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Review Article

Comparison of Different Approaches to Construct First Integrals for Ordinary Differential Equations

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Different approaches to construct first integrals for ordinary differential equations and systems of ordinary differential equations are studied here. These approaches can be grouped into three categories: direct methods, Lagrangian or partial Lagrangian formulations, and characteristic (multipliers) approaches. The direct method and symmetry conditions on the first integrals correspond to first category. The Lagrangian and partial Lagrangian include three approaches: Noether's theorem, the partial Noether approach, and the Noether approach for the equation and its adjoint as a system. The characteristic method, the multiplier approaches, and the direct construction formula approach require the integrating factors or characteristics or multipliers. The Hamiltonian version of Noether's theorem is presented to derive first integrals. We apply these different approaches to derive the first integrals of the harmonic oscillator equation. We also study first integrals for some physical models. The first integrals for nonlinear jerk equation and the free oscillations of a two-degree-of-freedom gyroscopic system with quadratic nonlinearities are derived. Moreover, solutions via first integrals are also constructed.

1. Introduction

The study of conserved quantities plays a great role in mathematical physics and in applied mathematics. For instance, a considerable number of phenomena have some kind of "conservation." Examples can be easily found from the hydrodynamics, electrodynamics, shallow water phenomena, and so forth. One can also mention the celebrated Kepler's third law or the conservation of energy in the classical mechanics, particularly the one-dimensional harmonic oscillator. In regard to these last two phenomena, the conserved quantity is called first integral, which is the analogous of conservation laws for ordinary differential equations models.

In a recent paper, Naz et al. [1] studied different approaches to construct conservation laws for partial differential equations. The purpose of this paper is to analyze all different approaches for construction of first integrals for ordinary differential equations. In fact, different approaches to derive first integrals can be grouped into three categories: direct methods, Lagrangian or partial Lagrangian formulations, and characteristic (multipliers) approaches. In 1798, Laplace [2] developed a method known as the direct method for the construction of first integrals. Although such a method does not originally require any symmetry of the considered equation, Kara and Mahomed [3] developed a relationship between symmetries and conservation laws. The joint conditions of symmetry and direct method are used to construct the first integrals.

Noether's theorem [4] is a powerful technique to derive first integrals for the differential equations having Lagrangian formulations using its symmetries, although it requires a suitable Lagrangian. Kara et al. [5] developed the partial Noether approach. The partial Noether approach is applicable to differential equations with or without a Lagrangian. The interested readers are referred to [6–11] for discussions on first integrals by the Noether approach and partial Noether approach. Ibragimov [12] introduced the concept of formal Lagrangian formulation for differential equations and its adjoint as a system. Atherton and Homsy [13] introduced the adjoint variational principle for such systems. Then Ibragimov [12] incorporated symmetry considerations and provided formulas to construct the first integrals similar to those provided by Noether's approach. The concepts of self-adjointness [14, 15], weak self-adjointness [16], nonlinear self-adjointness [17–20], and quasi-self-adjointness [21, 22] are used to construct first integrals by this approach.

The characteristic, multiplier, or integrating factor methods are also very powerful and elegant methods for construction of the first integrals. There are four different approaches based on the knowledge of the characteristics. The first method developed by Steudel [23] in 1962 involves writing a first integral in the characteristic form. The characteristics or integrating factors are the multipliers of the differential equations that makes them exact. To derive the first integrals by this method first of all the characteristics need to be determined. The second method is based on the first method and it involves the variational derivative (see Proposition 5.49 in Olver [24]). The reader is referred to [25–27] for a good account of understanding how to compute multipliers and first integrals. In the third approach, we compute the variational derivatives on the solution space of given differential equations and these characteristics sometimes may correspond to an adjoint symmetry not to first integral. The fourth approach according to Anco and Bluman [28] provides formulae for finding first integrals. In the last few decades, the researchers focused on the development of symbolic computational packages based on different approaches of first integrals and these packages are well documented in [29] and references therein.

The well-known Noether identity can be expressed in terms of Hamiltonian function and symmetry operators (see, e.g., [30, 31]). This is a simple and elegant way to construct first integrals of Hamiltonian equations consisting of a system of first-order differential equations. No integration is required here to construct solutions.

Lie approach as described, for example, by Ibragimov and Nucci [32] and Mahomed [33], is successfully applied to differential equations to derive the exact solutions. On the other hand, the knowledge of first integrals enables one to reduce or completely solve an ordinary differential equation. In fact, if one considers an nth-order scalar ordinary differential equation having n independent first integrals, one can obtain from those first integrals the general solution of the considered equation possessing n constants. Kara et al. [34] explored the solutions of differential equations using the Noether symmetries of a Lagrangian associated with the first integrals. Using the relationship between Noether symmetries and first integrals [35] the reductions and exact solutions of differential equations were derived. The generalization of this idea is the association of Lie-Bäcklund symmetries [3] and nonlocal symmetries [36, 37] with a first integral and it led to the development of the double reduction theory to find reductions and solutions [38–43].

The paper is organized in the following manner. The fundamental relations are defined in Section 2. We present the main ideas behind the mentioned methods in Section 3. Then, in Section 4, we apply these different approaches to the harmonic oscillator equation. In Section 5 some solutions are obtained via first integrals. Relations between Hamiltonian functions and first integrals are discussed in Section 6. In Section 7, the first integrals for nonlinear jerk equation are computed by the Noether approach for the equation and its adjoint as a system and by the multiplier approach. The exact solutions of jerk equation for different cases are also established via first integrals. The first integrals for the free oscillations of a two-degree-of-freedom gyroscopic system with quadratic nonlinearities are also derived. Finally, concluding remarks are presented in Section 8.

2. Fundamental Relations

The following definitions are taken from the literature (see, e.g., [44, 45]).

Consider a kth-order ordinary differential equation system

$$E_\alpha\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0, \quad \alpha = 1, \ldots, m, \tag{1}$$

where x is the independent variable and u^α, $\alpha = 1, 2, \ldots, m$, are the m dependent variables. We will adopt the summation convention and there is summation over repeated upper and lower indices.

The total derivative with respect to x is

$$D_x = \frac{\partial}{\partial x} + u_x^\alpha \frac{\partial}{\partial u^\alpha} + u_{xx}^\alpha \frac{\partial}{\partial u_x^\alpha} + \cdots. \tag{2}$$

The following are the basic operators defined in $\mathscr{A}$, the vector space of differential functions.

The Lie-Bäcklund operator X is defined as

$$X = \xi \frac{\partial}{\partial x} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta_s^\alpha \frac{\partial}{\partial u_s^\alpha}, \tag{3}$$

where

$$\zeta_s^\alpha = D_x\left(\zeta_{s-1}^\alpha\right) - u_s^\alpha D_x(\xi), \quad s \geq 1, \ \alpha = 1, \ldots, m, \tag{4}$$

in which $\zeta_0^\alpha \equiv \eta^\alpha$.

The Euler operator is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \left(-D_x\right)^s \frac{\partial}{\partial u_s^\alpha}, \quad \alpha = 1, 2, \ldots m. \tag{5}$$

The characteristic form of Lie-Bäcklund operator (3) is

$$X = \xi D_x + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_x^s\left(W^\alpha\right) \frac{\partial}{\partial u_s^\alpha}, \tag{6}$$

in which W^α is the *Lie characteristic function* defined by

$$W^\alpha = \eta^\alpha - \xi u_x^\alpha, \quad \alpha = 1, \ldots, m. \tag{7}$$

The *Noether operator* associated with a Lie-Bäcklund operator X is

$$N = \xi + W^\alpha \frac{\delta}{\delta u_x^\alpha} + \sum_{s\geq 1} D_x^s\left(W^\alpha\right) \frac{\delta}{\delta u_{s+1}^\alpha}, \tag{8}$$

where

$$\frac{\delta}{\delta u_x^\alpha} = \frac{\partial}{\partial u_x^\alpha} + \sum_{s\geq 1} (-D_x)^s \frac{\partial}{\partial u_{s+1}^\alpha}, \quad \alpha = 1, \ldots, m. \tag{9}$$

First Integral. A first integral of system (1) is a differential function $I \in \mathscr{A}$, such that

$$D_x I = 0 \tag{10}$$

for every solution of (1).

3. Approaches to Construct First Integrals

Now we present various approaches to construct first integrals taken from the literature.

3.1. Direct Method. The direct method was first used by Laplace [2] in 1798 to construct all local first integrals. The determining equations for the first integrals for the direct method are

$$D_x I\big|_{E_\alpha = 0} = 0. \tag{11}$$

3.2. Symmetry and First Integral Relation. Kara and Mahomed [3] added a symmetry condition to the direct method. The Lie-Bäcklund symmetry generator X and the first integral I are associated with the following equation:

$$X(I) + D_x(\xi) I = 0. \tag{12}$$

The first integrals are computed by the joint conditions (11) and (12).

