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Research Article

Limit Cycles Bifurcated from Some Z_4-Equivariant Quintic Near-Hamiltonian Systems

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We study the number and distribution of limit cycles of some planar Z_4-equivariant quintic near-Hamiltonian systems. By the theories of Hopf and heteroclinic bifurcation, it is proved that the perturbed system can have 24 limit cycles with some new distributions. The configurations of limit cycles obtained in this paper are new.

1. Introduction

In 1900, Hilbert proposed 23 open mathematical problems [1]; the second part of the 16th problem concerns the maximal number and relative position of limit cycles of the planar polynomial vector fields. Even though there have been many results of obtaining more limit cycles and various configuration patterns of their relative dispositions, it has not been solved completely. To reduce the difficulty one can study the systems with some symmetry. An important symmetry is the Z_q-equivariance which was first introduced in [2]. Here we mention some newer results; for more details, see summary work [3–5]. Li et al. [6] proved a cubic Z_2-equivariant system having 13 limit cycles; Zhao [7] proved that this system has 13 limit cycles with another distribution. Li and Liu [8] proved another cubic Z_2-equivariant system also having 13 limit cycles. Zhang et al. [9] found a quartic system having at least 15 limit cycles. Christopher [10] proved that a Z_2-equivariant system has 22 limit cycles. As to the case of quintic polynomial system, there are more results. Xu and Han [11] studied a cubic Z_4-equivariant system perturbed by quintic Z_4-equivariant polynomials having 13 limit cycles. Li et al. [12] studied a quintic system and obtained at least 23 limit cycles for Z_2-equivariant case and 17 limit cycles for Z_4-equivariant case. In [13], Wu et al. studied a Z_4-equivariant system and found 20 limit cycles. Li et al. [14] found that 24 limit cycles existing in a Z_6-equivariant quintic system. Yao and Yu [15] studied a Z_5-equivariant quintic planar vector fields by normal form theory and proved that the maximal number of small limit cycles bifurcated from such vector fields is 25. Wu et al. [16] proved that a quintic Z_6-equivariant near-Hamiltonian system has 28 limit cycles. In [17], 24 limit cycles are found and two different configurations of them were shown in a Z_3-equivariant quintic planar polynomial system.

Our main result is that there can be 24 limit cycles with other distributions for the perturbed quintic Z_4-equivariant systems which are different from the known results. Using the methods of Hopf and heteroclinic bifurcation theories, the number and location of limit cycles of the following Z_4-equivariant quintic near-Hamiltonian system will be investigated:

$$\begin{aligned} \dot{x} &= H_y + \varepsilon P_5(x, y), \\ \dot{y} &= -H_x + \varepsilon Q_5(x, y), \end{aligned} \tag{1}$$

where ε is nonnegative and small and the Hamiltonian system is

$$H(x, y) = 2x^2 + 2y^2 - \frac{5}{4}x^4 - \frac{5}{4}y^4 + \frac{1}{6}x^6 + \frac{1}{6}y^6 \tag{2}$$

with phase portrait of Figure 1. $(P_5(x, y), Q_5(x, y))$ is the five-degree polynomial vector invariant under rotation of $\pi/2$ with respect to the origin O. From [2] we know that

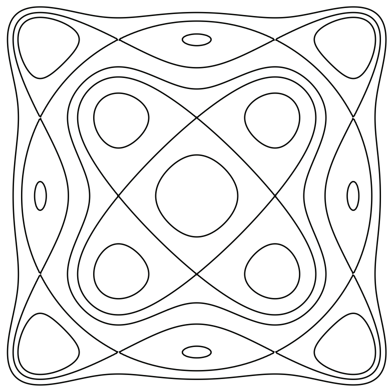


Figure 1: The phase portraits of $(1)_{(\varepsilon=0)}$.

$(P_5(x,y), Q_5(x,y))$ is, respectively, the real and imaginary parts of the following complex function:

$$F_{5,4}(z,\overline{z}) = \left(A_0 + A_1|z|^2 + A_3|z|^4\right)z + \left(A_3 + A_4|z|^2\right)\overline{z}^3 + A_5 z^5, \quad (3)$$

where $A_k = a_k + ib_k$, $k = 0, 1, 2, \ldots, 5$, $z = x + iy$, and $z = x - iy$. It is direct that

$$\begin{aligned} P_5 &= a_0 x + a_1\left(x^3 + xy^2\right) + a_2\left(x^5 + 2x^3y^2 + xy^4\right) \\ &+ a_3\left(x^3 - 3xy^2\right) + a_4\left(x^5 - 2x^3y^2 - 3xy^4\right) \\ &+ a_5\left(x^5 - 10x^3y^2 + 5xy^4\right) + b_0(-y) \\ &+ b_1\left(-x^2y - y^3\right) + b_2\left(-x^4y - 2x^2y^3 - y^5\right) \\ &+ b_3\left(3x^2y - y^3\right) + b_4\left(3x^4y + 2x^2y^3 - y^5\right) \\ &+ b_5\left(10x^2y^3 - 5x^4y - y^5\right), \\ Q_5 &= a_0 y + a_1\left(x^2y + y^3\right) + a_2\left(x^4y + 2x^2y^3 + y^5\right) \\ &+ a_3\left(y^3 - 3x^2y\right) + a_4\left(y^5 - 3x^4y - 2x^2y^3\right) \\ &+ a_5\left(5x^4y - 10x^2y^3 + y^5\right) + b_0 x \\ &+ b_1\left(x^3 + xy^2\right) + b_2\left(x^5 + 2x^3y^2 + xy^4\right) \\ &+ b_3\left(x^3 - 3xy^2\right) + b_4\left(x^5 - 2x^3y^2 - 3xy^4\right) \\ &+ b_5\left(x^5 - 10x^3y^2 + 5xy^4\right). \end{aligned} \quad (4)$$

Our result is the following.

