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Research Article

Complexity in Linear Systems: A Chaotic Linear Operator on the Space of Odd 2π-Periodic Functions

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Not just nonlinear systems but infinite-dimensional linear systems can exhibit complex behavior. It has long been known that twice the backward shift on the space of square-summable sequences l_2 displays chaotic dynamics. Here we construct the corresponding operator $\mathscr{C}$ on the space of 2π-periodic odd functions and provide its representation involving a Principal Value Integral. We explicitly calculate the eigenfunction of this operator, as well as its periodic points. We also provide examples of chaotic and unbounded trajectories of $\mathscr{C}$.

1. Introduction

Linear systems have commonly been thought to exhibit relatively simple behavior. Surprisingly, infinite-dimensional linear systems can have complex dynamics. In particular, Rolewicz in 1969 [1] showed that the backward shift B multiplied by 2 (i.e., $2B$) on the space of square-summable sequences l_2 exhibits transitivity (and thus gives rise to chaotic dynamics). A nice exposition of dynamics of infinite-dimensional operators is given in [2, 3] and the recent books [4, 5]. While chaoticity of linear operators is at first puzzling and the backward shift example seems contrived, these operators are not rare. In fact, Herrero [6] and Chan [7] showed that chaotic linear operators are dense (with respect to pointwise convergence) in the set of bounded linear operators. In addition to $2B$ there are many examples of chaotic linear operators including weighted shifts [8], composition operators [9], and differentiation and translations [10–12]. It has also been argued in [13, 14] that nonlinearity is not necessarily required for complex behavior; an infinite-dimensional state space can also provide the ingredients of chaotic dynamics.

Several recent papers explore chaotic behavior of linear systems (see, e.g., [15, 16]). Bernardes et al. [17], for example, obtain new characterizations of Li-Yorke chaos for linear operators on Banach and Fréchet spaces.

Here we construct a chaotic linear operator by "lifting" $2B$ to the space L_2 of square-integrable functions (more precisely to the Hilbert space $L_2(0,\pi)$ of 2π-periodic odd functions). Our main tool in finding the expression for the backward shift is utilizing a smidgen of distribution theory and Cauchy's principal value, a method for obtaining a finite result for a singular integral. The principal value (PV) integral (see, e.g., [18], p. 457) of a function f about a point $c \in [a,b]$ is given by

$$\mathrm{PV}\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+}\left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right). \tag{1}$$

The PV integral is commonly used in many fields of physics. A review of developments in the mathematics and methods for Principal Value Integrals is presented in [19]. Cohen et al. [20] examine first-order PV integrals and analyze several of their important properties. The structure of the paper is the following. In Section 2 we relate the backward shift on l_2 to a shift on $L_2(0,\pi)$. We state and prove a theorem about expressing this shift on $L_2(0,\pi)$ in terms of a PV integral. In Section 3 we define and analyze the corresponding chaotic operator $\mathscr{C}$ on $L_2(0,\pi)$, including finding its eigenvectors and periodic points. We provide examples of unbounded and chaotic trajectories of $\mathscr{C}$. In Section 4 we draw conclusions. We also show that utilizing the representation of operator $\mathscr{C}$ one can obtain principal values of certain integrals.

2. A Chaotic Linear Operator on the Space of 2π-Periodic Odd Functions

The backward shift B on the infinite-dimensional Hilbert space l_2 of square-summable sequence is defined as

$$B_a = (a_2, a_3, \ldots), \quad (2)$$

where $a = (a_1, a_2, \ldots)$, such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. The Hilbert space $L_2(0,\pi)$ of square-integrable functions is isomorphic with l_2 (by the Riesz-Fischer theorem) and is a natural functional representation of the sequence space l_2. By odd extension, elements of $L_2(0,\pi)$ can be viewed as odd 2π-periodic square-integrable functions so that $L_2(0,\pi)$ is also isomorphic with the space of odd 2π-periodic square-integrable functions. Now we "lift" $a \in l_2$ to $L_2(0,\pi)$ by the summation

$$f(t) = \sum_{n=1}^{\infty} a_n \sin nt. \quad (3)$$

Clearly, the nth Fourier coefficient of $f(t)$ is expressed as

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\xi) \sin n\xi \, d\xi. \quad (4)$$

We define the backward shift $\mathscr{B}$ acting on $L_2(0,\pi)$ as

$$\mathscr{B} f(t) = \sum_{n=1}^{\infty} a_{n+1} \sin nt = \sum_{n=1}^{\infty} a_n \sin(n-1)t. \quad (5)$$

Therefore

$$\begin{aligned} \mathscr{B} f(t) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(\xi) \sin(n+1)\xi \, d\xi \right) \sin nt \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(\xi) \sin n\xi \, d\xi \right) \sin(n-1)t. \end{aligned} \quad (6)$$

Our main result is the following.