3.3. Noether's Approach. In 1918, Noether developed a new approach to construct first integrals [4] and it is currently known as Noether's approach.

Euler-Lagrange Differential Equations. If there exists a function $L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) \in \mathscr{A}$, satisfying

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, \ldots, m, \tag{13}$$

then L is called a Lagrangian of (1) and relationship (13) yields the Euler-Lagrange differential equations. Equations (1) and (13) are equivalent.

Noether Symmetry Generator. A Lie-Bäcklund operator X is a Noether symmetry generator associated with a Lagrangian L of the Euler-Lagrange differential equations (13) if there exists a function B such that

$$X(L) + L D_x(\xi) = D_x(B). \tag{14}$$

Noether First Integral. For each Noether symmetry generator X associated with a given Lagrangian L corresponding to the Euler-Lagrange differential equations, there corresponds a function I known as a first integral and is defined by

$$I = B - N(L) \tag{15}$$

or

$$I = B - \xi L - W^\alpha \frac{\delta L}{\delta u_x^\alpha} - \sum_{s\geq 1} D_x^s (W^\alpha) \frac{\delta L}{\delta u_{s+1}^\alpha}, \tag{16}$$

where W^α are the characteristics of the first integral.

In the Noether approach we need to construct Lagrangian $L(x, u, \ldots, u_{(k-1)})$. The Noether symmetries are then computed from (14) and finally (16) provides the first integrals corresponding to each Noether symmetry. The reader is guided to [46] for further discussions about this technique and its relations with the so-called Noether symmetries.

3.4. Partial Noether Approach. The partial Noether approach for construction of first integrals was introduced by Kara et al. [5] and it can be useful for constructing first integrals when the differential equation does not have a known Lagrangian.

Partial Lagrangian. Suppose that the kth-order differential system (1) can be expressed as

$$E_\alpha = E_\alpha^0 + E_\alpha^1 = 0. \tag{17}$$

A function $L = L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(l)})$, $l \leq k$ is known as a partial Lagrangian of system (17) if

$$\frac{\delta L}{\delta u^\alpha} = f_\alpha^\beta E_\beta^1 \tag{18}$$

provided $E_\beta^1 \neq 0$ for some β. Here (f_α^β) is an invertible matrix.

Partial Noether Operator. The operator X satisfying

$$X(L) + L D_x(\xi) = D_x(B) + \left(\eta^\alpha - \xi u_x^\alpha\right) \frac{\delta L}{\delta u^\alpha}, \quad \alpha = 1, 2, \ldots, m \tag{19}$$

is a partial Noether operator corresponding to the partial Lagrangian L.

The first integrals of the system (1) associated with a partial Noether operator X corresponding to the partial Lagrangian L are determined from (16).

We can also use the partial Noether approach for equations arising from the variational principal and have the Lagrangian.

3.5. Noether Approach for a System and Its Adjoint

Adjoint Equations. Let $v = (v^1, v^2, \ldots, v^m)$ be the new dependent variables. The system of adjoint equations to the system of kth-order differential equations (1) is defined by (Atherton and Homsy [13], Ibragimov [12])

$$E_\alpha^*(x, u, v, \ldots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, 2, \ldots, m, \tag{20}$$

where

$$E_\alpha^*(x, u, v, \ldots, u_{(k)}, v_{(k)}) = \frac{\delta\left(v^\beta E_\beta\right)}{\delta u^\alpha}, \quad \alpha = 1, 2, \ldots, m, \ v = v(x). \tag{21}$$

Symmetries of Adjoint Equations. Suppose system (1) has the generator

$$X = \xi \frac{\partial}{\partial x} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}}. \tag{22}$$

Ibragimov [12] showed that the following operator is a Lie point symmetry for the system (1) and (20):

$$Y = \xi \frac{\partial}{\partial x} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \eta_{*}^{\alpha} \frac{\partial}{\partial v^{\alpha}}, \qquad \eta_{*}^{\alpha} = -\left(\lambda_{\beta}^{\alpha} v^{\beta} + v^{\alpha} D_x(\xi)\right). \tag{23}$$

The operator (23) is an extension of (22) to the variable v^{α} and

$$X(E_{\alpha}) = \lambda_{\alpha}^{\beta} E_{\beta} \tag{24}$$

yields λ_{β}^{α}.

Conservation Theorem. Every Lie point, Lie Bäcklund, and nonlocal symmetry of the system of kth-order differential equations (1) yields a first integral for the system consisting of (1) and the adjoint equations (21). Let L be the Lagrangian defined by

$$L = v^{\alpha} E_{\alpha}\left(x, u, \ldots, u_{(k)}\right). \tag{25}$$

Then the first integrals are given from the formula

$$I = \xi L + W^{\alpha} \frac{\delta L}{\delta u_{x}^{\alpha}} + \sum_{s \geq 1} D_x^{s}\left(W^{\alpha}\right) \frac{\delta L}{\delta u_{s+1}^{\alpha}}, \tag{26}$$

where ξ^{i}, η^{α} are the coefficient functions of the generator (22). The first integrals constructed from (26) contain the arbitrary solutions v of adjoint equation (21) and, thus, for each solution v one has first integrals.

The dependence on the nonlocal variable v provides a nonlocal first integral. One can eliminate such variable if the original system of ODEs is nonlinearly self-adjoint [14, 17] and to the equations admitting this remarkable property one can establish a first integral for the original system.

3.6. Characteristic Method. According to Steudel [23] and Olver [24], the first integral can be expressed in the characteristic form as

$$D_x I = Q^{\alpha} E_{\alpha}, \tag{27}$$

where Q^{α} are the characteristics or multipliers.

3.7. Variational Approach. The variational approach was developed by Olver [24]. The variational derivative of (27) yields all the multipliers for which the equation can be expressed as a local first integral. The multipliers determining equation is

$$\frac{\delta}{\delta u^{\beta}}\left(Q^{\alpha} E_{\alpha}\right) = 0, \tag{28}$$

and it holds for arbitrary functions $u(x)$.

3.8. Variational Approach on Solution Space of the Differential Equation. In this approach, the multiplier determining equation is obtained by taking the variational derivative of (27) on the solution space of the differential equation; that is,

$$\left.\frac{\delta}{\delta u^{\beta}}(Q^{\alpha} E_{\alpha})\right|_{E_{\alpha}=0} = 0. \tag{29}$$

Equations (29) are less overdetermined than (28). Sometimes the characteristics constructed from (29) may correspond to adjoint symmetries (see [28]) and not to a first integral.

3.9. Integrating Factor Method for First Integrals. Consider the system (1) and let $\phi[u] = \phi(x, u, u', \ldots, u_{k-1})$, $\Lambda^{\alpha}[u] = \Lambda^{\alpha}(x, u, u', \ldots, u_{k-1})$ and $E_{\alpha}[u] = E_{\alpha}(x, u, u', \ldots, u_{k-1})$ such that

$$\frac{d}{dx}\phi[u] = \Lambda^{\alpha}[u] E_{\alpha}[u]. \tag{30}$$

On the solutions u of system (1), it is concluded that $\phi[u] = \text{const}$, which means that $\phi[u]$ is a first integral of (1) and the functions Λ^{α} are integrating factors; see [28] for further details and deeper discussion.