Theorem 1. *There exist some $(a_0, a_1, a_2, a_3, a_4, a_5)$ such that system (1) can have 24 limit cycles with two different distributions, the distributions of these limit cycles are shown in Figure 2.*

The rest of this paper is organized as follows. Some useful preliminary theorems will be listed in Section 2. In Section 3, some related coefficients of asymptotic expansions are firstly computed; then using this coefficients and preliminary lemmas we prove the main result.

2. Preliminary Lemmas

Let $H(x,y)$, $p(x,y,\delta)$, and $q(x,y,\delta)$ be analytic functions, ε positive and small, and $\delta \in D \subset R^m$ with D compact; then the following system is a planar Hamiltonian system:

$$\dot{x} = H_y, \qquad \dot{y} = -H_x, \quad (5)$$

and the below system is usually called near-Hamiltonian system:

$$\begin{aligned} \dot{x} &= H_y(x,y) + \varepsilon p(x,y,\delta), \\ \dot{y} &= -H_x(x,y) + \varepsilon q(x,y,\delta). \end{aligned} \quad (6)$$

Let system (5) have at least one family of periodic orbits L_h defined by $H(x,y) = h$ which form a periodic annulus $\{L_h\}$; then the first-order approximation of the Poincaré map of system (6) is

$$M(h,\delta) = \oint_{L_h} (q\,dx - p\,dy) \quad (7)$$

which is called the Melnikov function or Abel integral. By the Poincaré-Pontryagin-Andronov theorem, an isolated zero of $M(h,\delta)$ corresponds a limit cycle of system (6). A popular method to find limit cycles of (6) is to find zeros of $M(h,\delta)$ and an efficient method to find zeros of $M(h,\delta)$ is to investigate the asymptotic expansion of $M(h,\delta)$ near the boundaries of $\{L_h\}$; see [18].

Let the outer boundary of $\{L_h\}$ be a homoclinic loop L_β defined by $H(x,y) = \beta$ passing through a hyperbolic saddle at the origin; we have the following.

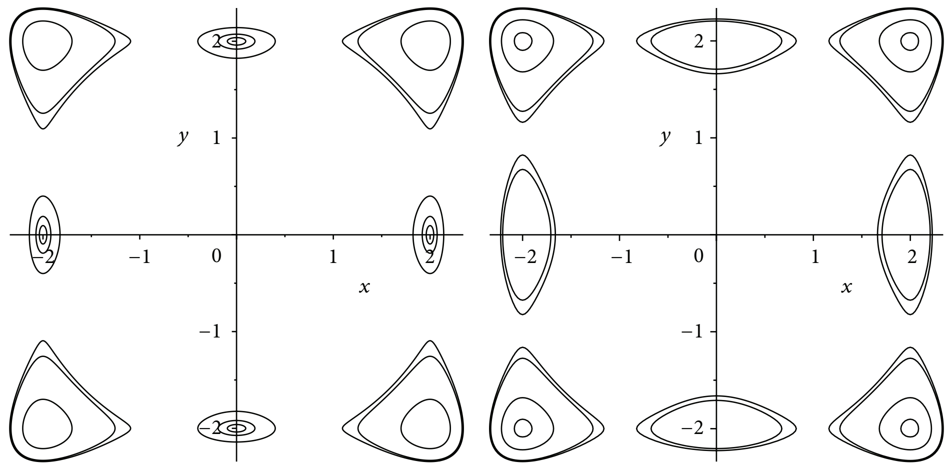
Lemma 2 (see [19]). *(i) The function $M(h,\delta)$ has the following expansion:*

$$\begin{aligned} M(h,\delta) &= c_0(\delta) + c_1(\delta)(h-\beta)\ln|h-\beta| \\ &+ c_2(\delta)(h-\beta) + c_3(\delta)(h-\beta)^2\ln|h-\beta| \\ &+ O\left(|h-\beta|^2\right), \end{aligned} \quad (8)$$

for $0 < \beta - h \ll 1$; $c_i(\delta)$ depends on the parameters of H, p, and q.

(ii) Further suppose that, for (x,y) near $(0,0)$,

$$\begin{aligned} H(x,y) &= \beta + \frac{\lambda}{2}\left(y^2 - x^2\right) + \sum_{i+j\geq 3} h_{ij}x^i y^j, \quad \lambda \neq 0, \\ p(x,y,\delta) &= \sum_{i+j\geq 0} a_{ij}x^i y^j, \\ q(x,y,\delta) &= \sum_{i+j\geq 0} b_{ij}x^i y^j. \end{aligned} \quad (9)$$



(a) 24 limit cycles

(b) 24 limit cycles

FIGURE 2: Two different distributions of 24 limit cycles of system (1).