Theorem 1. *$\mathscr{B} f(t)$ can be expressed as*

$$\mathscr{B} f(t) = f(t)\cos t - \frac{1}{\pi} \mathrm{PV} \int_0^{\pi} \frac{\sin t \sin \xi}{\cos t - \cos \xi} f(\xi) \, d\xi. \quad (7)$$

The strategy of the proof is the following: let us denote by $\mathscr{A} f(t)$ the right-hand side of (7) and by P_n the projection from $L_2(0,\pi)$ onto the linear span of $\{\sin t, \sin 2t, \ldots, \sin nt\}$. The sequence $\mathscr{B}P_n$ converges strongly to B. In particular, for every $\varphi \in \mathscr{D}(0,\pi)$ (this is the space of test functions, see Definition 2 in the Appendix), $\mathscr{B}P_n\varphi \to \mathscr{B}\varphi$ in $L_2(0,\pi)$. Then a subsequence tends to $\mathscr{B}\varphi$ almost everywhere. Hence if we prove that $\mathscr{B}P_n\varphi(t)$ tends to $\mathscr{A}\varphi(t)$ for all fixed t, then $\mathscr{A}\varphi = \mathscr{B}\varphi$ almost everywhere as functions in $L_2(0,\pi)$; that is, $\mathscr{A} = \mathscr{B}$ on $\mathscr{D}(0,\pi)$. Finally, $\mathscr{D}(0,\pi)$ is a dense set in $L_2(0,\pi)$; thus $\mathscr{A} = \mathscr{B}$ on the whole space $L_2(0,\pi)$.

Proof. We start from

$$\begin{aligned} \mathscr{B}P_n f(t) &= \frac{2}{\pi} \sum_{k=1}^{n} \left(\int_0^{\pi} f(\xi) \sin k\xi \, d\xi \right) \sin(k-1)t \\ &= \frac{2}{\pi} \int_0^{\pi} f(\xi) \left(\sum_{k=1}^{n} \sin k\xi \sin(k-1)t \right) d\xi. \end{aligned} \quad (8)$$

We first rewrite the "kernel" of (8) as

$$\begin{aligned} &\sin k\xi \sin(k-1)t \\ &\quad = \sin k\xi \sin kt \cos t - \sin k\xi \cos kt \sin t \\ &\quad = \frac{\cos k(\xi - t) - \cos k(\xi + t)}{2} \cos t \\ &\quad\quad - \frac{\sin k(\xi - t) + \sin k(\xi + t)}{2} \sin t. \end{aligned} \quad (9)$$

For test functions $\varphi \in \mathscr{D}(0,\pi)$ and $t \in (0,\pi)$ we get

$$\begin{aligned} \mathscr{B}P_n\varphi(t) &= \frac{1}{\pi} \cos t \\ &\quad \cdot \int_0^{\pi} \varphi(\xi) \sum_{k=1}^{n} (\cos k(\xi - t) - \cos k(\xi + t)) \, d\xi - \frac{1}{\pi} \\ &\quad \cdot \sin t \int_0^{\pi} \varphi(\xi) \sum_{k=1}^{n} (\sin k(\xi - t) + \sin k(\xi + t)) \, d\xi. \end{aligned} \quad (10)$$

Taking $n \to \infty$ limit and utilizing (A.1) and (A.2) yield

$$\begin{aligned} \lim_{n\to\infty} \mathscr{B}P_n\varphi(t) &= \cos t \, \langle (\delta(\xi - t) - \delta(\xi + t)), \varphi(\xi) \rangle \\ &\quad - \frac{1}{2\pi} \sin t \left\langle \left(\mathscr{P} \cot \frac{\xi - t}{2} + \mathscr{P} \cot \frac{\xi + t}{2} \right), \varphi(\xi) \right\rangle. \end{aligned} \quad (11)$$

Since

$$\begin{aligned} &\cot \frac{\xi - t}{2} + \cot \frac{\xi + t}{2} \\ &\quad = 2 \frac{\sin((\xi - t)/2 + (\xi + t)/2)}{2 \sin((\xi - t)/2) \sin((\xi + t)/2)} \\ &\quad = \frac{2 \sin \xi}{\cos t - \cos \xi}, \end{aligned} \quad (12)$$

the limit calculated in (11) is the same as $\mathscr{A} f(t)$. □

Our "chaotic" operator (twice the backward shift) is now defined as

$$\begin{aligned} \mathscr{C} f(t) &= 2\mathscr{B} f(t) \\ &= 2 f(t) \cos t - \frac{2}{\pi} \mathrm{PV} \int_0^{\pi} \frac{\sin t \sin \xi}{\cos t - \cos \xi} f(\xi) \, d\xi. \end{aligned} \quad (13)$$

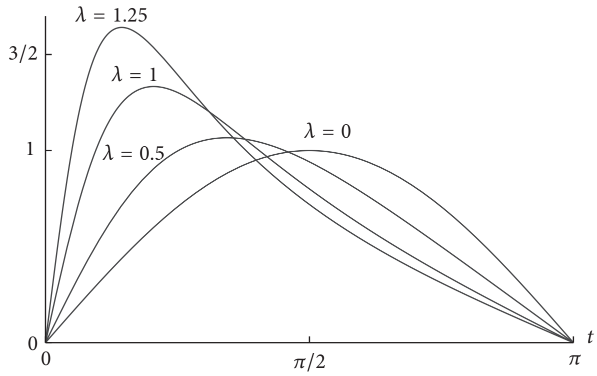

Figure 1: Eigenfunctions of $\mathscr{C}$ for $\lambda = 0, 0.5, 1,$ and 1.25.

