VOLTAGE COLLAPSE IN POWER SYSTEMS: DYNAMICAL STUDIES FROM A STATIC FORMULATION

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This paper addresses the problem of voltage collapse in power systems. More precisely, we exhibit a voltage collapse in a power system with two buses. This study is carried out with the help of two approaches. The first is a dynamical approach where a saddle-node bifurcation is analyzed and the second is an algebraic approach. Both approaches deal with the static behavior of the power system, but some dynamic aspects may be observed. An equivalence between the algebraic and dynamical approaches is obtained. The need to use both models comes from the fact that they are usually exploited in the literature, but a deep theoretical justification is still pending. Such a justification is meant in this work.

1. Introduction

Studying saddle-node in dynamical systems may help to understand and prevent some problems. This is because unlike other kinds of bifurcation, saddle-node is associated with an absence of equilibrium points beyond the bifurcation point. Therefore, analyzing a system during the bifurcation path may help to foresee a parameter value associated with the saddle-node point.

The study of saddle-node in power systems has increased in recent years. Such investigations help in understanding how a system may become unstable as a consequence of successive small parameter variations. This problem is known as voltage collapse, and deserves special attention from engineers and operators around the world [1]. Identifying the point of bifurcation plays a crucial role on power system analysis, since it may help the operator to avoid instability problems. For this purpose, continuation methods may be accurate and useful, since they identify the saddle-node point and trace the bifurcation diagram.

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Other methods are meant to determine this bifurcation point in a short computational time and interesting results have been obtained [5, 6]. In some cases, conflicting results have been reported, rendering some indices as better than others. For instance, the methods studied in [2] tend to produce different results, which seems to be a contradiction, since they are all based on the same set of equations. This problem has been investigated in [8], where it is shown that, under certain conditions, all the indices tend to provide the same behavior. This is particularly important when dealing with modal analysis, since a saddle-node is associated with a zero real eigenvalue.

Based on this knowledge, several works attempt to detect saddle-node bifurcations by tracking the least eigenvalue along the system loading. The literature shows, however, that monitoring the least eigenvalue may lead to frustrating results, since the bifurcation point is not predicted. This is because the least tracked eigenvalue may present a sharp variation at the bifurcation point. On the other hand, it is shown that tangent vector and the zero right eigenvector (the eigenvector associated with the vanishing eigenvalue, or the center manifold) provide an index associated with a quadratic behavior.

The literature shows that if a proper reduction of the set of equations is executed, the decoupled or normal form of a saddle-node may be obtained [7]. This kind of decomposition has not been derived for power systems, so far. On the other hand, it is possible to observe a saddle-node without the reduction of the set of equations to the normal form.

A set of algebraic equations models the power flow in electrical systems. It is important to mention that, in this paper, "algebraic" is just a substitute for a "nondifferential". However, those equations can seem to be differential equations, and analytical tools for those objects can be used.

This characteristic is exploited in this paper. In Section 2, we apply the saddle-node bifurcation theory to a simple power system. In Section 3, we present a detailed study of the saddle-node bifurcation in the analyzed power system. From a geometrical point of view of saddle-node bifurcation, we exhibit an algebraic approach for the initial problem in Section 4. An equivalence of the two approaches is obtained in Section 5.

2. Saddle-node bifurcation in a power system

Power systems are huge electrical systems. The name arises because of the huge amount of power required by the load. During the modeling, for simplicity, a set of generators is grouped in a single point, and the same is applied for a group of loads. This is called a bus, as illustrated in Figure 2.1. This kind of system is modeled by a set of differential-algebraic equations, and it is subject to several disturbances that tend to change its equilibrium points. In general, a power system may be driven to instability as a consequence of a transmission line tripping or a generator outage. Recently, it has been shown that a power system may also be driven to instability because of successive small load variations. In this case, the load variation is considered as the system parameter, and a power system may experiment Hopf or saddle-node bifurcations. Hopf bifurcations are characterized by the existence of a purely imaginary pair of eigenvalues. The effects in a power system are oscillations in the generator machines. In this paper, saddle-node is particularly focused on, and the consequences in a power system are discussed.