Then,

$$\begin{aligned} c_0(\delta) &= \oint_{L_\beta} q\,dx - p\,dy, \\ c_1(\delta) &= -\frac{1}{|\lambda|}(a_{10}+b_{01}), \\ c_2(\delta) &= \oint_{L_\beta} (p_x+q_y)\,dt \quad \text{if } c_1(\delta)=0, \\ c_3(\delta) &= \frac{-1}{2|\lambda|\lambda}\{(-3a_{30}-b_{21}+a_{12}+3b_{03}) \\ &\quad - \frac{1}{\lambda}[(2b_{02}+a_{11})(3h_{03}-h_{21}) \\ &\quad + (2a_{20}+b_{11})(3h_{30}-h_{12})]\}, \\ &\quad \text{if } c_1(\delta)=0. \end{aligned} \tag{10}$$

The values $c_1(\delta)$ and $c_3(\delta)$ are, respectively, called the first and second local Melnikov coefficients at the saddle O, denoted by $c_1(O,\delta)$ and $c_3(O,\delta)$, respectively.

Now let the outer boundary of $\{L_h\}$ be an 2-polycycle Γ^2:

$$\Gamma^2 = \bigcup_{i=1}^{2}(L_i \cup S_i) \tag{11}$$

with 2 hyperbolic saddles, S_1 and S_2, and 2 heteroclinic orbits, L_1 and L_2, connecting them, defined by $H(x,y)=\beta$. The following lemma was proved in [19].

Lemma 3 (see [19]). *Under the above assumptions, $M(h,\delta)$ has the form, for $0<\beta-h\ll 1$,*

$$\begin{aligned} M(h,\delta) &= \sum_{j\geq 0}\left[c_{2j}(\delta)+c_{2j+1}(\delta)(h-\beta)\ln|h-\beta|\right] \\ &\quad \times (h-\beta)^j, \end{aligned} \tag{12}$$

where

$$\begin{aligned} c_0(\delta) &= \sum_{i=1}^{2}\int_{L_i} q\,dx - p\,dy, \\ c_1(\delta) &= \sum_{i=1}^{2} c_1(S_i,\delta), \\ c_3(\delta) &= \sum_{i=1}^{2} c_3(S_i,\delta), \end{aligned} \tag{13}$$

where $c_1(S_i,\delta)$ and $c_3(S_i,\delta)$ are, respectively, the first and the second local Melnikov coefficient at the saddle S_i, $i=1,2$. In particular,

$$c_2(\delta) = \oint_{\Gamma^2}(p_x+q_y)\,dt = \sum_{i=1}^{2}\int_{L_i}(p_x+q_y)\,dt, \tag{14}$$

if $c_1(S_i,\delta)=0$, $i=1,2$.

When the inner boundary of $\{L_h\}$ is a elementary center (x_c,y_c) defined by $H(x_c,y_c)=\alpha$, the following lemma gives the asymptotic expansion of $M(h,\delta)$.

Lemma 4 (see [20]). *$M(h,\delta)$ has the form, for $0<h-\alpha\ll 1$,*

$$M(h,\delta) = \sum_{k\geq 0} B_k(h-\alpha)^{k+1}. \tag{15}$$

If for (x,y) near (x_c,y_c),

$$H(x,y) = \alpha + \frac{1}{2}\left((x-x_c)^2+(y-y_c)^2\right)$$

$$+\sum_{i+j\geq 3} h_{ij}(x-x_c)^i(y-y_c)^j,$$

$$p(x,y,\delta)=\sum_{i+j\geq 1} a_{ij}(x-x_c)^i(y-y_c)^j, \tag{16}$$

$$q(x,y,\delta)=\sum_{i+j\geq 1} b_{ij}(x-x_c)^i(y-y_c)^j,$$

the coefficients B_i can be obtained by the formulas in [20].

Remark 5. When the inner boundary is a nilpotent center, a new method of limit cycles bifurcated from the annulus near the center can be found in [21]. Lemma 3 has been developed in [22, 23].

In many cases the Hamiltonian function is not of the form presented in the above lemmas. Then to apply the lemmas we need first to introduce suitable linear change of variables which will cause a change in the first-order Melnikov function. The following lemma gives the relationship between the old and new Melnikov functions.

Lemma 6 (see [24]). *(i) Under the linear change of variables of the form:*

$$\begin{aligned} u&=a(x-x_0)+b(y-y_0),\\ v&=c(x-x_0)+d(y-y_0) \end{aligned} \tag{17}$$

and time rescaling $\tau=kt$, where $D=ad-bc\neq 0$, the system (6) *becomes*

$$\frac{du}{d\tau}=\widetilde{H}_v+\varepsilon\widetilde{p}, \qquad \frac{dv}{d\tau}=-\widetilde{H}_u+\varepsilon\widetilde{q}, \tag{18}$$

where $\widetilde{H}(u,v)=(D/k)H(x,y)$, $\widetilde{p}(u,v,\delta)=(1/k)[ap(x,y,\delta)+bq(x,y,\delta)]$, and $\widetilde{q}(u,v,\delta)=(1/k)[cp(x,y,\delta)+dq(x,y,\delta)]$.

