j \longrightarrow \infty. \tag{36}$$

Second, we show that $\widehat{x} \in S$.

(i) If $\rho_{n_j} = 0$. From (31), we get

$$\begin{aligned}\lambda(1-k-\lambda)\left\|x_j - Ux_j\right\|^2 &\le \alpha_j \left\langle f(x_j) - \widehat{x}, x_j - \widehat{x}\right\rangle - \alpha_j \left\|x_j - \widehat{x}\right\|^2 \\ &\le \alpha_j \left\langle f(x_j) - \widehat{x}, x_j - \widehat{x}\right\rangle.\end{aligned} \quad (37)$$

Hence

$$\left\|x_j - Ux_j\right\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \quad (38)$$

For the case $\rho_{n_j} = 0$, then it is clear we obtain

$$\left\|(I-T)Ax_j\right\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \quad (39)$$

From (37) and the demiclosedness of $I - U$ at zero, then

$$\widehat{x} \in \text{Fix}(U). \quad (40)$$

Since A is bounded linear operator, then A is weak continuity; then

$$\begin{aligned}&x_j \rightharpoonup \widehat{x} \Longrightarrow \\ &Ax_j \rightharpoonup A\widehat{x}, \quad \text{as } j \longrightarrow \infty.\end{aligned} \quad (41)$$

From (39) and the demiclosedness of $I - T$ at zero, then

$$A\widehat{x} \in \text{Fix}(T). \quad (42)$$

Hence, $\widehat{x} \in S$ by (40) and (42).

(ii) If $\rho_{n_j} \neq 0$. From (35), we get

$$\begin{aligned}&\lambda(1-k-\lambda)\left\|y_j - Uy_j\right\|^2 + \frac{(1-\tau)^2}{4}\frac{\left\|(I-T)Ax_j\right\|^4}{\left\|A^*(I-T)Ax_j\right\|^2} \\ &\le 2\frac{\alpha_j}{1-\alpha_j}\left\langle f(x_j) - \widehat{x}, x_j - \widehat{x}\right\rangle - \alpha_j\left\|x_j - \widehat{x}\right\|^2 \\ &\le 2\frac{\alpha_j}{1-\alpha_j}\left\langle f(x_j) - \widehat{x}, x_j - \widehat{x}\right\rangle.\end{aligned} \quad (43)$$

So, we have

$$\begin{aligned}0 &\le \lambda(1-k-\lambda)\left\|y_j - Uy_j\right\|^2 \le 2\frac{\alpha_j}{1-\alpha_j}\left\langle f(x_j) - \widehat{x}, x_j - \widehat{x}\right\rangle, \\ &\frac{(1-\tau)^2}{4}\frac{\left\|(I-T)Ax_j\right\|^4}{\left\|A^*(I-T)Ax_j\right\|^2} \le 2\frac{\alpha_j}{1-\alpha_j}\left\langle f(x_j) - \widehat{x}, x_j - \widehat{x}\right\rangle.\end{aligned} \quad (44)$$

Take $j \to \infty$, we have

$$\left\|y_j - Uy_j\right\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \quad (45)$$

$$\frac{\left\|(I-T)Ax_j\right\|^2}{\left\|A^*(I-T)Ax_j\right\|} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \quad (46)$$

Moreover,

$$\begin{aligned}\left\|(I-T)Ax_j\right\| &= \|A\| \cdot \frac{\left\|(I-T)Ax_j\right\|}{\|A\|} \\ &= \|A\| \cdot \left\|(I-T)Ax_j\right\| \frac{\left\|(I-T)Ax_j\right\|}{\|A\|\left\|(I-T)Ax_j\right\|} \\ &\le \|A\| \cdot \left\|(I-T)Ax_j\right\| \frac{\left\|(I-T)Ax_j\right\|}{\left\|A^*(I-T)Ax_j\right\|} \\ &= \|A\| \frac{\left\|(I-T)Ax_j\right\|^2}{\left\|A^*(I-T)Ax_j\right\|}.\end{aligned} \quad (47)$$

Hence, from (46)

$$\left\|(I-T)Ax_j\right\| \longrightarrow 0, \quad \text{as } j \longrightarrow \infty. \quad (48)$$

By $y_j := x_j - \rho_j A^*(I-T)Ax_j$, we have

$$\begin{aligned}\left\|x_j - y_j\right\| &= \rho_j \left\|A^*(I-T)Ax_j\right\| \\ &= \frac{1-\tau}{2}\frac{\left\|(I-T)Ax_j\right\|^2}{\left\|A^*(I-T)Ax_j\right\|}.\end{aligned} \quad (49)$$

So,

$$\left\|x_j - y_j\right\| \longrightarrow 0, \quad \text{as } j \longrightarrow \infty. \quad (50)$$

For $x_j \to \widehat{x}$, then $y_j \to \widehat{x}$ by (50).

From (45) and the demiclosedness of $I - U$ at zero, then

$$\widehat{x} \in \text{Fix}(U). \quad (51)$$

From (48) and the demiclosedness of $I - T$ at zero, then

$$A\widehat{x} \in \text{Fix}(T). \quad (52)$$

So, $\widehat{x} \in S$ by (51) and (52).