The linearized system to (1) is given by

$$L_{\sigma\rho}[u] v^{\rho} \equiv E_{\sigma\rho}[u] + E_{\sigma\rho}^{1} \frac{dv^{\rho}}{dx} + \cdots + E_{\sigma\rho}^{n} \frac{d^{n} v^{\rho}}{dx^{n}} = 0, \tag{31}$$

where

$$E_{\sigma\rho}[u] = \frac{\partial E_{\sigma}}{\partial y^{\rho}}, \ldots, E_{\sigma\rho}^{n}[u] = \frac{\partial E_{\sigma}}{\partial u^{(n)\rho}}. \tag{32}$$

The adjoint of the linearized system (31) is given by

$$\begin{aligned} L_{\rho\sigma}^{*}[u] w^{\sigma} &\equiv E_{\rho\sigma}[u] w^{\sigma} - \frac{d}{dx}\left(E_{\sigma\rho}^{1} w^{\sigma}\right) + \cdots + \frac{d^{n}}{dx^{n}}\left(E_{\sigma\rho}^{n} w^{\sigma}\right) \\ &= 0. \end{aligned} \tag{33}$$

Moreover, the operators $L_{\sigma\rho}$ and $L_{\rho\sigma}^{*}$ satisfy the identity

$$w^{\sigma} L_{\sigma\rho}[u] v^{\rho} - v^{\rho} L_{\rho\sigma}^{*}[u] w^{\sigma} = \frac{d}{dx} S[w, v; E[u]], \tag{34}$$

where

$$\begin{aligned} S[w, v, E[u]] = {} & v^{\rho} w^{\sigma} E_{\sigma\rho}^{1} + \left(\frac{dv^{\rho}}{dx} - v^{\rho} \frac{d}{dx}\right)\left(w^{\sigma} E_{\sigma\rho}^{2}\right) + \cdots \\ & + \left(\frac{d^{n-1}}{dx^{n-1} v^{\rho}} + \sum_{l=1}^{n-2} \frac{d^{n-l-1} v^{\rho}}{dx^{n-l-1}} \frac{d^{l}}{dx^{l}} \right. \\ & \left. + (-1)^{n-1} v^{\rho} \frac{d^{n-1}}{dx^{n-1}}\right)\left(w^{\sigma} E_{\sigma\rho}^{n}\right). \end{aligned} \tag{35}$$

If Λ^{σ} satisfy the condition

$$\begin{aligned} L_{\rho\sigma}^{*}[u] \Lambda^{\sigma}[u] = {} & -\Lambda_{\rho}^{\sigma}[u] E_{\sigma}[u] + \frac{d}{dx}\left(\Lambda_{\rho}^{1\sigma} E_{\sigma}\right) + \cdots \\ & + (-1)^{n-2} \frac{d^{n-1}}{dx^{n-1}}\left(\Lambda_{\rho}^{(n-1)\sigma}[u] E_{\sigma}\right), \end{aligned} \tag{36}$$

where

$$\Lambda_{\rho}^{\sigma} = \frac{\partial \Lambda^{\sigma}}{\partial u^{\rho}}, \ldots, \Lambda_{\rho}^{(n-1)\sigma} = \frac{\partial \Lambda^{\sigma}}{\partial u^{(n-1)\rho}}, \tag{37}$$

then the first integral ϕ is

$$\phi[u] = \phi_1 + \phi_2 = \text{const.} \tag{38}$$

In (38)

$$\begin{aligned}
\phi_1 &= \int_0^1 \left\{ S\left[\Lambda[u(x;\lambda)], \frac{\partial u(x;\lambda)}{\partial \lambda}; E[u(x;\lambda)]\right] \right.\\
&\quad \left. + N\left[\Lambda[u(x;\lambda)], \frac{\partial u(x;\lambda)}{\partial \lambda}; E[u(x;\lambda)]\right]\right\} d\lambda, \\
\phi_2 &= \int k(x)\, dx,
\end{aligned} \tag{39}$$

$$\begin{aligned}
N&\left[\Lambda[u(x;\lambda)], \frac{\partial u(x;\lambda)}{\partial \lambda}; E[u(x;\lambda)]\right] \\
&= \frac{\partial u^{\rho}(x;\lambda)}{\partial \lambda} E_{\sigma}[u(x;\lambda)]\, \Lambda_{\rho}^{1\sigma}[u(x;\lambda)] \\
&\quad + \left(\frac{d}{dx}\left(\frac{\partial u^{\rho}(x;\lambda)}{\partial \lambda}\right) - \frac{\partial u^{\rho}(x;\lambda)}{\partial \lambda}\frac{d}{dx}\right) \\
&\quad \times \left(E_{\sigma}[u(x;\lambda)]\, \Lambda_{\rho}^{2\sigma}[u(x;\lambda)]\right) \\
&\quad + \cdots + \frac{d^{n-2}}{dx^{n-2}}\left(\frac{\partial u^{\rho}(x;\lambda)}{\partial \lambda}\right) \\
&\quad + \sum_{l=1}^{n-2} \frac{d^{n-l-2}}{dx^{n-l-2}}\left(\frac{\partial u^{\rho}(x;\lambda)}{\partial \lambda}\right)\frac{d^{l}}{dx^{l}} \\
&\quad + \cdots + (-1)^{n-2}\frac{\partial u^{\rho}(x;\lambda)}{\partial \lambda}\frac{d^{n-2}}{dx^{n-2}}, \\
k(x) &= \Lambda^{\sigma}[\tilde{u}(x)]\, E_{\sigma}[\tilde{u}(x)],
\end{aligned} \tag{40}$$

and $\tilde{u}(x) = (\tilde{u}_1(x), \ldots, \tilde{u}_n(x))$ are any fixed functions such that the function $k(x)$ is finite, while

$$u(x;\lambda) = \lambda u(x) + (1-\lambda)\,\tilde{u}(x). \tag{41}$$

We finish with the following definition.

Definition 1. The system (1) is said to be self-adjoint if and only if $L^{*}_{\sigma\rho}[u] = L_{\sigma\rho}[u]$.

4. First Integrals of Simple Harmonic Oscillator

We compute the first integrals of simple harmonic oscillator by utilizing different approaches. Consider

$$u'' + u = 0. \tag{42}$$

4.1. Direct Method. Equation (11) with $I(x,u,u')$ becomes

$$D_x I\big|_{u''+u=0} = 0, \tag{43}$$

where

$$D_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \cdots. \tag{44}$$

Equation (43) after expansion results in

$$I_x + I_u u' + I_{u'} u''\big|_{u''+u=0} = 0 \tag{45}$$

or

$$I_x + I_u u' - I_{u'} u = 0. \tag{46}$$

If we further restrict I to be

$$I = a(x,u)\frac{u'^2}{2} + b(x,u)\,u' + c(x,u), \tag{47}$$

then (46) becomes

$$\frac{a_u}{2}u'^3 + \left(\frac{a_x}{2} + b_u\right)u'^2 + \left(b_x + c_u - au\right)u' + c_x - bu = 0. \tag{48}$$

Splitting (47) according to derivatives of u, we obtain

$$\begin{aligned}
u'^3 &: a_u = 0, \\
u'^2 &: \frac{a_x}{2} + b_u = 0, \\
u' &: b_x + c_u - au = 0, \\
\text{rest} &: c_x - bu = 0.
\end{aligned} \tag{49}$$

The system of (49) is solved for a, b, and c to obtain

$$\begin{aligned}
I_1 &= \frac{u'^2}{2} + \frac{u^2}{2}, \\
I_2 &= -u'\sin x + u\cos x, \\
I_3 &= u'\cos x + u\sin x, \\
I_4 &= \frac{u'^2}{2}\sin 2x - uu'\cos 2x - \frac{u^2}{2}\sin 2x, \\
I_5 &= \frac{u'^2}{2}\cos 2x + uu'\sin 2x - \frac{u^2}{2}\cos 2x.
\end{aligned} \tag{50}$$

4.2. Symmetry and First Integral Relation. The first-order prolongation of the Lie point symmetry generators of (42) is

$$\begin{aligned}
X &= \left[c_1 + c_5\sin 2x + c_6\cos 2x + c_7 u\sin x + c_8 u\cos x\right]\frac{\partial}{\partial x} \\
&\quad + \left[c_2 u + c_3\sin x + c_4\cos x + c_5 u\cos 2x - c_6 u\sin 2x \right.\\
&\quad \left. + c_7\cos x - c_8 u^2\sin x\right]\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial u'},
\end{aligned} \tag{51}$$

where

$$\zeta = D_x\eta - u' D_x(\xi). \quad (52)$$

The first integrals are computed by the joint conditions (11) and (12).

The second important aspect of this approach is that we can associate a symmetry with a first integral. The relationship (12) holds for symmetry X_3 and first integral I_2 and, thus, symmetry X_3 is associated with I_2. Similarly X_4 is associated with I_3. This association of symmetries with a first integral helps in finding a solution via double reduction theory [38–43].

4.3. Noether's Approach. Equation (42) admits the standard Lagrangian

$$L = \frac{u'^2}{2} - \frac{u^2}{2}, \quad (53)$$

which satisfies the Euler Lagrange equation $\delta L/\delta u = 0$. Now we show how to compute the Noether symmetries corresponding to a Lagrangian (53).