(ii) Let

$$\widetilde{M}(h,\delta)=\oint_{L_h}\widetilde{q}\,du-\widetilde{p}\,dv \tag{19}$$

which is the Melnikov function of the system (18)*; then*

$$M(h,\delta)=\frac{|k|}{D}\widetilde{M}\left(\frac{D}{k}h,\delta\right). \tag{20}$$

When systems (5) and (6) are Z_4-equivariant, (5) has a compound cycled denoted by Γ^4, which consists of 8 hyperbolic saddles $S_1,\ldots,S_8$ and 16 heteroclinic orbits L_{12}, L_{21}, L_{23}, L_{32}, L_{34}, L_{43}, L_{45}, L_{54}, L_{56}, L_{65}, L_{67}, L_{76}, L_{78}, L_{87}, L_{81}, and L_{18} satisfying $\alpha(L_{ij})=S_i,\omega(L_{ij})=S_j$. Γ^4 contains 8 two-polycycles L_i $(i=1,\ldots,8)$, where $L_1=L_{12}\cup L_{21}$, $L_2=L_{23}\cup L_{32}$, $L_3=L_{34}\cup L_{43}$, $L_4=L_{45}\cup L_{54}$, $L_5=L_{56}\cup L_{65}$, $L_6=L_{67}\cup L_{76}$, $L_7=L_{78}\cup L_{87}$, and $L_8=L_{81}\cup L_{18}$. See Figure 3. We suppose that Γ^4 is defined by $H(x,y)=H(S_i)=\beta$, $i=1,\ldots,8$. There are 8 centers $C_i(x_i,y_i)$ inside the 2-polycycle L_1, with $H(C_1)=H(C_3)=H(C_5)=H(C_7)=\alpha_1$ and $H(C_2)=H(C_4)=H(C_6)=H(C_8)=\alpha_2$. There are 4 families of periodic orbits L_i^h inside the 2-polycycle L_i, defined by $H(x,y)=h$ for $h\in(\alpha_1,\beta)$, $i=1,3,5,7$, and 4 families of periodic orbits L_j^h inside the 2-polycycle L_j, defined by $H(x,y)=h$ for $h\in(\alpha_2,\beta)$, $j=2,4,6,8$. Then we have 8 Melnikov functions below:

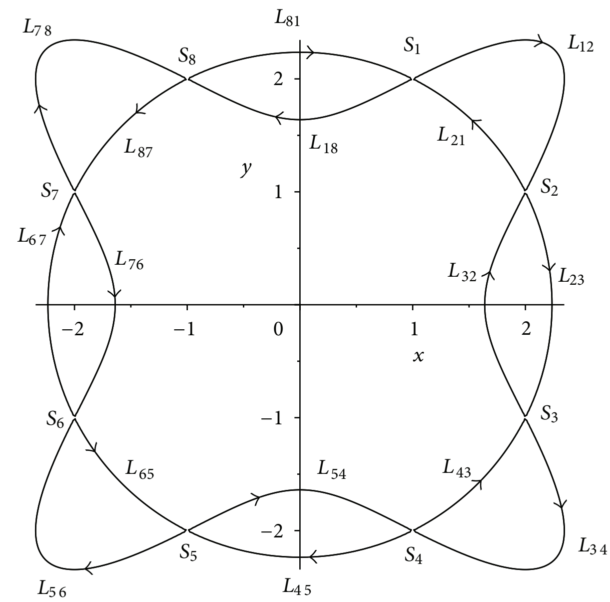


Figure 3

$$\begin{aligned} &M_i(h,\delta)=\oint_{L_i^h}(q\,dx-p\,dy)\big|_{\varepsilon=0},\\ &\quad\text{for } h\in(\alpha_1,\beta)\ \ i=1,3,5,7.\\ &M_j(h,\delta)=\oint_{L_i^h}(q\,dx-p\,dy)\big|_{\varepsilon=0},\\ &\quad\text{for } h\in(\alpha_2,\beta)\ \ j=2,4,6,8. \end{aligned} \tag{21}$$

By Z_4-equivariance, $M_1(h,\delta)=M_3(h,\delta)=M_5(h,\delta)=M_7(h,\delta)$ and $M_2(h,\delta)=M_4(h,\delta)=M_6(h,\delta)=M_8(h,\delta)$, we can only study $M_1(h,\delta)$ and $M_8(h,\delta)$. For convenience, the notations are introduced as follows:

$$\begin{aligned} c_{01}(\delta)&=\int_{L_{12}}q\,dx-p\,dy,\\ c_{02}(\delta)&=\int_{L_{21}}q\,dx-p\,dy,\\ e_{01}(\delta)&=\int_{L_{81}}q\,dx-p\,dy,\\ e_{02}(\delta)&=\int_{L_{18}}q\,dx-p\,dy, \end{aligned} \tag{22}$$

and $d_i(\delta) = (p_x + q_y)(S_i, \delta)$, $1 \le i \le 8$, where $d_1 = d_3 = d_5 = d_7$ and $d_2 = d_4 = d_6 = d_8$. Letting $d_1 = d_2 = 0$, we introduce

$$\begin{aligned} c_{21}(\delta) &= \int_{L_{12}} \left(p_x + q_y\right) dt, \\ c_{22}(\delta) &= \int_{L_{21}} \left(p_x + q_y\right) dt, \\ e_{21}(\delta) &= \int_{L_{81}} \left(p_x + q_y\right) dt, \\ e_{22}(\delta) &= \int_{L_{18}} \left(p_x + q_y\right) dt. \end{aligned} \tag{23}$$

The following is directly from Lemma 3.