3. Analysis of $\mathscr{C}$

The eigenfunctions of $\mathscr{C}$ can be found from the eigenvalue relation

$$\mathscr{C} f^*(t) = \lambda f^*(t). \tag{14}$$

Instead of using (13), we revert to (5) to write

$$\sum_{n=1}^{\infty} 2a_{n+1} \sin nt = \sum_{n=1}^{\infty} \lambda a_n \sin nt. \tag{15}$$

From this we have

$$a_{n+1} = \left(\frac{\lambda}{2}\right) a_n, \tag{16}$$

and thus

$$f^*(t) = a_1 \sum_{n=1}^{\infty} \left(\frac{\lambda}{2}\right)^{n-1} \sin nt = a_1 \frac{4 \sin t}{4 + \lambda^2 - 4\lambda \cos t}. \tag{17}$$

The functions corresponding to eigenvalue $\lambda = 1$ are

$$f^*(t) = a_1 \frac{4 \sin(t)}{5 - 4\cos(t)}; \tag{18}$$

that is, the functions $a_1(4\sin(t)/(5 - 4\cos(t)))$ are left invariant under the action of $\mathscr{C}$. In other words $f^*(t)$'s are fixed points of operator $\mathscr{C}$. A family of eigenfunctions is displayed in Figure 1 (we set $a_1 = 1$).

To better characterize the action of $\mathscr{C}$ we want to understand how a given function is "shaped" under the repeated application of $\mathscr{C}$. For $f \in L_2(0,\pi)$ the orbit of f is defined as $\mathrm{Orb}(\mathscr{C}, f) = \{f, \mathscr{C}f, \mathscr{C}^2 f, \ldots\}$, where $\mathscr{C}^k = \underbrace{\mathscr{C} \circ \cdots \circ \mathscr{C}}_{k}$ is the kth composition of $\mathscr{C}$ with itself. The k-fold composition acts on $f(t)$ as

$$\mathscr{C}^k f(t) = 2^k \sum_{n=1}^{\infty} a_{n+k} \sin nt. \tag{19}$$

A given $f \in L_2(0,\pi)$ is a T-periodic point of $\mathscr{C}$ if $\mathscr{C}^T f = f$ for some $T \geq 1$ (a fixed point is a 1-periodic point; i.e., $T = 1$).

We are now in the position to construct T-periodic points of $\mathscr{C}$.

Introducing $y = \{y_1, \ldots, y_T\}$, a T-periodic point of $2B$ (acting on l_2) can be written as [3]

$$a = \left(y_1, \ldots, y_T, \frac{y_1}{2^T}, \ldots, \frac{y_T}{2^T}, \frac{y_1}{2^{2T}}, \ldots, \frac{y_T}{2^{2T}}, \ldots\right), \tag{20}$$

whose nth component is given by

$$a_n = \frac{y_{(n-1 \bmod T)+1}}{2^{[(n-1)/T]T}}. \tag{21}$$

Using the linearity of $\mathscr{C}$ we can easily find a period-2 point of $\mathscr{C}$, that is, a function $g(t)$ such that $\mathscr{C}^2 g(t) = g(t)$:

$$\begin{aligned} g(t) &= y_1 \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{2^{2n-2}} + y_2 \sum_{n=1}^{\infty} \frac{\sin(2n)t}{2^{2n-2}} \\ &= y_1 \frac{20 \sin t}{17 - 8\cos 2t} + y_2 \frac{16 \sin 2t}{17 - 8\cos 2t}. \end{aligned} \tag{22}$$

In general, we find a period-T point of $\mathscr{C}$ as

$$\begin{aligned} f(t) = {} & y_1 \left(\sin t + \frac{\sin(T+1)t}{2^T} + \cdots + \frac{\sin(nT+1)t}{2^{nT}} + \cdots\right) \\ & + y_2 \left(\sin 2t + \frac{\sin(T+2)t}{2^T} + \cdots + \frac{\sin(nT+2)t}{2^{nT}} + \cdots\right) + \cdots + \\ & + y_T \left(\sin Tt + \frac{\sin(2T)t}{2^T} + \cdots + \frac{\sin((n+1)T)t}{2^{nT}} + \cdots\right). \end{aligned} \tag{23}$$

By defining the "basis functions"

$$\begin{aligned} \phi(l,T) &= \sum_{n=1}^{\infty} \frac{\sin((n-1)T + 1 + l)t}{2^{(n-1)T}} \\ &= \frac{2^T \left(2^T \sin(l+1)t + \sin((T-1-l)t)\right)}{1 + 4^T - 2^{T+1}\cos Tt}, \end{aligned} \tag{24}$$