The Noether symmetry determining (14) results in

$$X(L) + D_x(\xi) = D_x B, \quad (54)$$

where $\xi = \xi(x,u)$, $\eta = \eta(x,u)$, lie symmetry operators and $B = B(x,u)$ is the gauge terms. Expansion of (54) gives Noether symmetry determining equation

$$\begin{aligned} &-u\eta + u'\left[\eta_x + \eta_u u' - u'\left(\xi_x + \xi_u u'\right)\right] \\ &\quad + \left(\frac{u'^2}{2} - \frac{u^2}{2}\right)\left(\xi_x + \xi_u u'\right) = B_x + B_u u'. \end{aligned} \quad (55)$$

The separation of (55) with respect to powers of derivatives of u gives rise to

$$\begin{aligned} &u'^3 : \xi_u = 0, \\ &u'^2 : \eta_u - \frac{1}{2}\xi_x = 0, \\ &u' : \eta_x = B_u, \\ &u'^0 : 0 = u\eta + \frac{u^2}{2}\xi_x + B_x. \end{aligned} \quad (56)$$

The solution of system (56) is

$$\begin{aligned} &B = c_2 u\cos x - c_3 u\sin x - c_4 u^2 \sin 2x - c_5 u^2 \cos 2x, \\ &\xi = c_1 + c_4 \sin 2x + c_5 \cos 2x, \\ &\eta = c_2 \sin x + c_3 \cos x + u\left(c_4 \cos 2x - c_5 \sin 2x\right). \end{aligned} \quad (57)$$

Formula (16) with ξ, η, and B from (57) yields the first integrals (50).

4.4. Partial Noether Approach. Equation (42) admits the partial Lagrangian $L = u'^2/2$ and the corresponding partial Euler-Lagrange equation is

$$u = \frac{\delta L}{\delta u}, \quad (58)$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u'} + D_x^2\frac{\partial}{\partial u''} - \cdots. \quad (59)$$

The partial Noether operators $X = \xi\partial/\partial x + \eta\partial/\partial u$ corresponding to L satisfy (19); that is,

$$\begin{aligned} &\left[\eta_x + \eta_u u' - u'\xi_x - u'^2\xi_u\right]u' + \left(\xi_x + \xi_u u'\right)\frac{1}{2}u'^2 \\ &\quad = \eta u - u'\xi u + B_x + u' B_u. \end{aligned} \quad (60)$$

The usual separation by derivatives of u gives

$$\begin{aligned} &u'^3 : \xi = a(x), \\ &u'^2 : \eta = \frac{1}{2}a' u + b(x), \\ &u' : \eta_x = -\xi u + B_u, \\ &u'^0 : \eta u + B_x = 0. \end{aligned} \quad (61)$$

System (61) yields

$$\begin{aligned} &B = \frac{1}{2}c_1 u^2 + c_2 u\cos x - c_3 u\sin x \\ &\quad - \frac{1}{2}c_4 u^2 \sin 2x - \frac{1}{2}c_5 u^2 \cos 2x, \\ &\xi = \frac{1}{4}c_1 + c_4 \sin 2x + c_5 \cos 2x, \\ &\eta = c_2 \sin x + c_3 \cos x + u\left(c_4 \cos 2x - c_5 \sin 2x\right). \end{aligned} \quad (62)$$

Formula (16) with ξ, η, and B from (62) yields the first integrals (50). Hence the first integrals in each case are $D_x I = W(-u'' - u) = 0$ with respective characteristic W. Here the partial Noether's approach yields all nontrivial first integrals as obtained by Noether's approach. The difference here lies in the forms of B and L which are distinct from the ones used in the Noether approach.

4.5. Noether Approach for a System and Its Adjoint. The adjoint equation for (42) is

$$E_\alpha^*\left(x,u,v,u',v',u'',v''\right) \equiv \frac{\delta}{\delta u}\left[v\left(u'' + u\right)\right] = 0 \quad (63)$$

and this yields

$$v'' + v = 0. \quad (64)$$

Let $v = Q(x,u)$, then $v' = Q_x + Q_u u'$, and $v'' = Q_{xx} + 2Q_{xu} + Q_{uu}(u')^2 + Q_u u''$. Substituting these expressions of

v and v'' into (64) and equating the coefficients of u u' and u'' to zero, one obtains $v = u + c_1 \cos x + c_2 \sin x$. Thus, (42) is nonlinearly self-adjoint.

The Lagrangian for the system consisting of (42) and (64) is

$$L = v\left(u'' + u\right). \tag{65}$$

The Lagrangian L satisfies

$$\frac{\delta L}{\delta u} = v'' + v = 0, \qquad \frac{\delta L}{\delta v} = u'' + u = 0. \tag{66}$$

The formula for first integrals from (26) is

$$I = \xi L + W\left[\frac{\partial L}{\partial u'} - D_x\left(\frac{\partial L}{\partial u''}\right)\right] + D_x(W)\frac{\partial L}{\partial u''}. \tag{67}$$

Equation (67) with L from (65) results in

$$I = \xi v\left(u'' + u\right) - v'W + vD_x(W), \qquad W = \eta - \xi u'. \tag{68}$$

According to the conservation theorem, every Lie point, Lie-Bäcklund, and nonlocal symmetry of the system of second-order differential equation (42) yields a first integral for the system consisting of (42) and the adjoint equation (64). For the Lie symmetry $X_1 = \partial/\partial x$, the first integrals are

$$I = uv + u'v', \tag{69}$$

where v is the solution of adjoint equation (64). Note that $v = u$ yields I_1 and I_2, I_3 can be obtained from the substitution $v = \sin x$ and $v = \cos x$, respectively. One can use the other Lie symmetries given in (51) to derive the first integrals but one requires the solution of adjoint equation to construct first integrals. In order to illustrate this fact, let us consider the generator

$$X = u\frac{\partial}{\partial u}. \tag{70}$$

It is easy to check that (70) is not a Noether symmetry generator. Substituting $\xi = 0$ and $\eta = u$ into (68), one arrives at

$$I = -uv' + u'v. \tag{71}$$

The substitution $v = u$ in (71) provides the trivial first integral $I = 0$. However, setting $v = \cos x$ and $v = \sin x$, respectively, into (71), then the first integrals I_2 and I_3 are obtained again.

4.6. Characteristic Method. For (42) assume the characteristics $Q(x, u, u')$ and first integrals of form $I(x, u, u')$; then formula (27) yields

$$I_x + u'I_u + u''I'_u = Q\left(u'' + u\right). \tag{72}$$

Equation (72), after separating, with respect to u'' gives

$$\begin{aligned} I'_u &= Q, \\ I_x + u'I_u &= Q + u. \end{aligned} \tag{73}$$

Equations (73) finally result in (46) and, hence, following the same procedure as we did for the direct method, five first integrals (50) are obtained.

4.7. Variational Approach. For the variational approach with multiplier of form $Q = Q(x, u, u')$, we have

$$\frac{\delta}{\delta u}\left[Q\left(x, u, u'\right)\left(u'' + u\right)\right] = 0. \tag{74}$$

Equation (74), after expansion, takes the following form:

$$\begin{aligned} &Q + Q_u\left(u'' + u\right) - Q_{u'}\left(u''' + u'\right) \\ &\quad - \left(u'' + u\right)\left(Q_{u'x} + u'Q_{uu'} + u''Q_{u'u'}\right) + Q_{xx} \\ &\quad + u'Q_{xu} + u''Q_{xu'} + Q_u u'' + u'Q_{ux} + u'^2 Q_{uu} \\ &\quad + u'u''Q_{uu'} + u'''Q_{u'} + u''Q_{u'x} + u''u'Q_{uu'} + u''^2 Q_{u'u'} \\ &\quad = 0. \end{aligned} \tag{75}$$

Separation of (75) with respect to u'' yields

$$\begin{gathered} u'Q_{uu'} - uQ_{u'u'} + Q_{u' \cdot x} + 2Q_u = 0, \\ Q + Q_{xx} + u'^2 Q_{uu} + 2u'Q_{ux} + uQ_u - uQ_{xu'} - uu'Q_{uu'} = 0. \end{gathered} \tag{76}$$

The solution of (76) gives rise to the following characteristics:

$$\begin{aligned} Q = {} & \left(c_1 + c_2 \sin 2x + c_3 \cos 2x\right) u' \\ & + \left[\left(c_3 \sin 2x - c_2 \cos 2x\right) u + c_4 \sin x + c_5 \cos x\right]. \end{aligned} \tag{77}$$

A multiplier has the property

$$Q\left(u'' + u\right) = D_x I. \tag{78}$$

Equation (78) with multipliers from (77) and $I = I(x, u, u')$ yields the first integrals as given in (50).