Lemma 7. *Under the above assumptions, we have the following expansions:*

$$\begin{aligned} M_i(h,\delta) = {} & c_0(\delta) + c_1(\delta)(h-\beta)\ln|h-\beta| \\ & + c_2(\delta)(h-\beta) \\ & + c_3(\delta)(h-\beta)^2\ln|h-\beta| + \cdots, \end{aligned} \tag{24}$$

for $0 < h - \beta \ll 1$, $i = 1, 3, 5, 7$,

$$\begin{aligned} M_j(h,\delta) = {} & e_0(\delta) + e_1(\delta)(h-\beta)\ln|h-\beta| \\ & + e_2(\delta)(h-\beta) \\ & + e_3(\delta)(h-\beta)^2\ln|h-\beta| + \cdots, \end{aligned} \tag{25}$$

for $0 < h - \beta \ll 1$, $j = 2, 4, 6, 8$, *where*

$$\begin{aligned} c_0(\delta) &= c_{01}(\delta) + c_{02}(\delta), \\ e_0(\delta) &= e_{01}(\delta) + e_{02}(\delta), \\ c_1(\delta) &= c_1(S_1,\delta) + c_1(S_2,\delta) = -\frac{d_1(\delta)}{|\lambda_1|} - \frac{d_2(\delta)}{|\lambda_2|}, \\ e_1(\delta) &= c_1(S_1,\delta) + c_1(S_8,\delta) \\ &= c_1(S_1,\delta) + c_1(S_2,\delta) = c_1(\delta), \\ c_3(\delta) &= c_3(S_1,\delta) + c_3(S_2,\delta), \\ e_3(\delta) &= c_3(S_1,\delta) + c_3(S_8,\delta) \\ &= c_3(S_1,\delta) + c_3(S_2,\delta) = c_3(\delta), \\ c_2(\delta) &= c_{21}(\delta) + c_{22}(\delta), \\ e_2(\delta) &= e_{21}(\delta) + e_{22}(\delta) \end{aligned} \tag{26}$$

if $d_1(\delta) = d_2(\delta) = 0$. λ_i *denotes an eigenvalue of* S_i *for* (6) *and* $c_1(S_i,\delta)$ *and* $c_3(S_i,\delta)$ *are the first and the second Melnikov coefficients at the saddle* S_i $(i = 1, 2)$ *as defined after Lemma 2.*

By Lemma 4, for the expansions of $M_i(h,\delta)$ near the center, we have

$$\begin{aligned} M_1(h,\delta) &= \sum_{k\ge 0} B_k(\delta)(h-\alpha_1)^{k+1} \\ &\qquad \text{for } 0 < h - \alpha_1 \ll 1, \\ M_8(h,\delta) &= \sum_{k\ge 0} d_k(\delta)(h-\alpha_2)^{k+1} \\ &\qquad \text{for } 0 < h - \alpha_2 \ll 1. \end{aligned} \tag{27}$$

To obtain more limit cycles, we have the following.

Theorem 8. *Let* (24), (25), *and* (27) *hold.*

(1) Suppose that there exists $\delta_0 \in D$ *such that*

$$\begin{gathered} c_0(\delta_0) = c_1(\delta_0) = d_0(\delta_0) = d_1(\delta_0) = 0, \\ c_2(\delta_0) \ne 0, \qquad d_2(\delta_0) \ne 0, \\ \operatorname{rank} \frac{\partial(c_0, c_1, d_0, d_1)}{\partial(\delta_1, \delta_2, \ldots, \delta_m)} = 4. \end{gathered} \tag{28}$$

Then there exist some (ε, δ) *near* $(0, \delta_0)$ *such that* (5) *has*

$$\begin{aligned} 4 &+ \frac{1 - \operatorname{sgn}(M_1(h_1,\delta_0) M_1(h_2,\delta_0))}{2} \\ &+ \frac{1 - \operatorname{sgn}(M_8(h_3,\delta_0) M_8(h_4,\delta_0))}{2} \end{aligned} \tag{29}$$

limit cycles in the 2-polycycles L_1 *and* L_8, *where* $h_1 = \beta - \varepsilon_0$, $h_2 = \alpha_1 + \varepsilon_0$, $h_3 = \beta - \varepsilon_0$, *and* $h_4 = \alpha_2 + \varepsilon_0$ *with* ε_0 *being positive and very small, and the location of these limit cycles is as follows: 2 limit cycles near the 2-polycycle* L_1, *2 limit cycles near the center* C_8, $(1 - \operatorname{sgn}(M_1(h_1,\delta_0)M_1(h_2,\delta_0)))/2$ *limit cycles between the center* C_1 *and the polycycle* L_1, *and* $(1 - \operatorname{sgn}(M_8(h_3,\delta_0)M_8(h_4,\delta_0)))/2$ *limit cycles between the center* C_8 *and the polycycle* L_8.