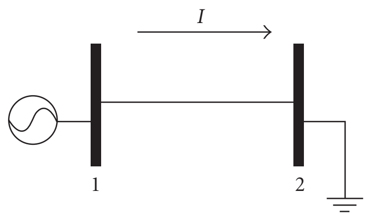

Figure 2.1. Two-bus system.

Consider the sample lossless power system shown in Figure 2.1.

For this system, the power flow algebraic equations associated with Bus 2 are

$$P_2 = \frac{V_2 \sin\delta_2}{X},\qquad Q_2 = -\frac{V_2\cos\delta_2}{X} + \frac{V_2^2}{X}, \tag{2.1}$$

where P_2 and Q_2 are the active and reactive powers at Bus 2, whereas V_2 and δ_2 are the voltage level and the phase angle at the same bus, respectively. Considering that the power factor is equals to 1, P_2 is equal to V_2. Taking $P_2 = Q_2 = \lambda$, and setting $\alpha = \lambda X$, $x = V_2$, $y = \delta_2$ one has

$$x\sin y + \alpha = 0,\qquad -x\cos y + x^2 + \alpha = 0, \tag{2.2}$$

where $x \in (0,\infty)$, $y \in [-\pi/2, \pi/2]$ and $\alpha \in [0,\infty)$.

Equations (2.2) define the equilibrium points of a system of differential equations, according to [3]. Thus

$$\dot{x} = x\sin y + \alpha,\qquad \dot{y} = x^2 - x\cos y + \alpha. \tag{2.3}$$

We have the following theorem.

Theorem 2.1. *The one-parameter family of ordinary differential equations (2.3) has a saddle-node bifurcation.*

For the proof of Theorem 2.1, we use the following theorem due to J. Sotomayor (see [4, page 148]).

Theorem 2.2. *Let $x' = f(x,\alpha)$ be a system of differential equations in $\mathbb{R}^n$ depending on the single parameter α. When $\alpha = \alpha_*$, assume that there is an equilibrium point p_* satisfying the following:*

(SN1) *the Jacobian matrix at (p_*, α_*) presents a zero eigenvalue (only one) with right and left eigenvectors v_* and w_*, respectively;*

(SN2) $w_* \cdot (d/d\alpha) f(p_*, \alpha_*) \neq 0$;

(SN3) $w_* \cdot D_x^2 f(p_*, \alpha_*)(v_*, v_*) \neq 0$.

Then, there is a smooth curve of equilibrium points in $\mathbb{R}^n \times \mathbb{R}$ *passing through* (p_*, α_*)*, tangent to the hyperplane* $\mathbb{R}^n \times \{\alpha_*\}$*. Depending on the signs in (SN2) and (SN3), there is no equilibrium point near* (p_*, α_*) *when* $\alpha < \alpha_*$ $(\alpha > \alpha_*)$*, and there are two equilibrium points near* (p_*, α_*) *for each value* $\alpha > \alpha_*$ $(\alpha < \alpha_*)$*. Both equilibrium points of* $x' = f(x,\alpha)$ *near* (p_*, α_*) *are hyperbolic. Such equilibrium points coalesce at* $\alpha = \alpha_*$*.*

Theorem 2.2 states that the saddle-node bifurcation behaves qualitatively as

$$x' = \pm(x - x_*)^2 \pm (\alpha - \alpha_*) \tag{2.4}$$

along the direction given by the eigenvector associated with the zero eigenvalue. For other directions, it presents a hyperbolic behavior.

Hence, the analysis of $x' = f(x,\alpha)$, where $x \in \mathbb{R}^n$, near (p_*, α_*) may be reduced to the study of the equation $x' = \pm(x - x_*)^2 \pm (\alpha - \alpha_*)$, as long as x belongs to the eigenspace associated with the vanishing eigenvalue.