a period-T point of $\mathscr{C}$ can be expressed as the linear combination

$$f(t) = \sum_{l=1}^{T} y_l \phi(l-1, T). \tag{25}$$

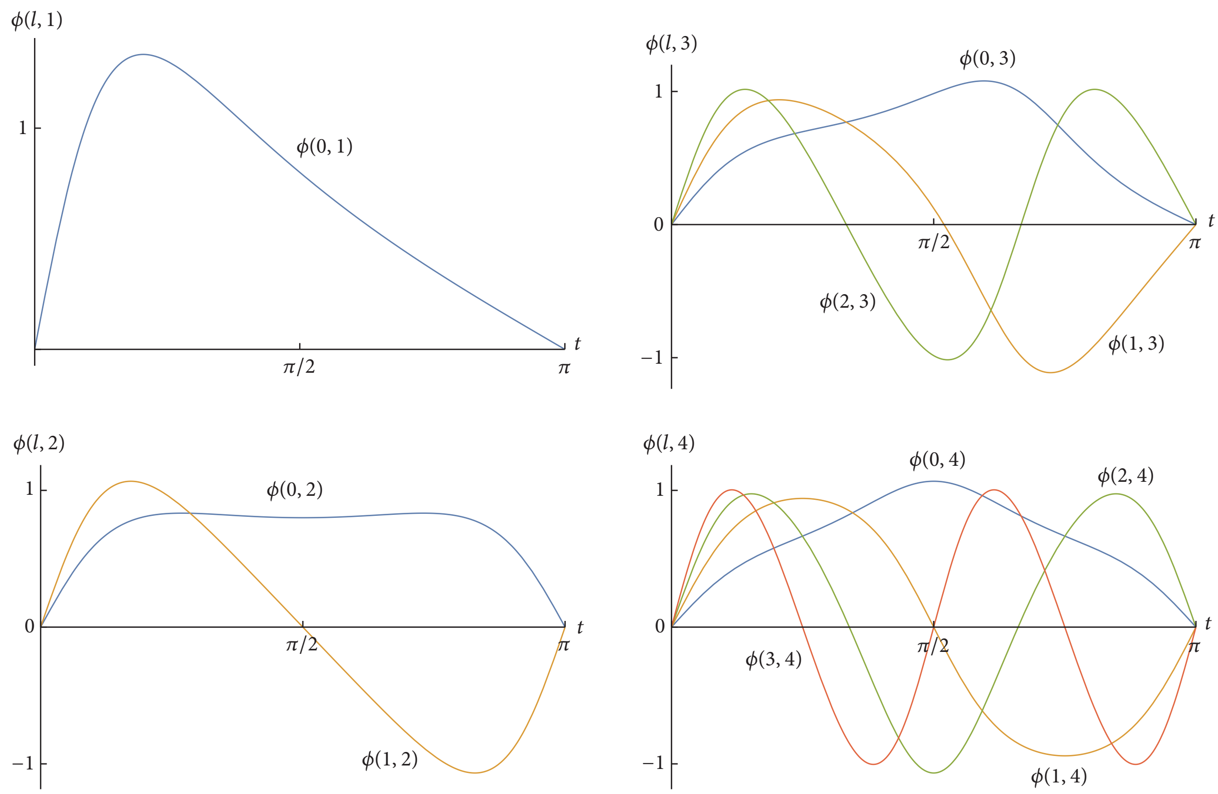

FIGURE 2: Basis functions $\phi(l,T)$ for $T = 1,\dots,4$ and $l = 0,\dots,T-1$.

The first few basis functions are (shown in Figure 2)

$$\phi(0,1) = \frac{4\sin(t)}{5-4\cos(t)} \qquad T = 1,$$

$$\phi(0,2) = \frac{20\sin(t)}{17-8\cos(2t)},$$

$$\phi(1,2) = \frac{16\sin(2t)}{17-8\cos(2t)} \qquad T = 2,$$

$$\phi(0,3) = \frac{64\sin(t)+8\sin(2t)}{65-16\cos(3t)},$$

$$\phi(1,3) = \frac{8\sin(t)+64\sin(2t)}{65-16\cos(3t)},$$

$$\phi(2,3) = \frac{64\sin(3t)}{65-16\cos(3t)} \qquad T = 3,$$

$$\phi(0,4) = \frac{256\sin(t)+16\sin(3t)}{257-32\cos(4t)},$$

$$\phi(1,4) = \frac{272\sin(2t)}{257-32\cos(4t)},$$

$$\phi(2,4) = \frac{16\sin(t)+256\sin(3t)}{257-32\cos(4t)},$$

$$\phi(3,4) = \frac{256\sin(4t)}{257-32\cos(4t)} \qquad T = 4. \tag{26}$$

Now we turn to creating a function that gives rise to a chaotic orbit under the action of $\mathscr{C}$. First, we note that for $2B$ (on l_2) the point

$$y = \left(\frac{y_1}{1}, \frac{y_2}{2}, \frac{y_3}{4}, \dots\right), \tag{27}$$

where y_i is the ith digit of a normal irrational number (whose digits are uniformly distributed) and generates a chaotic orbit. π is believed to be normal, so we take y_i to be the ith digit of π. We now lift this point to $L_2(0,\pi)$ using (3):

$$\Psi(t) = \sum_{i=1}^{\infty} \frac{y_i}{2^{i-1}} \sin nt. \tag{28}$$

Figure 3 shows the first 10 elements of the orbit of Ψ under the action of $\mathscr{C}$, that is, $\mathrm{Orb}(\mathscr{C},\Psi)$. The first element of the orbit is Ψ itself.

Figure 4 shows the orbit $\mathrm{Orb}(\mathscr{C},\Psi)$ evaluated at three different t's ($\pi/20, \pi/2, 19\pi/20$) for 200 iterations.

Engineering applications of chaotic orbits include design of fuel efficient space missions [21] and efficient mixing protocols for microfluids [22].

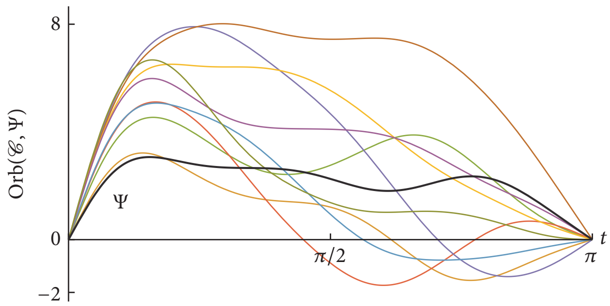

FIGURE 3: The first 10 elements of $\mathrm{Orb}(\mathscr{C}, \Psi)$.