4.8. Variational Approach on Space of Solutions of the Differential Equation. For (42), condition (29) results in

$$\left.\frac{\delta}{\delta u}[Q(x, u, u')(u'' + u)]\right|_{u'' + u = 0} = 0. \tag{79}$$

Expanding (79), we have

$$\begin{aligned} &Q + Q_{xx} + 2u'Q_{xu} - uQ_u + u'^2 Q_{uu} - 2uu'Q_{uu'} \\ &\quad - u'Q_{u'} - 2uQ_{u'x} - u^2 Q_{u'u'} = 0 \end{aligned} \tag{80}$$

and this yields

$$\begin{aligned} Q = {} & \left(c_1 + c_2 \sin 2x + c_3 \cos 2x\right) u' \\ & + \left[\left(c_3 \sin 2x - c_2 \cos 2x\right) u + c_4 \sin x + c_5 \cos x\right] + c_6 u, \end{aligned} \tag{81}$$

where $c_1, \ldots, c_6$ are constants. The multipliers with respect to constants $c_1, \ldots, c_5$ are the same as obtained in Section 4.7 and yield the first integrals obtained in (50). The multiplier associated with c_6 is u which does not correspond to any first integral. It might correspond to an adjoint symmetry.

4.9. Integrating Factor Method for First Integrals. Applying the results of Section 3.9 to our considered equation means to follow closely the first example presented in [28].

Substituting $E = u'' + u$ into (33) and (36), it is, respectively, obtained

$$\Lambda_{xx} + 2u'\Lambda_{xu} - 2u\Lambda_{xu'} + \left(u'\right)^2\Lambda_{uu} - 2uu'\Lambda_{uu'} + u^2\Lambda_{u'u'} - u'\Lambda_{u'} - u\Lambda_{u} + \Lambda = 0, \tag{82}$$

$$\Lambda_{xu'} + u'\Lambda_{uu'} - u\Lambda_{u'u'} + 2\Lambda_{u} = 0. \tag{83}$$

Noticing that $\Lambda = u'$ is a solution of both (82) and (83), one can take $\tilde{u} = 0$ into (41). Therefore, from (39) and (38), it is obtained $S + N = \lambda[(u')^2 + u^2]$ and

$$\phi = \int_0^1 \lambda\left[\left(u'\right)^2 + u^2\right] d\lambda = \frac{\left(u'\right)^2 + u^2}{2} = c_1. \tag{84}$$

5. Exact Solutions via First Integrals

The Noether symmetries associated with the first integrals can be utilized to derive the exact solutions of ordinary differential equations [34].

If X is a Noether symmetry and I is a first integral of (1) corresponding to a first-order Lagrangian $L = L(x, u^{[1]})$, then the following properties are satisfied [34]:

$$X^{[1]}\left(I_j\right) = S_j^1 - S_j^2 - S_j^3, \quad 1 \le j \le m, \tag{85}$$

where

$$\begin{aligned} S_j^1 &= \sum_{i\ne j}^{m} \xi_i \frac{\partial f_j}{\partial x_i} - \xi_j \sum_{i\ne j}^{m} D_i f_i + \frac{\partial f_j}{\partial u}\sum_{i\ne j}^{m}\xi_i u_i, \\ S_j^2 &= L\left[\sum_{i\ne j}^{m} \xi_i \frac{\partial \xi_j}{\partial x_i} - \xi_j \sum_{i\ne j}^{m} D_i \xi_i + \frac{\partial \xi_j}{\partial u}\sum_{i\ne j}^{m}\xi_i u_i\right], \\ S_j^3 &= \frac{\partial L}{\partial u_j}\left[X^{[1]}\left(\eta\sum_{i=1}^{m}\xi_i u_i\right) - \left(\eta - \sum_{i=1}^{m}\xi_i u_i\right)\operatorname{Div}\xi\right] \\ &\quad - \left(\eta - \sum_{i=1}^{m}\xi_i u_i\right)\sum_{i=1}^{m}\frac{\partial L}{\partial u_i}\frac{\partial \zeta_i}{\partial u_j}. \end{aligned} \tag{86}$$

Proposition 2. *Suppose X is a symmetry of $I_j = B_j$, where $B_j = B_j(x, u)$; then it satisfies*

$$\sum_{i=1}^{m} \xi \frac{\partial B_j}{\partial x_i} + \eta \frac{\partial B_j}{\partial u} = S_j^1 - S_j^2 - S_j^3. \tag{87}$$

Proposition 3. *In* (87) *if $m = 1$ and $X^{[1]}(I) = 0$, then X is a point symmetry of reduced equation $I(x, u, u') = c$, in which c is an arbitrary constant.*

Now we will compute the exact solutions of (42) using its first integrals which are reduced forms of the equation under consideration. The first integrals reduce an nth-order ODE to $(n-1)$th-order ODE. For scalar first-order ODE, the first integrals transform to quadrature whereas for scalar second-order ODE the first integrals result in the first-order ODEs. Some of these reduced forms (first integrals) can be solved directly. The other reduced form can be transformed to quadrature by using the Noether symmetries with its associated first integrals which yield the exact solutions. The first three integrals I_1, I_2, and I_3 of (42) yield a solution directly. Since $D_x I = 0$ which implies $I = c$, the first integral I_1 in (50) can be written as

$$\frac{u^2}{2} + \frac{u'^2}{2} = c_1 \tag{88}$$

which can also be expressed as

$$\frac{du}{\sqrt{c_1 - u^2}} = \pm dx. \tag{89}$$

Equation (153) is a variable separable and yields

$$\frac{u}{\sqrt{c_1 - u^2}} = \tan\left(\pm x + c_2\right), \tag{90}$$

and this comprises the exact solution of ODE (42).

A similar procedure is adapted to get the following exact solution of (42) using I_2 or I_3:

$$u = c_1 \cos x + c_2 \sin x. \tag{91}$$

Now we show how one can find the exact solution of (42) using Noether symmetries associated with the first integral. The Noether symmetry

$$X = \sin 2x \frac{\partial}{\partial x} + u\cos 2x \frac{\partial}{\partial u} \tag{92}$$

is associated with the first integral I_4 in (50). The induced equation $I_4 = c_1$ can be expressed as

$$\frac{u'^2}{2}\sin 2x - uu'\cos 2x - \frac{u^2}{2}\sin 2x = c_1. \tag{93}$$

Using (92) one can easily find the invariant

$$u = A(x)\sqrt{\sin 2x}, \tag{94}$$

and it reduces (93) to

$$-A^2 + A'^2\sin^2(2x) = 2c_1 + A^2. \tag{95}$$

Equation (95) is expressible as a variable separable and it finally yields

$$A + \sqrt{2c_1 + A^2} = c_2\sqrt{\csc 2x - \cot 2x} \tag{96}$$

or

$$A + \sqrt{2c_1 + A^2} = \frac{c_2}{\sqrt{\csc 2x - \cot 2x}}. \tag{97}$$

The solution in (94) is an exact solution of (42) with A which can be determined from (96) or (97).

Similarly, the Noether symmetry $X = \cos(2x)(\partial/\partial x) - u\sin(2x)(\partial/\partial u)$ associated with I_5 in (50) provides the exact solution

$$u = B(x)\sqrt{\cos 2x}, \tag{98}$$

where $B(x)$ satisfies

$$B + \sqrt{2c_1 + B^2} = c_2\sqrt{\sec 2x + \tan 2x} \tag{99}$$

or

$$B + \sqrt{2c_1 + B^2} = \frac{c_2}{\sqrt{\sec 2x + \tan 2x}}. \tag{100}$$

6. Hamiltonian Functions and First Integrals

Suppose t is the independent variable and $(q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ are the phase space coordinates. The derivatives of q^i, p_i with respect to t are given by

$$\dot{p}_i = D(p_i), \quad \dot{q}_i = D(q_i), \quad i = 1, 2, \ldots, n, \tag{101}$$

where

$$D = \frac{\partial}{\partial t} + \dot{q}_i\frac{\partial}{\partial q_i} + \dot{p}_i\frac{\partial}{\partial p_i} + \cdots \tag{102}$$

is known as the total derivative operator with respect to t. Here we present the basic operators needed in the sequel after introducing the necessary notations.

The Euler operator, for each α, is

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i}, \quad i = 1, 2, \ldots, n, \tag{103}$$

and the associated variational operator is

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D\frac{\partial}{\partial \dot{p}_i}, \quad i = 1, 2, \ldots, n. \tag{104}$$

Applying operators (103) and (104) on

$$L(t, q, \dot{q}) = p_i\dot{q}^i - H(t, q, p) \tag{105}$$

equated to zero results in the following canonical Hamilton equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \tag{106}$$

Equations (106) are obtained using $\delta L/\delta q^i = 0$ and $\delta L/\delta p_i = 0$. Equation (105) is the well-known Legendre transformation which relates the Hamiltonian and Lagrangian, where $p_i = \partial L/\partial \dot{q}^i$ and $\dot{q}^i = \partial H/\partial \dot{p}_i$.