(2) Suppose that there exists $\delta_0 \in D$ *such that*

$$\begin{gathered} c_0(\delta_0) = c_1(\delta_0) = B_0(\delta_0) = e_0(\delta_0) = 0, \\ c_2(\delta_0) \ne 0, \qquad e_2(\delta_0) \ne 0, \qquad B_1(\delta_0) \ne 0, \\ \operatorname{rank} \left.\frac{\partial(c_0, c_1, B_0, e_0)}{\partial(\delta_1, \delta_2, \ldots, \delta_m)}\right|_{\delta=\delta_0} = 4. \end{gathered} \tag{30}$$

Then there exist some (ε, δ) *near* $(0, \delta_0)$ *such that* (5) *has*

$$\begin{aligned} 4 &+ \frac{1 - \operatorname{sgn}(M_1(h_1,\delta_0) M_1(h_2,\delta_0))}{2} \\ &+ \frac{1 - \operatorname{sgn}(M_8(h_3,\delta_0) M_8(h_4,\delta_0))}{2} \\ &+ \frac{1 + \operatorname{sgn}(c_2(\delta_0) e_2(\delta_0))}{2} \end{aligned} \tag{31}$$

limit cycles in the 2-polycycles L_1 *and* L_8, *where* $h_1 = \beta - \varepsilon_0$, $h_2 = \alpha_1 + \varepsilon_0$, $h_3 = \beta - \varepsilon_0$, *and* $h_4 = \alpha_2 + \varepsilon_0$ *with* ε_0 *being positive* *and very small, and the location of these limit cycles is the following:* 2 *limit cycles near the* 2*-polycycle* L_1, 1 *limit cycles near the center* C_1, $1+((1+\operatorname{sgn}(c_2(\delta_0)e_2(\delta_0)))/2)$ *limit cycles near the* 2*-polycycle* L_8, $(1-\operatorname{sgn}(M_1(h_1,\delta_0)M_1(h_2,\delta_0)))/2$ *limit cycles between the center* C_1, *and the polycycle* L_1 *and* $(1-\operatorname{sgn}(M_8(h_3,\delta_0)M_8(h_4,\delta_0)))/2$ *limit cycles between the center* C_8 *and the polycycle* L_8.

Proof. Because of the similarity, we only prove the last conclusion. For $\delta = \delta_0$, by continuity, there exist $(1-\operatorname{sgn}(M_1(h_1,\delta_0)M_1(h_2,\delta_0)))/2$ zeros of $M_1(h,\delta_0)$ between h_1 and h_2 and $(1-\operatorname{sgn}(M_8(h_3,\delta_0)M_8(h_4,\delta_0)))/2$ zeros of $M_8(h,\delta_0)$ between h_3 and h_4. Thus, for all δ near δ_0 or $\delta \in U_0 = \{\delta \mid |\delta| < \varepsilon^*,\ \varepsilon^* \text{ is very small}\}$ there exist $(1-\operatorname{sgn}(M_1(h_1,\delta_0)M_1(h_2,\delta_0)))/2$ zeros of $M_1(h,\delta)$ between h_1 and h_2 and $(1-\operatorname{sgn}(M_8(h_3,\delta_0)M_8(h_4,\delta_0)))/2$ zeros of $M_8(h,\delta)$ between h_3 and h_4.

According to the condition, we can take c_0, c_1, e_0, and B_0 as free parameters. Hence, we first take c_1 satisfying

$$|c_1| \ll |c_2|, \qquad |c_1| \ll |e_2|, \qquad c_1c_2 > 0, \tag{32}$$

so that there is a zero of $M_1(h,\delta)$ denoted by $\widetilde{h}_1$ near β satisfying $h_1 < \widetilde{h}_1 < \beta$. On the other hand, if $c_2e_2 > 0$, considering $c_1 = e_1$, we have $|e_1| \ll |e_2|$ and $e_1e_2 > 0$ which implies that there is a zero of $M_8(h,\delta)$ denoted by $\overline{h}_1$ near β satisfying $h_3 < \overline{h}_1 < \beta$. If $c_2e_2 < 0$, we are not sure if $M_8(h,\delta)$ has a zero. Thus, so far we obtain 1 zero of $M_1(h,\delta)$ and $(1+\operatorname{sgn}(c_2(\delta_0)e_2(\delta_0)))/2$ zeros of $M_8(h,\delta)$ for $\delta \in U_1 = \{\delta \mid |c_1| \ll |c_2|,\ |c_1| \ll |e_2|, c_1c_2 > 0\}$.

Next, we take c_0, e_0, and B_0 satisfying

$$\begin{aligned} &|c_0| \ll |c_1|, \qquad c_0c_1 < 0, \qquad |e_0| \ll |e_1|, \\ &e_0e_1 < 0, \qquad |B_0| \ll |B_1|, \qquad B_0B_1 < 0. \end{aligned} \tag{33}$$