Now we examine the effect of $\mathscr{C}$ on some commonly used periodic functions, namely, the ramp, the square-wave, and the triangle. The Fourier series of these functions are the following:

$$
\begin{aligned}
\mathrm{Ramp}(t) &= t = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kt \quad (-\pi < t < \pi), \\
\mathrm{Sqw}(t) &= \begin{cases} -\pi & -\pi < t < 0 \\ \pi & 0 < t < \pi \end{cases} \\
&= \sum_{k=1}^{\infty} \frac{2}{k} \sin^2\left(\frac{k\pi}{2}\right) \sin kt \\
&= \sum_{k=1}^{\infty} \frac{2}{2k+1} \sin(2k+1)t, \\
\mathrm{Triangle}(t) &= \begin{cases} t & 0 < t < \dfrac{\pi}{2} \\ \dfrac{\pi}{2} - t & \dfrac{\pi}{2} < t < \pi \end{cases} \\
&= \sum_{k=1}^{\infty} \frac{4}{k^2\pi} \sin\left(\frac{k\pi}{2}\right) \sin kt \\
&= \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{(2k+1)^2\pi} \sin(2k+1)t.
\end{aligned} \tag{29}
$$

Figure 5 shows the first 4 elements of the orbits of these functions. First, we note that the norm of the iterates grows (moreover, each Fourier coefficient tends to infinity); that is, these functions have unbounded orbits under the action of $\mathscr{C}$. Second, the graphs of the even iterates ($\mathscr{C}^{2n}$) of $\mathrm{Ramp}(t)$, $\mathrm{Sqw}(t)$, and $\mathrm{Triangle}(t)$ are similar to the graph of $\tan(t/2)$, $-\tan(t+\pi/2)$, and $\pm(-1)^{[t/\pi-1/2]}\tan t$, respectively. This is not too surprising, since the Fourier expansion of $\tan(t/2)$ is $2\sum_{k=1}^{\infty}(-1)^{k+1}\sin kt$ which is close in some sense to

$$
\frac{2n}{2^{2n}} \mathscr{C}^{2n} \mathrm{Ramp}(t) = 2\sum_{k=1}^{\infty} (-1)^{k+1} \left(1 - \frac{k}{2n+k}\right) \sin kt \tag{30}
$$

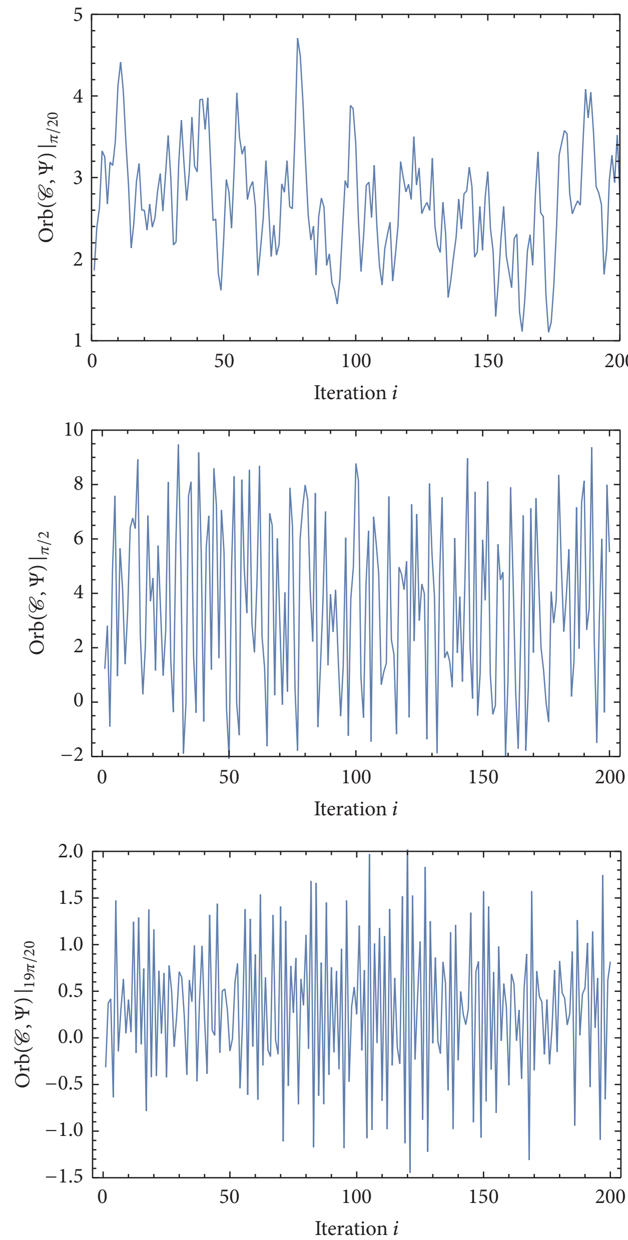

FIGURE 4: The orbit $\mathrm{Orb}(\mathscr{C}, \Psi)$ for 200 iterations evaluated at $t = \pi/20, \pi/2, 19\pi/20$.

for large enough n. Unbounded orbits of differential equations (the so-called escape orbits) play an important role in Newtonian gravitation [23].