Proof of Theorem 2.1. Write the ordinary differential equations (2.3) in the form

$$\dot{x} = f(x,\alpha) = (f_1(x,\alpha), f_2(x,\alpha)) = (x_1 \sin x_2 + \alpha, x_1^2 - x_1 \cos x_2 + \alpha), \tag{2.5}$$

where $x = (x_1, x_2)$. When $\alpha = \alpha_* = 0.207\,106$, (2.5) has only one equilibrium point at

$$p_* = (0.541\,196, -0.392\,699). \tag{2.6}$$

The Jacobian matrix of f at (p_*, α_*) is

$$J(p_*, \alpha_*) = \begin{bmatrix} -0.382\,683 & 0.500\,000 \\ 0.158\,512 & -0.207\,106 \end{bmatrix} \tag{2.7}$$

whose eigenvalues are

$$\lambda_* = 0, \qquad \eta_* = -0.589\,790. \tag{2.8}$$

The right and the left eigenvectors belonging to λ_* are

$$\begin{aligned} v_* &= (1, 0.765\,366), \\ w_* &= (1, 2.414\,213), \end{aligned} \tag{2.9}$$

respectively. As

$$\frac{d}{d\alpha} f(p_*, \alpha_*) = (1,1), \tag{2.10}$$

we have

$$w_* \cdot \frac{d}{d\alpha} f(p_*, \alpha_*) = (1, 2.414\,213) \cdot (1,1) = 3.414\,213 \neq 0. \tag{2.11}$$

Now

$$D_x^2 f(p_*,\alpha_*)(v_*,v_*) = \left(\sum_{j,k=1}^{2} \frac{\partial^2 f_1}{\partial x_k \partial x_j}(p_*,\alpha_*)\beta_j\beta_k, \sum_{j,k=1}^{2} \frac{\partial^2 f_2}{\partial x_k \partial x_j}(p_*,\alpha_*)\beta_j\beta_k\right), \tag{2.12}$$

where

$$(\beta_1,\beta_2) = v_* = (1, 0.765366). \tag{2.13}$$

Thus, we have

$$D_x^2 f(p_*,\alpha_*)(v_*,v_*) = (1.535533, 1.707106). \tag{2.14}$$

Therefore,

$$w_* \cdot D_x^2 f(p_*,\alpha_*)(v_*,v_*) = (1, 2.414213) \cdot (1.535533, 1.707106) \neq 0. \tag{2.15}$$

The conditions (SN1), (SN2), and (SN3) are satisfied in (2.8), (2.11) and (2.15), respectively. Theorem 2.1 is proved. □

3. Study of the saddle-node bifurcation

In this section, we exhibit a detailed study of the saddle-node bifurcation in a power system given by (2.3).

For $0 \leq \alpha < 0.207106$, there are two equilibrium points of (2.3) determined by

$$\begin{aligned} p_0 = (x_0, y_0) &= \left(\sqrt{-\frac{2\alpha+1-\sqrt{\Delta}}{2}}, \sin^{-1}\left(-\frac{\sqrt{2}\alpha}{\sqrt{-2\alpha+1-\sqrt{\Delta}}}\right)\right), \\ p_1 = (x_1, y_1) &= \left(\sqrt{-\frac{2\alpha+1+\sqrt{\Delta}}{2}}, \sin^{-1}\left(-\frac{\sqrt{2}\alpha}{\sqrt{-2\alpha+1+\sqrt{\Delta}}}\right)\right), \end{aligned} \tag{3.1}$$

where $\Delta = -4\alpha^2 - 4\alpha + 1$.