Let

$$X = \xi(t, q, p)\frac{\partial}{\partial t} + \eta^i(t, q, p)\frac{\partial}{\partial q^i} + \zeta_i(t, q, p)\frac{\partial}{\partial p_i} \tag{107}$$

be the operator in the space (t, q, p). The operator in (107) is a generator of a point symmetry of the canonical Hamiltonian system (106) if it satisfies [24]

$$\dot{\eta}^i - \dot{q}^i\dot{\xi} - X\left(\frac{\partial H}{\partial p_i}\right) = 0, \qquad \dot{\zeta}_i - \dot{p}_i\dot{\xi} + X\left(\frac{\partial H}{\partial q^i}\right) = 0, \quad i = 1, \ldots, n \tag{108}$$

on the system (106).

In [24], the authors have studied the Hamiltonian symmetries in evolutionary or canonical form. The symmetry properties of the Hamiltonian action have been considered in the space (t, q, p) in [30, 31]. They presented the Hamiltonian version of Noether's theorem considering the general form of the symmetries (107).

The following important results which are analogs of Noether symmetries and the Noether theorem (see [24, 30, 44, 47] for a discussion) were established.

Theorem 4 (Hamilton action symmetries). *A Hamiltonian action*

$$p_i dq^i - H dt = 0 \tag{109}$$

is said to be invariant up to gauge $B(t, q, p)$ associated with a group generated by (107) *if*

$$\zeta_i\frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - HD(\xi) - D(B) = 0. \tag{110}$$

Theorem 5 (Hamiltonian version of Noether's theorem). *The canonical Hamilton system* (106) *which is invariant has the first integral*

$$I = p_i\eta^i - \xi H - B \tag{111}$$

for some gauge function $B = B(t, q, p)$ if and only if the Hamiltonian action is invariant up to divergence with respect to the operator X given in (107) *on the solutions to* (106).

6.1. First Integrals of Harmonic Oscillator in Hamiltonian Framework. Let us transfer the preceding example into the Hamiltonian framework and define

$$q = u, \qquad p = \frac{\partial L}{\partial \dot{u}} = \dot{u}, \qquad x = t. \tag{112}$$

The Hamiltonian function for this problem is

$$H(t, p, q) = \dot{u}\frac{\partial L}{\partial \dot{u}} - L = \frac{1}{2}\left(p^2 + q^2\right). \tag{113}$$

The canonical Hamiltonian equations (106) for Hamiltonian function (113) result in

$$\dot{q} = p, \qquad \dot{p} = -q. \tag{114}$$

The Hamiltonian operator determining equation (110), after expansion, yields

$$p\left(\eta_t + \dot{q}\eta_q\right) - \eta q - \frac{1}{2}\left(p^2 + q^2\right)\left(\xi_t + \dot{q}\xi_q\right) - B_t - \dot{q}B_q = 0, \tag{115}$$

in which we assume that $\xi = \xi(t, q)$, $\eta = \eta(t, q)$, and $B = B(t, q)$. One can also assume these functions to be dependent on p. We have chosen (t, q) dependence to simplify the calculations here and this leads to at least one Hamiltonian Noether operator. Equation (115) with the help of (114) can be written as

$$\dot{q}\left(\eta_t + \dot{q}\eta_q\right) - \eta q - \frac{1}{2}\left(\dot{q}^2 + q^2\right)\left(\xi_t + \dot{q}\xi_q\right) - B_t - \dot{q}B_q = 0. \tag{116}$$

One can separate (116) with respect to powers of derivatives of $\dot{q}$ and finally arrive at the following Hamiltonian Noether operators and the gauge terms:

$$\begin{aligned} B &= c_2 q\cos t - c_3 q\sin t - c_4 q^2 \sin 2t - c_5 q^2\cos 2t, \\ \xi &= c_1 + c_4 \sin 2t + c_5 \cos 2t, \\ \eta &= c_2 \sin t + c_3\cos t + q\left(c_4\cos 2t - c_5 \sin 2t\right). \end{aligned} \tag{117}$$

The first integrals from formula (111) are

$$\begin{aligned} I_1 &= \frac{p^2}{2} + \frac{q^2}{2}, \\ I_2 &= -p\sin t + q\cos t, \\ I_3 &= p\cos t + q\sin t, \\ I_4 &= \frac{p^2}{2}\sin 2t - qp\cos 2t - \frac{q^2}{2}\sin 2t, \\ I_5 &= \frac{p^2}{2}\cos 2t + qp\sin 2t - \frac{q^2}{2}\cos 2t. \end{aligned} \tag{118}$$

It is worthy to notice here that no integration is required to derive solutions of (114).

7. Applications to Some Models from Real World

In this section we apply the considered techniques to some equations arising from concrete problems, namely, the jerk equation and free oscillations with two-degree-of-freedom gyroscopic system with quadratic nonlinearities.

7.1. Jerk Equation. According to Gottlieb [48, 49], the most general nonlinear jerk equation is

$$E_1 = y''' + \alpha y'^3 + \beta y^2 y' + \gamma y' - \delta y y' y'' + \epsilon y' y''^2 = 0, \tag{119}$$

where the prime denotes differentiation with respect to x and $\alpha, \beta, \gamma, \delta, \epsilon$ are constants. In (119), at least one of β, δ, ϵ should be different from zero and if $\epsilon = 0$, then $\delta \neq 2\alpha$ so that the jerk equation is not a derivative of a second-order ODE [50].

7.1.1. First Integrals for Nonlinear Jerk Equation by Noether Approach for a System and Its Adjoint. Let us look for first integrals for (119) using Noether approach for a system and its adjoint. The adjoint equation for (119) is

$$\begin{aligned} E_1^* = {} & v\left[-6\alpha y'\left(y''\right)^2 - 3\alpha\left(y'\right)^2 y''' - 2\epsilon y'' y''' \right. \\ & \left. -3\delta y' y'' + 6\epsilon y'' y''' + 2\epsilon y' y''''\right] \\ & + v'\left[-3\alpha\left(y'\right)^2 y'' - \beta y^2 - \gamma + \delta y y'' - \epsilon\left(y''\right)^2 \right. \\ & \left. -2\delta\left(y'\right)^2 - 2\delta y y'' + 4\epsilon\left(y''\right)^2 + 4\epsilon y'' y'''\right] \\ & + v''\left[-\delta y y' + 2\epsilon y' y''\right] - v'''. \end{aligned} \tag{120}$$

The Lagrangian for system $E_1 = 0$ and $E_1^* = 0$ is $L = vE_1$. The system $E_1 = 0$ and $E_1^* = 0$ possesses a first integral

$$\begin{aligned} I &= \xi L + W\frac{\delta L}{\delta y'} + D(W)\frac{\delta L}{\delta y''} + D^2(W)\frac{\delta L}{\delta y''}, \\ W &= \eta - \xi y', \end{aligned} \tag{121}$$

where

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{122}$$

is any Lie point symmetry of (119). Now we use strictly self-adjointness for eliminating the nonlocal variable v in the first integral (121).

Using the self-adjoint condition

$$E_1^*\big|_{v=u} = \lambda E_1, \tag{123}$$

we conclude that $\lambda = -1$ and $\epsilon = \alpha = \delta = 0$, $\beta = 1$, $\gamma = 1$. Then we conclude that the strictly self-adjoint subclass of (119) is given by the family

$$y''' + y^2 y' + y' = 0. \tag{124}$$

The only admitted Lie point symmetry generator of (124) is

$$X_1 = \frac{\partial}{\partial x}. \tag{125}$$

If we take $\epsilon = \alpha = \delta = 0$, $\beta = 1$, $\gamma = 0$, (119) admits not only (125), but also

$$X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{126}$$

The Lie point symmetry generator (125) provides the trivial first integral $I = 0$. However, from (126) one can construct a nontrivial one. In fact, using (126), we obtain $W = -y - xy'$. Substituting this expression for W, $\xi = x$, and $\eta = -y$ into (121) and setting $v = y$, after reckoning, we have

$$I = 4yy'' - 2y'^2 + y^4. \tag{127}$$

7.1.2. First Integrals for Nonlinear Jerk Equation by Multipliers Approach. Now we will derive first integrals for nonlinear jerk equation by multipliers approach. Assume multipliers of form $Q(x, y, y')$. The multipliers determining equation (29) becomes

$$\frac{\delta}{\delta y}\left[Q\left(y''' + \alpha y'^3 + \beta y^2 y' + \gamma y' - \delta y y' y'' + \epsilon y' y''^2\right)\right]\Big|_{E_1=0} = 0. \tag{128}$$

After expansion of (128), the multipliers and first integrals are computed for specific values of parameters.