Then $M_1(h,\delta)$ has two new zeros $\widetilde{h}_2$ near β and $\widetilde{h}_3$ near α_2 satisfying $h_1 < \widetilde{h}_1 < \widetilde{h}_2 < \beta$ and $\alpha_2 < \widetilde{h}_3 < h_4$, and $M_8(h,\delta)$ has a new zero $\overline{h}_2$ near β satisfying $h_3 < \overline{h}_2 < \beta$. In this step, we get 2 more zeros of $M_1(h,\delta)$ and 1 more zero of $M_1(h,\delta)$ for $\in U_2 = \{\delta \mid |c_0| \ll |c_1|,\ c_0c_1 < 0,\ |e_0| \ll |e_1|,\ e_0e_1 < 0,\ |B_0| \ll |B_1|, \text{and } B_0B_1 < 0\}$. Then totally we have $4 + (1-\operatorname{sgn}(M_1(h_1,\delta_0)M_1(h_2,\delta_0)))/2 + (1-\operatorname{sgn}(M_8(h_3,\delta_0)M_8(h_4,\delta_0)))/2 + (1+\operatorname{sgn}(c_2(\delta_0)e_2(\delta_0)))/2$ zeros for $\delta \in U_0 \cap U_1 \cap U_2$. Therefore, there are $4 + (1-\operatorname{sgn}(M_1(h_1,\delta_0)M_1(h_2,\delta_0)))/2 + (1-\operatorname{sgn}(M_8(h_3,\delta_0)M_8(h_4,\delta_0)))/2 + (1+\operatorname{sgn}(c_2(\delta_0)e_2(\delta_0)))/2$ limit cycles for some δ near δ_0. This completes the proof. □

Remark 9. The signs of $M_1(h_1,\delta_0)$, $M_1(h_2,\delta_0)$, $M_8(h_3,\delta_0)$, and $M_8(h_4,\delta_0)$ can be determined by using the first nonzero coefficients in their expansions.

3. Main Result

In this section, we investigate the distributions of limit cycles of system (1). For $\varepsilon = 0$, (1) has two level sets, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, defined by $H(x,y) = -5/12$ and $H(x,y) = -5/12$ respectively. $\widetilde{\Gamma}_1$ consists of 8 saddles, $S_1 = (1,2)$, $S_2 = (2,1)$, $S_3 = (2,-1)$, $S_4 = (1,-2)$, $S_5 = (-1,-2)$, $S_6 = (-2,-1)$, $S_7 = (-2,1)$, and $S_8 = (-1,2)$, as shown in Figure 4. Let L_{ij}, L_i, L_j^h, c_{01}, c_{02}, e_{01}, e_{02}, c_{21}, c_{22}, e_{21}, e_{22}, and so forth are the same as those for Lemma 7. We can write the compound cycles $\widetilde{\Gamma}_1$ as

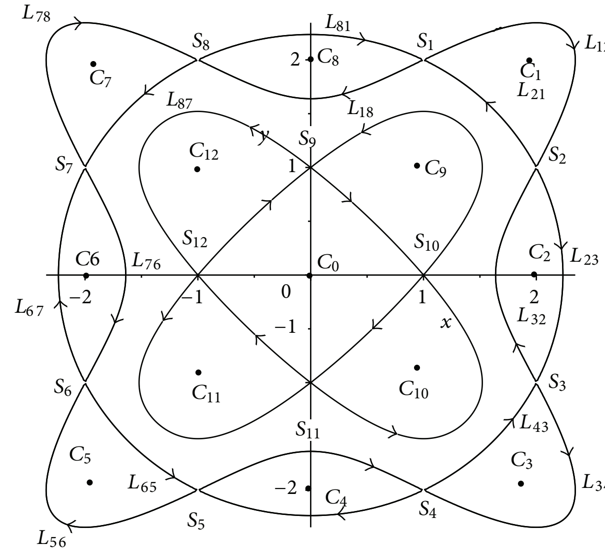


Figure 4

$$\begin{aligned} \widetilde{\Gamma}_1 &= \bigcup_{i=1}^{8}\{S_i\} \\ &\quad \cup \{L_{12}, L_{21}, L_{23}, L_{32}, \ldots, L_{78}, L_{87}, L_{81}, L_{18}\}. \end{aligned} \tag{34}$$

The equation $H(x,y) = -8/3$ defines four centers, $C_1 = (2,2)$, $C_3 = (2,-2)$, $C_5 = (-2,-2)$, and $C_7 = (-2,2)$, where the equation $H(x,y) = -4/3$ defines four centers of $C_2 = (2,0)$, $C_4 = (0,-2)$, $C_6 = (-2,0)$, and $C_8 = (0,2)$. The center C_i is inside the 2-polycycle L_i for $i = 1,\ldots,8$.

We first investigate the 2-polycycles L_1 and L_8. Here, L_1^h denotes the periodic orbit defined by $H(x,y) = h$ surrounding the unique center C_1 and L_8^h denotes the periodic orbit defined by $H(x,y) = h$ surrounding the unique center C_8. Then

$$\begin{aligned} M_1(h,\delta) &= \oint_{L_1^h} Q_5\,dx - P_5\,dy \\ &= \oint_{L_1^h}\left(Q_5 - P_5\frac{dy}{dx}\right)dx, \quad h \in \left(-\frac{8}{3}, -\frac{5}{12}\right), \\ M_8(h,\delta) &= \oint_{L_8^h} Q_5\,dx - P_5\,dy \\ &= \oint_{L_8^h}\left(Q_5 - P_5\frac{dy}{dx}\right)dx, \quad h \in \left(-\frac{4}{3}, -\frac{5}{12}\right), \end{aligned} \tag{35}$$