For $0.194788 \leq \alpha < 0.207106$, p_0 is a stable node and p_1 is a saddle point. Setting

$$\beta(\alpha) = \sqrt{-\frac{2\alpha+1-\sqrt{\Delta}}{2}}, \qquad \gamma(\alpha) = \sqrt{-\frac{2\alpha+1+\sqrt{\Delta}}{2}}, \tag{3.2}$$

the Jacobian matrices at p_0 and p_1 become

$$J(p_0) = \begin{bmatrix} -\dfrac{\alpha}{\beta(\alpha)} & \sqrt{(\beta(\alpha))^2 - \alpha^2} \\ 2\beta(\alpha) - \dfrac{\sqrt{(\beta(\alpha))^2 - \alpha^2}}{\beta(\alpha)} & -\alpha \end{bmatrix},$$
$$J(p_1) = \begin{bmatrix} -\dfrac{\alpha}{\gamma(\alpha)} & \sqrt{(\gamma(\alpha))^2 - \alpha^2} \\ 2\gamma(\alpha) - \dfrac{\sqrt{(\gamma(\alpha))^2 - \alpha^2}}{\gamma(\alpha)} & -\alpha \end{bmatrix}. \tag{3.3}$$

From (3.3), one can see that $J(p_0)$ presents two negative eigenvalues,

$$\lambda_{01} = -\frac{\left(\alpha + \alpha/\sqrt{(-2\alpha+1-\sqrt{\Delta})/2}\right) + \sqrt{\Delta'}}{2},$$
$$\lambda_{02} = -\frac{\left(\alpha + \alpha/\sqrt{(-2\alpha+1-\sqrt{\Delta})/2}\right) - \sqrt{\Delta'}}{2}, \tag{3.4}$$

where

$$\Delta' = \left(\alpha + \frac{\alpha}{\sqrt{(-2\alpha+1-\sqrt{\Delta})/2}}\right)^2 - 4\sqrt{-\frac{2\alpha+1-\sqrt{\Delta}}{2}}\sqrt{-\frac{2\alpha+1-\sqrt{\Delta}}{2} - \alpha^2}, \tag{3.5}$$

whereas $J(p_1)$ presents one positive and another negative,

$$\lambda_{11} = -\frac{\left(\alpha + \alpha/\sqrt{(-2\alpha+1+\sqrt{\Delta})/2}\right) + \sqrt{\Delta''}}{2},$$
$$\lambda_{12} = -\frac{\left(\alpha + \alpha/\sqrt{(-2\alpha+1+\sqrt{\Delta})/2}\right) - \sqrt{\Delta''}}{2}, \tag{3.6}$$

where

$$\Delta'' = \left(\alpha + \frac{\alpha}{\sqrt{(-2\alpha+1+\sqrt{\Delta})/2}}\right)^2 - 4\sqrt{-\frac{2\alpha+1+\sqrt{\Delta}}{2}}\sqrt{-\frac{2\alpha+1+\sqrt{\Delta}}{2} - \alpha^2}. \tag{3.7}$$

For $\alpha = \alpha_* = 0.207\,106$, (2.3) have only one equilibrium point at

$$p_* = (x_*, y_*) = (0.541\,196, -0.392\,699), \tag{3.8}$$

which is a saddle-node point. This is confirmed by the analysis developed in the previous section.

There is no equilibrium point for $\alpha > 0.207106$.

4. The significance of the use of differential equations

In Sections 2 and 3, the saddle-node bifurcation of the one-parameter family of ordinary differential equations (2.3) was analyzed. Recall that these differential equations are obtained from the one-parameter family of algebraic equations (2.2), which models the two-bus power system of Figure 2.1.

Note that we are not interested in the dynamical properties of (2.3). We are only interested in the evolution of the equilibrium points of (2.3) as the parameter varies.

An *operating point* of the power system is defined as the point that satisfies (2.2). From the geometrical point of view, an operating point of the power system is a point where the curves $A = A(x,y,\alpha) = 0$ and $B = B(x,y,\alpha) = 0$, defined by

$$A(x,y,\alpha) = x\sin y + \alpha, \qquad B(x,y,\alpha) = -x\cos y + x^2 + \alpha, \tag{4.1}$$

have intersection.