4. Conclusions

Contrary to common belief, linear systems can display complicated dynamics. Starting from twice the backward shift on l_2 we constructed the corresponding shift operator C on $L_2(0, \pi)$ (the space of odd, 2π-periodic functions) and provided its representation using a modicum of distribution theory and Cauchy's Principal Value Integral. We explicitly calculated the periodic points of the operator (including its

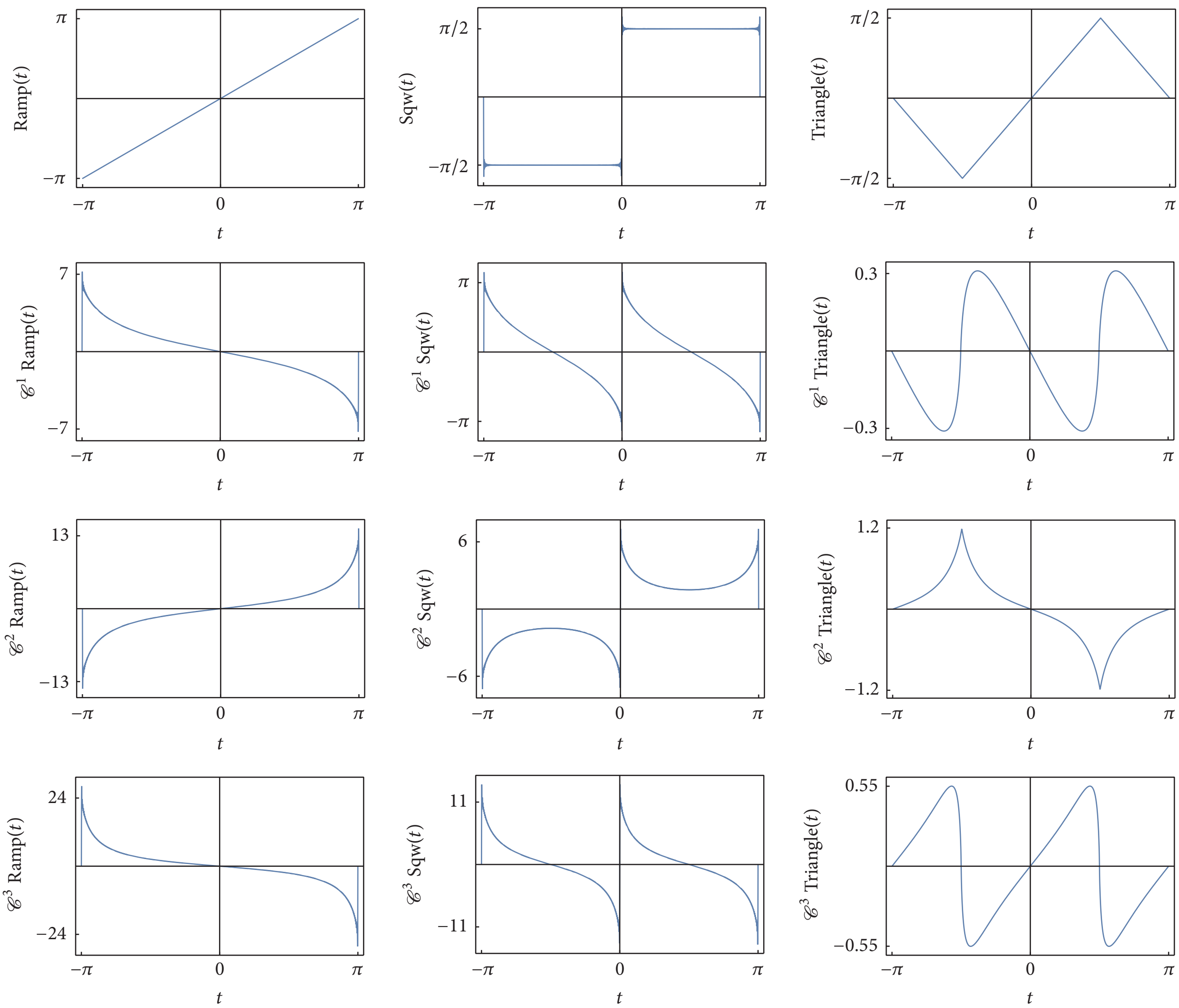

Figure 5: Iterates of the ramp, square-wave, and triangle functions.

nontrivial fixed point) and provided examples of chaotic and unbounded trajectories of $\mathscr{C}$.

We note here that utilizing representation (7) of operator $\mathscr{C}$ one can actually calculate principal values. To wit, rearranging (7) yields

$$\mathrm{PV}\int_0^{\pi}\frac{\sin t\sin\xi}{\cos t-\cos\xi}f(\xi)\,d\xi = \pi f(t)\cos t-\frac{\pi}{2}\mathscr{C}f(t). \tag{31}$$

For the simplest case, when $f(t)$ is the eigenfunction of operator $\mathscr{C}$, that is, $\mathscr{C}f(t)=\lambda f(t)$, we have (cf. (17))

$$\mathrm{PV}\int_0^{\pi}\frac{\sin t\sin\xi}{\cos t-\cos\xi}\frac{4\sin\xi}{4+\lambda^2-4\lambda\cos\xi}d\xi = \frac{2\pi\sin t\,(2\cos t-\lambda)}{4+\lambda^2-4\lambda\cos t}. \tag{32}$$

The basis functions $\phi(l,T)$ can similarly be used to obtain PV integrals. The Principal Value Integral is a tool commonly used in physics, but not in engineering-related fields. We hope that this connection between chaotic operators and Principal Value Integrals will stimulate further research.