Case 1 ($\alpha = 0,\ \beta = 0,\ \gamma = 1,\ \delta = 0,\ \epsilon = 1$). Equation (119) reduces to

$$y''' + y' + y' y''^2 = 0. \tag{129}$$

The multipliers and first integrals are

$$\begin{aligned} Q_1\left(x, y, y'\right) &= \sin(y)\, e^{(1/2)y'^2}, \\ Q_2\left(x, y, y'\right) &= e^{(1/2)y'^2}\cos(y), \\ I_1 &= e^{(1/2)y'^2}\left(y''\sin(y) - \cos(y)\right), \\ I_2 &= e^{(1/2)y'^2}\left(y''\cos(y) + \sin(y)\right). \end{aligned} \tag{130}$$

Case 2 ($\alpha = 1,\ \beta = 1,\ \gamma = 0,\ \delta = 0,\ \epsilon = 0$). Equation (119) reduces to

$$y''' + y'^3 + y^2 y' = 0. \tag{131}$$

The multipliers and first integrals are

$$\begin{aligned} Q_1\left(x, y, y'\right) &= \sin\left(\sqrt{2}y\right), \\ Q_2\left(x, y, y'\right) &= \cos\left(\sqrt{2}y\right), \\ I_1 &= y''\sin\left(\sqrt{2}y\right) - \frac{1}{2}y'^2\sqrt{2}\cos\left(\sqrt{2}y\right) \\ &\quad - \frac{1}{2}\sqrt{2}y^2\cos\left(\sqrt{2}y\right) + \frac{1}{2}\cos\left(\sqrt{2}y\right)\sqrt{2} \\ &\quad + y\sin\left(\sqrt{2}y\right), \\ I_2 &= y''\cos\left(\sqrt{2}y\right) + \frac{1}{2}y'^2\sqrt{2}\sin\left(\sqrt{2}y\right) \\ &\quad + \frac{1}{2}\sqrt{2}y^2\sin\left(\sqrt{2}y\right) - \frac{1}{2}\sin\left(\sqrt{2}y\right)\sqrt{2} \\ &\quad + y\cos\left(\sqrt{2}y\right). \end{aligned} \tag{132}$$

Case 3 ($\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 0,\ \epsilon = 0$). Equation (119) reduces to

$$y''' + y' + y'^3 + y^2 y' = 0. \tag{133}$$

The multipliers and first integrals are

$$\begin{aligned} Q_1\left(x, y, y'\right) &= \sin\left(\sqrt{2}y\right), \\ Q_2\left(x, y, y'\right) &= \cos\left(\sqrt{2}y\right), \\ I_1 &= y''\sin\left(\sqrt{2}y\right) - \frac{1}{2}y'^2\sqrt{2}\cos\left(\sqrt{2}y\right) \\ &\quad - \frac{1}{2}\sqrt{2}y^2\cos\left(\sqrt{2}y\right) + y\sin\left(\sqrt{2}y\right), \\ I_2 &= y''\cos\left(\sqrt{2}y\right) + \frac{1}{2}y'^2\sqrt{2}\sin\left(\sqrt{2}y\right) \\ &\quad + \frac{1}{2}\sqrt{2}y^2\sin\left(\sqrt{2}y\right) + y\cos\left(\sqrt{2}y\right). \end{aligned} \tag{134}$$

Case 4 ($\alpha = 0,\ \beta = 0,\ \gamma = 1,\ \delta = 1,\ \epsilon = 0$). Equation (119) reduces to

$$y''' + y' - y y' y'' = 0. \tag{135}$$

Equation (128) yields two multipliers

$$\begin{aligned} Q_1\left(x, y, y'\right) &= e^{-(1/2)y^2}\operatorname{erf}\left(\frac{1}{2}I\sqrt{2}y\right), \\ Q_2\left(x, y, y'\right) &= e^{-(1/2)y^2}. \end{aligned} \tag{136}$$

The first integrals are given by

$$\begin{aligned} I_1 &= \frac{1}{2}\frac{1}{\sqrt{\pi}}\left(-Iy'^2\sqrt{2} + 2e^{-(1/2)y^2}y''\sqrt{\pi}\operatorname{erf}\left(\frac{1}{2}I\sqrt{2}y\right)\right. \\ &\qquad \left. +2\left(\int e^{-(1/2)y^2}\operatorname{erf}\left(\frac{1}{2}I\sqrt{2}y\right)dy\right)\sqrt{\pi}\right), \\ I_2 &= \frac{1}{2}\frac{2e^{-(1/2)y^2}y''\sqrt{\pi} + \pi\sqrt{2}\operatorname{erf}\left((1/2)\sqrt{2}y\right)}{\sqrt{\pi}}. \end{aligned} \tag{137}$$

Case 5 ($\alpha = 1,\ \beta = 1,\ \gamma = 1,\ \delta = 1,\ \epsilon = 0$). Equation (119) reduces to

$$y''' + y' + y'^3 + y^2 y' - y y' y'' = 0. \tag{138}$$

The multipliers and first integrals are

$$\begin{aligned} Q_1\left(x, y, y'\right) &= -\sqrt{2}\sqrt{\pi}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right)e^{-(1/2)y^2} \\ &\qquad + \sqrt{2}\sqrt{\pi}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right)e^{-(1/2)y^2}y^2 - 2y, \\ Q_2\left(x, y, y'\right) &= e^{-(1/2)y^2}\left(-1 + y^2\right), \end{aligned}$$

$$\begin{aligned}
I_1 = & \int \Big(e^{-(1/2)y^2}\sqrt{2}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right) y^4\sqrt{\pi} - 2y^3 \\
& -\sqrt{2}\sqrt{\pi}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right) e^{-(1/2)y^2} - 2y \Big) dy \\
& - y'^2 e^{-(1/2)y^2}\sqrt{2}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right) y\sqrt{\pi} \\
& - y''\sqrt{2}\sqrt{\pi}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right) e^{-(1/2)y^2} \\
& + y''\sqrt{2}\sqrt{\pi}\operatorname{erfi}\left(\frac{1}{2}\sqrt{2}y\right) e^{-(1/2)y^2} y^2 + 2y'^2 - 2yy'', \\
I_2 = & -y^3 e^{-(1/2)y^2} - 3ye^{-(1/2)y^2} + \sqrt{2}\sqrt{\pi}\operatorname{erf}\left(\frac{1}{2}\sqrt{2}y\right) \\
& + y'' e^{-(1/2)y^2} y^2 - y'' e^{-(1/2)y^2} - y'^2 y e^{-(1/2)y^2}.
\end{aligned} \tag{139}$$

Case 6 ($\epsilon = \alpha = \delta = 0,\ \beta = 1, \gamma = 0$). The multipliers and first integrals for this case are

$$Q_1 = 1, \qquad Q_2 = y, \qquad I_1 = y'' + \frac{y^3}{3} \tag{140}$$

and I_2 is the same as the first integral (127) derived by Noether approach for system and its adjoint.

7.1.3. Reduction of Order and Implicit Solution to Jerk Equation. We now utilize first integrals to compute the exact solution of the jerk equation (119).

We firstly consider the first integrals I_1 and I_2 obtained in Case 2. Setting $I_1 = c_1$ and $I_2 = c_2$, straightforward calculations yield

$$y'' + y = c_1 \sin\left(\sqrt{2}y\right) + c_2 \cos\left(\sqrt{2}y\right). \tag{141}$$

The routine calculations show that (141) finally results in

$$\pm \int \frac{1}{\sqrt{1 - y^2 - c_1\cos\left(\sqrt{2}y\right)\sqrt{2} + c_2\sin\left(\sqrt{2}y\right)\sqrt{2}}} dy = x + c_3, \tag{142}$$

which satisfies $I_1 = c_1$, $I_2 = c_2$, and

$$y''' + y'^3 + y^2 y' = 0. \tag{143}$$

Now we obtain an implicit solution to

$$y''' + y' + y' y''^2 = 0 \tag{144}$$

using the first integrals given in Case 1. Again, assuming that $I_1 = c_1$ and $I_2 = c_2$, where c_1 and c_2 are arbitrary constants and I_1 and I_2 are the first integrals established in Case 1, we finally arrive at

$$c_1 \sin y + c_2 \cos y = e^{(y')^2/2} y''. \tag{145}$$

From (145), we obtain

$$\int \frac{1}{\sqrt{2\ln\left(-c_1\cos y + c_2 \sin y\right)}} dy = x + c_3, \tag{146}$$

which comprises the solution of jerk equation for Case 1.