Let $p_0 = (x_0, y_0)$ be an operating point of the power system. We say that p_0 is a *transversal operating point* at $\alpha = \alpha_0$ if

$$J(A,B)(p_0,\alpha_0) = \frac{\partial(A,B)}{\partial(x,y)}(p_0,\alpha_0) = \det\begin{bmatrix} \dfrac{\partial A}{\partial x}(p_0,\alpha_0) & \dfrac{\partial A}{\partial y}(p_0,\alpha_0) \\ \dfrac{\partial B}{\partial x}(p_0,\alpha_0) & \dfrac{\partial B}{\partial y}(p_0,\alpha_0) \end{bmatrix} \neq 0. \tag{4.2}$$

This condition means that the curves $A = 0$ and $B = 0$ are regular and meet transversally at p_0 when $\alpha = \alpha_0$. It follows that transversal operating points are isolated. Therefore, p_0 is a transversal operating point at $\alpha = \alpha_0$ if and only if the vectors

$$\begin{aligned} \nabla A(p_0,\alpha_0) &= \left(\frac{\partial A}{\partial x}(p_0,\alpha_0), \frac{\partial A}{\partial y}(p_0,\alpha_0)\right), \\ \nabla B(p_0,\alpha_0) &= \left(\frac{\partial B}{\partial x}(p_0,\alpha_0), \frac{\partial B}{\partial y}(p_0,\alpha_0)\right) \end{aligned} \tag{4.3}$$

are linearly independent.

We say that an operating point p_0 is a *tangential operating point* at $\alpha = \alpha_0$ if the curves $A = 0$ and $B = 0$ are regular and the vectors $\nabla A(p_0,\alpha_0)$ and $\nabla B(p_0,\alpha_0)$ are linearly dependent. This implies that the matrix

$$\begin{bmatrix} \dfrac{\partial A}{\partial x}(p,\alpha) & \dfrac{\partial A}{\partial y}(p,\alpha) \\ \dfrac{\partial B}{\partial x}(p,\alpha) & \dfrac{\partial B}{\partial y}(p,\alpha) \end{bmatrix} \tag{4.4}$$

evaluated at (p_0,α_0) has a zero eigenvalue.

A *tangential bifurcation* occurs at a tangential operating point $p = p_0$ for the parameter value $\alpha = \alpha_0$ if the following hold:

(TB1) there are numbers t_0 and $\varepsilon > 0$ and a smooth curve $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$ in $\mathbb{R}^2 \times \mathbb{R}$ such that $\gamma(t_0) = (p_0, \alpha_0)$, $A(\gamma(t)) \equiv 0$, and $B(\gamma(t)) \equiv 0$, for $|t - t_0| < \varepsilon$;

(TB2) the curve γ has a quadratic tangency with $\mathbb{R}^2 \times \{\alpha_0\}$ at (p_0, α_0);

(TB3) if $0 < |t - t_0| < \varepsilon$, then $\nabla A(\gamma(t))$ and $\nabla B(\gamma(t))$ are linearly independent;

(TB4) if $\mu(t)$ is the eigenvalue of the matrix (4.4) evaluated at $\gamma(t)$ such that $\mu(t_0) = 0$, then $\mu(t)$ crosses the imaginary axis with nonzero speed at $t = t_0$.

We have the following theorem.