Appendix

In this section we give a short introduction on distributional derivatives. The following definitions together with Proposition 5 can be found in standard textbooks on partial differential equations; see, for example, [24, 25].

Notation 1 (multi-index). Let Ω be an open subset of $\mathbb{R}^n$ and let $\varphi:\Omega\to\mathbb{R}$ be a smooth function. Then, for a vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, $\partial^{\alpha}\varphi$ denotes the partial derivative $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}\varphi$ of order $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$.

Definition 2 (test functions). Let $\mathscr{D}(\Omega)$ denote the space of test functions on the open set $\Omega \subset \mathbb{R}^n$, that is, $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$, endowed with the following convergence: $\varphi_n \to \varphi$ if there is a compact subset K of Ω containing the support of φ_n for all n and for every multi-index α $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly.

Definition 3 (distributions). Let $\mathscr{D}'(\Omega)$ denote the space of continuous linear functionals on $\mathscr{D}(\Omega)$. The convergence on $\mathscr{D}(\Omega)$ induces a weak or pointwise convergence on $\mathscr{D}'(\Omega)$; namely, $u_n \to u$ if $\langle u_n, \varphi\rangle \to \langle u, \varphi\rangle$ for all $\varphi \in \mathscr{D}(\Omega)$.

Note that every function h locally integrable on Ω acts as a distribution via $\langle h, \varphi\rangle = \int_\Omega h\varphi$. Then h is called a *regular distribution*.

Definition 4 (derivatives of distributions). Let $u \in \mathscr{D}'(\Omega)$. Then $\langle \partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle$.

Note the similarity with the integration by parts formula for regular distributions.

Proposition 5. *Differentiation is a continuous operation with respect to the pointwise convergence of distributions. As a consequence, derivatives of infinite series of distributions can be calculated by term-by-term differentiation.*

To aid the proof of Theorem 1 we state and prove the following.

Proposition 6. *Consider the following:*

$$\sum_{k=1}^{\infty} \cos kx = \pi \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi) - \frac{1}{2}, \quad \text{(A.1)}$$

$$-\sum_{k=1}^{\infty} \sin kx = -\frac{1}{2}\mathscr{P}\cot\frac{x}{2}, \quad \text{(A.2)}$$

where the distribution $\mathscr{P}\cot(x/2)$ *acts on a function* φ *as*

$$\left\langle \mathscr{P}\cot\frac{x}{2}, \varphi(x)\right\rangle = \sum_{k=-\infty}^{\infty} \text{PV}\int_{(2k-1)\pi}^{(2k+1)\pi} \cot\frac{x}{2}\varphi(x)\,dx, \quad \text{(A.3)}$$

where the principal values are given in the sense of (1) with $a = (2k-1)\pi$, $b = (2k+1)\pi$, *and* $c = 2k\pi$.

Proof. We start with the identities [26]

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \quad (0 < x < 2\pi), \quad \text{(A.4)}$$

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2}\ln 2(1 - \cos x) \quad (0 < x < 2\pi). \quad \text{(A.5)}$$

By periodicity, both equalities extend to $\mathbb{R}$. Term-by-term differentiation of the left hand sides of (A.4) and (A.5) (using Proposition 5) results immediately in the left hand sides of (A.1) and (A.2). Concerning the right-hand side of (A.4), let us denote the 2π-periodic extension of $(\pi - x)/2$ by $h(x)$. Then

$$\begin{aligned}\left\langle h'(x), \varphi(x)\right\rangle &= -\left\langle h(x), \varphi'(x)\right\rangle \\ &= -\int_{-\infty}^{\infty} h(x)\varphi'(x)\,dx \\ &= -\sum_{k=-\infty}^{\infty}\int_{2k\pi}^{2(k+1)\pi} h(x)\varphi'(x)\,dx.\end{aligned} \quad \text{(A.6)}$$

Note that this is actually a finite sum as the test function φ is compactly supported. One partial integration in all terms leads to

$$\begin{aligned}&-\sum_{k=-\infty}^{\infty}\int_{2k\pi}^{2(k+1)\pi} h(x)\varphi'(x)\,dx \\ &= -\sum_{k=-\infty}^{\infty}\left\{\left[h(x)\varphi(x)\right]_{2k\pi}^{2(k+1)\pi}\right. \\ &\left.-\int_{2k\pi}^{2(k+1)\pi}\left(-\frac{1}{2}\varphi(x)\right)dx\right\} = \sum_{k=-\infty}^{\infty}\pi\varphi(2k\pi) \\ &-\int_{-\infty}^{\infty}\frac{1}{2}\varphi(x)\,dx,\end{aligned} \quad \text{(A.7)}$$

which is the right-hand side of (A.1) applied to $\varphi(x)$. Similarly, let us define $h(x)$ as the 2π-periodic extension of $-(1/2)\ln 2(1 - \cos x)$ (the right-hand side of (A.5)). The ordinary derivative of $h(x)$ is $\cot(x/2)$, but, in contrast with the previous case, this is not locally integrable near the integer multiples of 2π. Cutting the singularities first and then integrating by parts,

$$\begin{aligned}&-\sum_{k=-\infty}^{\infty}\int_{2k\pi}^{2(k+1)\pi} h(x)\varphi'(x)\,dx \\ &= -\lim_{\varepsilon\to 0+}\sum_{k=-\infty}^{\infty}\int_{2k\pi+\varepsilon}^{2(k+1)\pi-\varepsilon} h(x)\varphi'(x)\,dx \\ &= \lim_{\varepsilon\to 0+}\sum_{k=-\infty}^{\infty}\int_{2k\pi+\varepsilon}^{2(k+1)\pi-\varepsilon}\cot\frac{x}{2}\varphi(x)\,dx.\end{aligned} \quad \text{(A.8)}$$

This is the same as (A.3), that is, the distribution in (A.2) applied to $\varphi(x)$. □

Competing Interests

The authors declare that they have no competing interests.