Third, we show that $\widehat{x} \in S$ solves the variational inequality (24).

Indeed, from (22), we get

$$\begin{aligned}(I-f)x_j &= -\frac{1-\alpha_j}{\alpha_j}\left[x_j - U_\lambda\left(x_j - \rho_n A^*(I-T)Ax_j\right)\right] \\ &= -\frac{1-\alpha_j}{\alpha_j}\left(x_j - U_\lambda y_j\right).\end{aligned} \quad (53)$$

The above equality and (30) imply that

$$\begin{aligned}\left\langle (I-f)x_j, x_j - z\right\rangle &= -\frac{1-\alpha_j}{\alpha_j}\left\langle x_j - U_\lambda y_j, x_j - z\right\rangle \\ &= -\frac{1-\alpha_j}{\alpha_j}\left\langle x_j - z - \left(U_\lambda y_j - z\right), x_j - z\right\rangle \le 0.\end{aligned} \tag{54}$$

Since

$$\begin{aligned}&\left\langle x_j - z - \left(U_\lambda y_j - z\right), x_j - z\right\rangle \\ &\quad = \left\|x_j - z\right\|^2 - \left\langle U_\lambda y_j - z, x_j - z\right\rangle \\ &\quad \ge \left\|x_j - z\right\|^2 - \left\|U_\lambda y_j - z\right\| \cdot \left\|x_j - z\right\| \\ &\quad \ge \left\|x_j - z\right\|^2 - \left\|x_j - z\right\| \cdot \left\|x_j - z\right\| = 0,\end{aligned} \tag{55}$$

take the limit through $j \to \infty$ and we obtain

$$\langle \hat{x} - f(\hat{x}), \hat{x} - z\rangle \le 0, \quad \forall z \in S. \tag{56}$$

Step 3. We show that $x_n \to \hat{x}$ as $n \to \infty$.

To show that $x_n \to \hat{x}$ as $n \to \infty$, we only need to show that any subsequence of $\{x_n\}$ converges strongly to $\hat{x}$.

Assuming the above conclusion does not hold, that is to say, there exists another subsequence $\{x_{n_k}\} \subseteq \{x_n\}$, which converges strongly to $\tilde{x} \neq \hat{x}$ as $k \to \infty$. Similarly, we know $\tilde{x} \in S$ solves the variational inequality

$$\langle \tilde{x} - f(\tilde{x}), \tilde{x} - z\rangle \le 0, \quad \forall z \in S. \tag{57}$$

Replacing $z \in S$ with $\tilde{x} \in S$ in (56) and replacing $z \in S$ with $\hat{x} \in S$ in (57), we obtain

$$\begin{aligned}&\langle \hat{x} - f(\hat{x}), \hat{x} - \tilde{x}\rangle \le 0, \\ &\langle \tilde{x} - f(\tilde{x}), \tilde{x} - \hat{x}\rangle \le 0.\end{aligned} \tag{58}$$

Adding up the above variational inequality yields

$$\begin{aligned}(1-\alpha)\|\hat{x} - \tilde{x}\|^2 &\le \langle \hat{x} - \tilde{x}, (I-f)\hat{x} - (I-f)\tilde{x}\rangle \\ &\le 0.\end{aligned} \tag{59}$$

Thus $\hat{x} = \tilde{x}$. This is contradicting with the assumption $\hat{x} \neq \tilde{x}$, so $\{x_n\}$ converges strongly to $\hat{x}$.

The proof is completed. □

4. Applications

In this section, we consider some special cases as the applications of Theorem 10.

Based on the relations of k-demicontractive operators, directed operators, and quasi-nonexpansive operators (Proposition 8), the following corollaries are obtained easily.

Corollary 11. *Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive operators and $I - U$ and $I - T$ be demiclosed at zero. Assume the SCFP (4) is consistent ($S \neq \emptyset$). If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by implicit algorithm (22) converges strongly to a point $\hat{x} \in S$, and $\hat{x} = P_S f(\hat{x})$; that is, $\hat{x}$ satisfies the following variational inequality (24).*

Corollary 12. *Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be directed operators and $I - U$ and $I - T$ be demiclosed at zero. Assume the SCFP (4) is consistent ($S \neq \emptyset$). If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by implicit algorithm (22) converges strongly to a point $\hat{x} \in S$, and $\hat{x} = P_S f(\hat{x})$; that is, $\hat{x}$ satisfies the following variational inequality (24).*

Corollary 13. *Let $U : H_1 \to H_1$ be a directed operator, $T : H_2 \to H_2$ be a quasi-nonexpansive operator, and $I - U$ and $I - T$ be demiclosed at zero. Assume the SCFP (4) is consistent ($S \neq \emptyset$). If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by implicit algorithm (22) converges strongly to a point $\hat{x} \in S$, and $\hat{x} = P_S f(\hat{x})$; that is, $\hat{x}$ satisfies the following variational inequality (24).*