Theorem 4.1. *The one-parameter family of algebraic equations (2.2) has a tangential bifurcation.*

Proof. The point

$$p_0 = (0.541\,196, -0.392\,699) \tag{4.5}$$

is a tangential operating point at $\alpha_0 = 0.207\,106$. From $A = 0$ and $B = 0$, we have

$$\alpha = -x \sin y, \qquad \alpha = -x^2 + x \cos y, \tag{4.6}$$

respectively. Thus,

$$\alpha = -x \sin y = -x^2 + x \cos y, \qquad x = \sin y + \cos y. \tag{4.7}$$

Therefore,

$$\alpha = -x \sin y = -(\sin y + \cos y) \sin y. \tag{4.8}$$

Setting $t = y$, we have from the above equations

$$x(t) = \sin t + \cos t, \qquad \alpha(t) = -(\sin t + \cos t) \sin t. \tag{4.9}$$

Define the curve

$$\gamma(t) = (x(t), y(t), \alpha(t)) = (\sin t + \cos t, t, -\sin t(\sin t + \cos t)), \tag{4.10}$$

for $|t - t_0| < \varepsilon$, where $t_0 = -0.392\,699$ and $\varepsilon > 0$ is small. Thus, we have

$$\gamma(t_0) = (p_0, \alpha_0), \qquad A(\gamma(t)) \equiv 0, \qquad B(\gamma(t)) \equiv 0, \tag{4.11}$$

for $|t - t_0| < \varepsilon$. Now,

$$\gamma_2(t_0) = 0.207\,106, \qquad \gamma_2'(t_0) = 0, \qquad \gamma_2''(t_0) = -2.828\,427 \neq 0. \tag{4.12}$$

This implies that the curve γ has a quadratic tangency with $\mathbb{R}^2 \times \{\alpha_0\}$ at (p_0, α_0). As the matrix (4.4), evaluated at $\gamma(t)$, is given by

$$\begin{bmatrix} \sin t & \cos t(\sin t + \cos t) \\ 2\sin t + \cos t & \sin t(\sin t + \cos t) \end{bmatrix}, \tag{4.13}$$

we have

$$J(A(\gamma(t)),B(\gamma(t))) = (\sin t+\cos t)(\sin^2 t - 2\sin t\cos t - \cos^2 t) \neq 0, \tag{4.14}$$

for $0 < |t - t_0| < \varepsilon$. This implies that the vectors $\nabla A(\gamma(t))$ and $\nabla B(\gamma(t))$ are linearly independent for $0 < |t - t_0| < \varepsilon$. Let $\mu(t)$ be the eigenvalue of the matrix (4.13) satisfying $\mu(t_0) = 0$. We have

$$\begin{aligned}\mu(t) &= \frac{1}{2}\Big[\sin t+\sin^2 t+\sin t\cos t \\ &\quad + \sqrt{\sin^2 t-2\sin^3 t+6\sin^2 t\cos t+\sin^4 t+2\sin^3 t\cos t+\sin^2 t\cos^2 t+12\sin t\cos^2 t+4\cos^3 t}\,\Big].\end{aligned} \tag{4.15}$$

Thus

$$\mu'(t_0) = 2.595386 \neq 0. \tag{4.16}$$

The theorem is proved. □

From Theorems 2.1 and 4.1, we have the following main theorem, whose proof is immediate.

THEOREM 4.2. *The one-parameter family of algebraic equations (2.2) has a tangential bifurcation if and only if the one-parameter family of ordinary differential equations (2.3) has a saddle-node bifurcation.*

Hence, studying the loss of transversality of the curves $A = 0$ and $B = 0$ may provide some important pieces of information about voltage collapse in power systems. This occurs exactly at the saddle-node point of the dynamical model. This is observed with the help of (4.2), whose rows are the components of the gradient vectors of the functions A and B.

At the saddle-node point, the Jacobian matrix is singular (has a zero eigenvalue). But this implies that the matrix shown in (4.2) is also singular. Thus the gradients of the functions A and B are parallels and this point is a tangential operating point of the algebraic model.

5. Tangential versus saddle-node bifurcation

Consider two C^1 functions $F,G : U\times I \subset \mathbb{R}^2\times\mathbb{R} \to \mathbb{R}$, where $(x,y)\in U$ and $\alpha \in I$. The following theorem generalizes Theorem 4.2.