7.2. First Integrals for a Gyroscopic System with Quadratic Nonlinearities. Consider the free oscillations of a two-degree-of-freedom gyroscopic system with quadratic nonlinearities [51]

$$\begin{aligned} y'' + z' + 2y &= 2yz, \\ z'' - y' + 2z &= y^2. \end{aligned} \tag{147}$$

The system (147) satisfies the partial Euler-Lagrange equations $\delta L/\delta y = z' + 2y - 2yz$ and $\delta L/\delta z = -y' + 2z - y^2$ and has a partial Lagrangian

$$L = \frac{1}{2}\left(y'^2 + z'^2\right). \tag{148}$$

The partial Noether operator X of system (147) corresponding to the partial Lagrangian (148) satisfies

$$\begin{aligned}
&\left[\eta^1_x + y'\eta^1_y + z'\eta^1_z - y'\left(\xi_x + y'\xi_y + z'\xi_z\right)\right] y' \\
&\quad + \left[\eta^2_x + y'\eta^2_y + z'\eta^2_z - z'\left(\xi_x + y'\xi_y + z'\xi_z\right)\right] z' \\
&\quad + \left(\xi_x + y'\xi_y + z'\xi_z\right)\left[\frac{y'^2}{2} + \frac{z'^2}{2}\right] \\
&= \left(\eta^1 - y'\xi\right)\left[z' + 2y - 2yz\right] \\
&\quad + \left(\eta^2 - z'\xi\right)\left[-y' + 2z - y^2\right] + B_x + y'B_y + z'B_z.
\end{aligned} \tag{149}$$

Equation (149) splits into the following by comparing the coefficients of powers of y' and z':

$$\xi_y = 0, \qquad \xi_z = 0, \tag{150}$$

$$\eta^1_y - \frac{1}{2}\xi_x = 0, \qquad \eta^2_z - \frac{1}{2}\xi_x = 0, \qquad \eta^1_z + \eta^2_y = 0, \tag{151}$$

$$\eta^1_x = -2y(1-z)\xi - \eta^2 + B_y, \tag{152}$$

$$\eta^2_x = -\left(2z - y^2\right)\xi + \eta^1 + B_z, \tag{153}$$

$$\eta^1\left(2y - 2yz\right) + \eta^2\left(2z - y^2\right) + B_x = 0. \tag{154}$$

Thus (150)–(154) finally yield

$$\begin{aligned} \xi = -2c_1, \qquad \eta^1 = 0, \qquad \eta^2 = 0, \\ B = -2c_1\left(y^2 + z^2\right) + 2c_1 y^2 z. \end{aligned} \tag{155}$$

Choosing $c_1 = 1$, we find that the partial Noether operator and gauge term of system (147) are

$$X = -2\frac{\partial}{\partial x}, \qquad B = -2\left(y^2 + z^2\right) + 2y^2 z. \tag{156}$$

The only first integral for the two-degree-of-freedom gyroscopic system by formula (16) is

$$I = -2\left(y^2 + z^2\right) + 2y^2 z - y'^2 - z'^2. \qquad (157)$$

8. Concluding Remarks

The first integrals for simple harmonic oscillator were constructed using different approaches. All different approaches to compute first integrals for ordinary differential equations and systems of ordinary differential equations were explained with the ample example. The systematic way to compute the first integrals is by Noether's approach but it depends upon the existence of a standard Lagrangian. The Noether symmetries and the corresponding first integrals were constructed for simple harmonic oscillator. In the absence of a standard Lagrangian one can use the partial Noether approach which works with or without Lagrangian and the framework for this approach is similar to the Noether approach. The direct method and its use with the symmetry condition were explained in detail. We commented on some other approaches: the characteristic method, the multiplier approach for arbitrary functions as well as on the solution space, and the direct construction formula approach based on the knowledge of characteristics or multipliers which work without regard to a standard Lagrangian. The multipliers or characteristics can be easily constructed taking the variational derivative of $D_i T^i = Q^\alpha E_\alpha$ not only for solutions but also for arbitrary functions. We have shown how one can compute the first integrals for ordinary differential equations using the Hamiltonian version of Noether theorem.

Furthermore, some solutions of ordinary differential equations using first integrals with its associated Noether symmetries were obtained. The first integrals are the reduced form of the given differential equation. Some of these reduced forms can be solved directly whereas the other form can be used to further reduce the order of a differential equation. The harmonic oscillator yielded five first integrals. Three first integrals were used to compute the solutions directly. Two first integrals were written as the first-order equations and the Noether symmetries were used to find the invariants which completely transform the reduced equation to quadrature.

First integrals for nonlinear jerk equation were derived by using Ibragimov's and multipliers' approach. Then using these first integrals, some exact solutions of jerk equation for different cases were also established. The partial Noether approach is used to derive the first integrals of the free oscillations of a two-degree-of-freedom gyroscopic system with quadratic nonlinearities.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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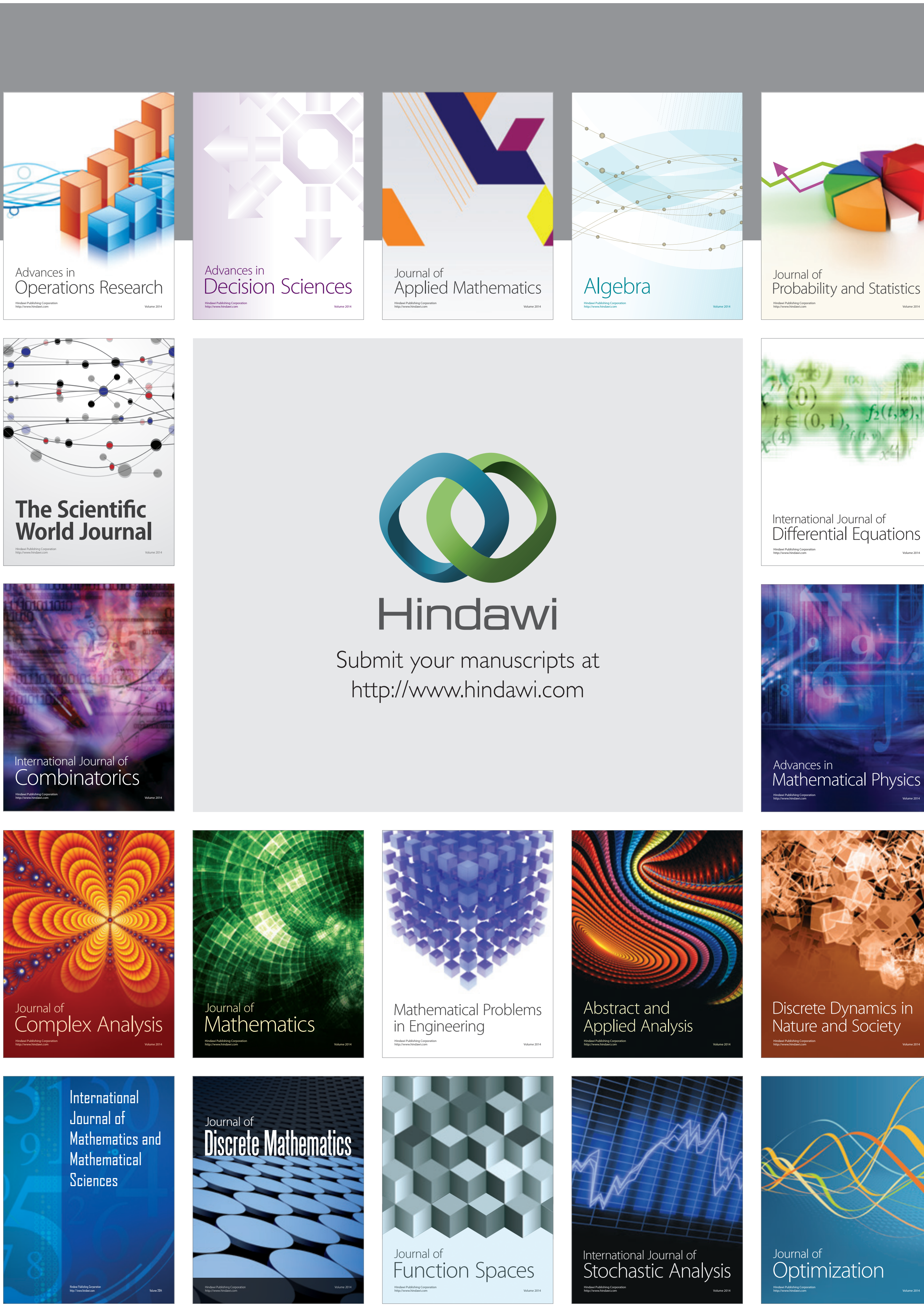
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