Corollary 14. *Let $U : H_1 \to H_1$ be a directed operator, $T : H_2 \to H_2$ be a τ-demicontractive operator, and $I - U$ and $I - T$ be demiclosed at zero. Assume the SCFP (4) is consistent ($S \neq \emptyset$). If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by implicit algorithm (22) converges strongly to a point $\hat{x} \in S$, and $\hat{x} = P_S f(\hat{x})$; that is, $\hat{x}$ satisfies the following variational inequality (24).*

Corollary 15. *Let $U : H_1 \to H_1$ be a quasi-nonexpansive operator, $T : H_2 \to H_2$ be a τ-demicontractive operator, and $I - U$ and $I - T$ be demiclosed at zero. Assume the SCFP (4) is consistent ($S \neq \emptyset$). If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by implicit algorithm (22) converges strongly to a point $\hat{x} \in S$, and $\hat{x} = P_S f(\hat{x})$; that is, $\hat{x}$ satisfies the following variational inequality (24).*

5. Conclusions

In this paper, the research highlights that the strong convergence of the SCFP (4) is constructed. We construct a novel implicit algorithm for demicontractive operator to solve the split common fixed points problem SCFP, and we prove that the sequence $\{x_n\}$ strongly converges to a solution of the SCFP. These results further complete the theory of the SCFP, and some relevant work can be extended in the future.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

This work was carried out by the three authors, in collaboration. Moreover, the three authors have read and approved the final manuscript.

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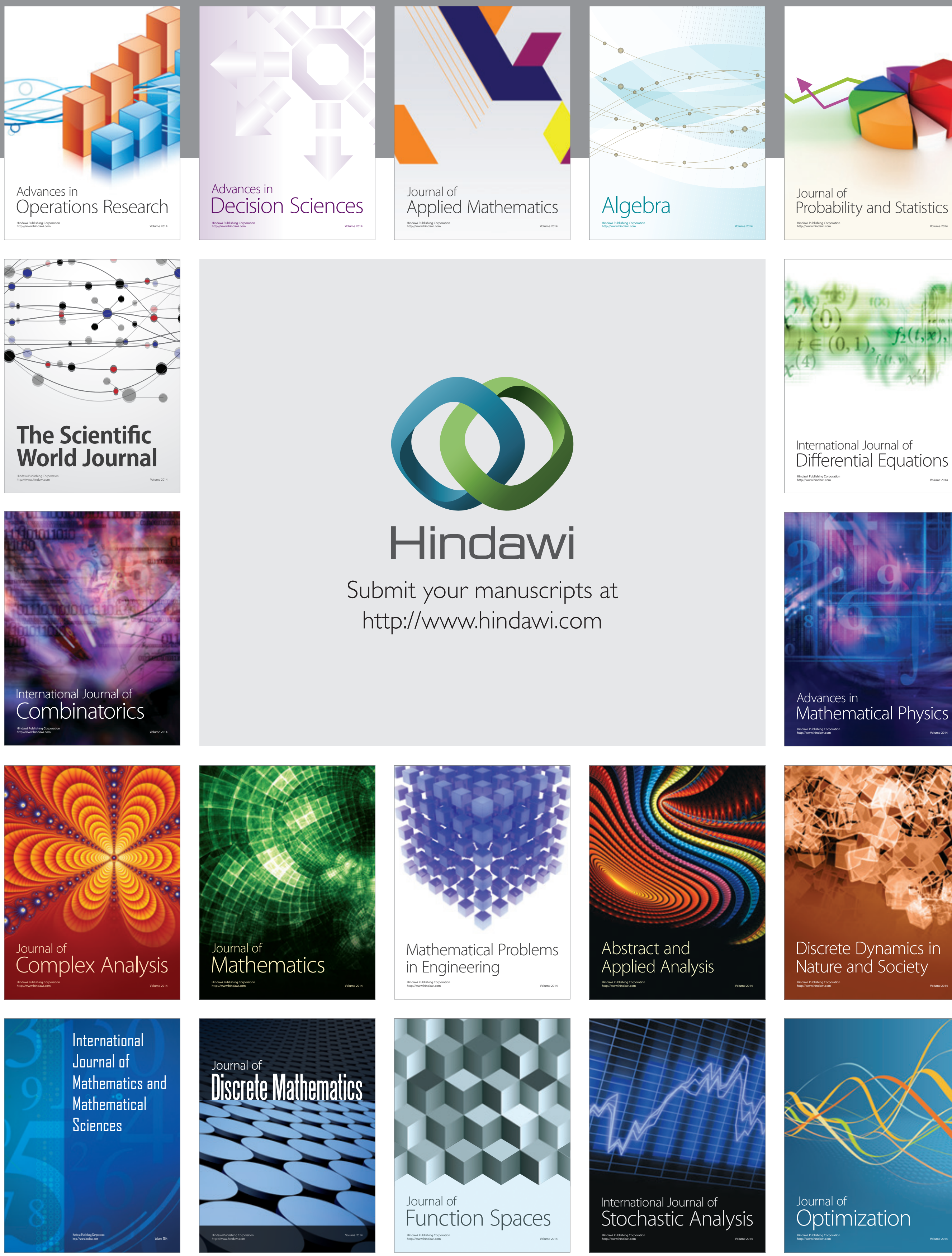
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