THEOREM 5.1. *The one-parameter family of algebraic equations*

$$F(x,y,\alpha) = 0, \qquad G(x,y,\alpha) = 0 \tag{5.1}$$

has a tangential bifurcation at $p_0 = (x_0,y_0)\in U$ *for* $\alpha = \alpha_0 \in I$ *if and only if the* *one-parameter family of ordinary differential equations*

$$x' = F(x,y,\alpha), \qquad y' = G(x,y,\alpha) \tag{5.2}$$

has a saddle-node bifurcation at p_0 for α_0.

Proof. Suppose that (5.2) has a saddle-node bifurcation at p_0 for α_0. Without loss of generality, we can take (5.2) in the saddle-node normal form

$$x' = x^2 + \alpha, \qquad y' = -y, \tag{5.3}$$

which has a saddle-node bifurcation at $p_0 = (0,0)$ for $\alpha_0 = 0$. Therefore, $F(x,y,\alpha) = x^2 + \alpha$ and $G(x,y,\alpha) = -y$. Define $\gamma(t) = (t,0,-t^2)$, $t \in \mathbb{R}$. Let $t_0 = 0$. Thus,

$$\begin{aligned} \gamma(t_0) &= \gamma(0) = (0,0,0) = (p_0,\alpha_0), \\ F(\gamma(t)) &= F(t,0,-t^2) = t^2 - t^2 \equiv 0, \\ G(\gamma(t)) &= G(t,0,-t^2) \equiv 0. \end{aligned} \tag{5.4}$$

Condition (TB1) is satisfied. As $\gamma_2(t) = -t^2$,

$$\gamma_2(0) = 0, \qquad \gamma_2'(0) = 0, \qquad \gamma_2''(0) = -2 \neq 0, \tag{5.5}$$

and condition (TB2) is satisfied. Now,

$$J(A(\gamma(t)),B(\gamma(t))) = \det\begin{bmatrix} 2t & 0 \\ 0 & -1 \end{bmatrix} = -2t \neq 0, \tag{5.6}$$

for $t \neq t_0 = 0$. This implies condition (TB3). Condition (TB4) is immediate since $\mu(t) = 2t$, $\mu(0) = 0$, and $\mu'(0) = 2 \neq 0$. Therefore, (5.1) has a tangential bifurcation at p_0 for α_0.

On the other hand, as the curve $\gamma(t)$ satisfies $F(\gamma(t)) \equiv 0$ and $G(\gamma(t)) \equiv 0$, (TB1), each operating point $\gamma(t)$ of (5.1) is an equilibrium point of (5.2). As the curve γ has a quadratic tangency with $\mathbb{R}^2 \times \{\alpha_0\}$ at (p_0,α_0), (TB2), depending on the sign of $\gamma_2''(t_0) \neq 0$, there is no equilibrium point near (p_0,α_0) when $\alpha < \alpha_0$ ($\alpha > \alpha_0$), and there are two equilibrium points near (p_0,α_0) for each value $\alpha > \alpha_0$ ($\alpha < \alpha_0$). Both equilibrium points of (5.2) near (p_0,α_0) are hyperbolic, (TB3). Such equilibrium points coalesce at $\alpha = \alpha_0$. Note that (TB2) and (TB4) imply nondegeneracy conditions of saddle-node bifurcation. The theorem is proved. □

6. Conclusions

This paper dealt with the important problem of saddle-node bifurcation in power systems. This problem is usually studied in power systems with the help of an algebraic set of equations. Such a model may be justified by comparing the results obtained by both algebraic and differential-algebraic methods.

In this paper, it is shown that saddle-node takes place when the transversality of the curves obtained from the system model no longer occurs. This is described by the parallelism of the gradient vectors associated with a reduced set of equations.

The results obtained for the sample system analyzed are easily extended to bigger power systems, since the reduction to the equations of interest is straightforward. The theorem proposed here may also be applied for similar dynamical systems, enabling one to employ simplified system models, while understanding the pieces of information obtained.

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