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References

[1] S. Rolewicz, "On orbits of elements," *Studia Mathematica*, vol. 32, no. 1, pp. 17–22, 1969.

[2] J. H. Shapiro, *Notes on the Dynamics of Linear Operators*, Lecture Notes, 2001, http://users.math.msu.edu/users/mshapiro/.

[3] N. S. Feldman, "Linear chaos," 2001, http://home.wlu.edu/~feldmann/Papers/LinearChaos.html.

[4] F. Bayart and É. Matheron, *Dynamics of Linear Operators*, vol. 179, Cambridge University Press, 2009.

[5] K.-G. Grosse-Erdmann and A. P. Manguillot, *Linear Chaos*, Springer Science & Business Media, 2011.

[6] D. A. Herrero, "Hypercyclic operators and chaos," *Journal of Operator Theory*, vol. 28, no. 1, pp. 93–103, 1992.

[7] K. C. Chan, "The density of hypercyclic operators on a Hilbert space," *Journal of Operator Theory*, vol. 47, no. 1, pp. 131–143, 2002.

[8] H. N. Salas, "Hypercyclic weighted shifts," *Transactions of the American Mathematical Society*, vol. 347, no. 3, pp. 993–1004, 1995.

[9] P. Bourdon and J. H. Shapiro, *Cyclic Phenomena for Composition Operators*, vol. 596, American Mathematical Society, 1997.

[10] R. M. Gethner and J. H. Shapiro, "Universal vectors for operators on spaces of holomorphic functions," *Proceedings of the American Mathematical Society*, vol. 100, no. 2, pp. 281–288, 1987.

[11] X.-C. Fu and J. Duan, "Infinite-dimensional linear dynamical systems with chaoticity," *Journal of Nonlinear Science*, vol. 9, no. 2, pp. 197–211, 1999.

[12] F. Martínez-Giménez and A. Peris, "Chaos for backward shift operators," *International Journal of Bifurcation and Chaos*, vol. 12, no. 8, pp. 1703–1715, 2002.

[13] V. Protopopescu, "Linear vs nonlinear and infinite vs finite: an interpretation of chaos," Tech. Rep. Oak Ridge, Tenn, USA, Oak Ridge National Lab, 1990.

[14] V. Protopopescu and Y. Y. Azmy, "Topological chaos for a class of linear models," *Mathematical Models and Methods in Applied Sciences*, vol. 2, no. 1, pp. 79–90, 1992.

[15] B. Hou, G. Tian, and S. Zhu, "Approximation of chaotic operators," *Journal of Operator Theory*, vol. 67, no. 2, pp. 469–493, 2012.

[16] F. Martínez-Giménez, P. Oprocha, and A. Peris, "Distributional chaos for operators with full scrambled sets," *Mathematische Zeitschrift*, vol. 274, no. 1-2, pp. 603–612, 2013.

[17] J. Bernardes, A. Bonilla, V. Müller, and A. Peris, "Li-yorke chaos in linear dynamics," *Ergodic Theory and Dynamical Systems*, vol. 35, no. 6, pp. 1723–1745, 2015.

[18] I. N. Bronshtein and K. A. Semendyayev, *Handbook of Mathematics*, Springer-Verlag, Berlin, Germany, 2013.

[19] K. T. Davies, M. L. Glasser, V. Protopopescu, and F. Tabakin, "The mathematics of principal value integrals and applications to nuclear physics, transport theory, and condensed matter physics," *Mathematical Models and Methods in Applied Sciences*, vol. 6, no. 6, pp. 833–885, 1996.

[20] S. M. Cohen, K. T. R. Davies, R. W. Davies, and M. H. Lee, "Principal-value integrals—revisited," *Canadian Journal of Physics*, vol. 83, no. 5, pp. 565–575, 2005.

[21] E. Belbruno, *Fly Me to the Moon: An Insider's Guide to the New Science of Space Travel*, Princeton University Press, Princeton, NJ, USA, 2007.

[22] C.-Y. Lee, C.-L. Chang, Y.-N. Wang, and L.-M. Fu, "Microfluidic mixing: a review," *International Journal of Molecular Sciences*, vol. 12, no. 5, pp. 3263–3287, 2011.

[23] F. G. Gascon and D. Peralta-Salas, "Escape to infinity in a Newtonian potential," *Journal of Physics A: Mathematical and General*, vol. 33, no. 30, pp. 5361–5368, 2000.

[24] V. S. Vladimirov, *Methods of the Theory of Generalized Functions*, vol. 6 of *Analytical Methods and Special Functions*, Taylor & Francis, London, UK, 2002.

[25] R. P. Kanwal, *Generalized Functions: Theory and Applications*, Springer, Boston, Mass, USA, 3rd edition, 2004.

[26] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, San Diego, Calif, USA, 2014.