where $\delta = (a_0, a_1, \ldots, a_5) \in R^6$. By Lemma 7, we have

$$M_1(h,\delta) = c_0(\delta) + c_1(\delta)\left(h+\frac{5}{12}\right)\ln\left|h+\frac{5}{12}\right| + c_2(\delta)\left(h+\frac{5}{12}\right) + c_3(\delta)\left(h+\frac{5}{12}\right)^2\ln\left|h+\frac{5}{12}\right| + \cdots, \tag{36}$$

for $0 < -(h+5/12) \ll 1$,

$$M_8(h,\delta) = e_0(\delta) + e_1(\delta)\left(h+\frac{5}{12}\right)\ln\left|h+\frac{5}{12}\right| + e_2(\delta)\left(h+\frac{5}{12}\right) + e_3(\delta)\left(h+\frac{5}{12}\right)^2\ln\left|h+\frac{5}{12}\right| + \cdots, \tag{37}$$

for $0 < -(h+5/12) \ll 1$.

By Lemma 4, for $0 < h + 8/3 \ll 1$, $i = 1,3,5,7$,

$$M_i(h,\delta) = \sum_{k\ge 0} B_k(\delta)\left(h+\frac{8}{3}\right)^{k+1}, \tag{38}$$

and, for $0 < h + 4/3 \ll 1$, $j = 2,4,6,8$,

$$M_j(h,\delta) = \sum_{k\ge 0} d_k(\delta)\left(h+\frac{4}{3}\right)^{k+1}. \tag{39}$$

To find the zeros of $M_1(h,\delta)$ and $M_8(h,\delta)$, the coefficients in these asymptotic expansions need to be calculated. In order to calculate the coefficients $c_{01}(\delta)$, $c_{02}(\delta)$, $e_{01}(\delta)$, $e_{02}(\delta)$, and so forth, the expressions of heteroclinic orbits are found as follows:

$$\begin{aligned}
L_{12}:\ & y_1(x) = \frac{1}{2}\sqrt{2x^2+5+\sqrt{-12x^4+60x^2+33}}, \quad 1 \le x \le \frac{1}{2}\sqrt{22},\\
& y_2(x) = \frac{1}{2}\sqrt{2x^2+5-\sqrt{-12x^4+60x^2+33}}, \quad \frac{1}{2}\sqrt{22} \le x \le 2,\\
L_{21}:\ & y_3(x) = \sqrt{5-x^2}, \quad 2 \ge x \ge 1,\\
L_{81}:\ & y_4(x) = \sqrt{5-x^2}, \quad -1 \le x \le 1,\\
L_{18}:\ & y_5(x) = \frac{1}{2}\sqrt{2x^2+5+\sqrt{-12x^4+60x^2+33}}, \quad 1 \ge x \ge -1.
\end{aligned} \tag{40}$$

Figure 5 may be helpful to understand the step to calculate the coefficients in the folowing. By (4), $P_5(x,y)$ and $P_5(x,y)$ are written as follows:

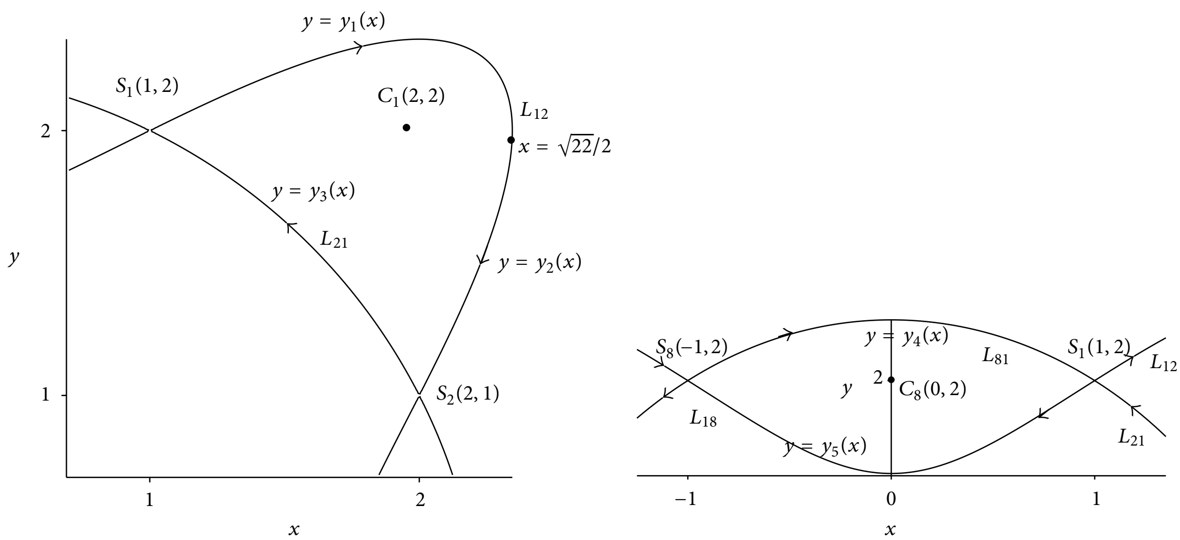
$$P_5(x,y) = \sum_{i=0}^{5} p_{1i}a_i + p_{2i}b_i, \qquad Q_5(x,y) = \sum_{i=0}^{5} q_{1i}a_i + q_{2i}b_i, \tag{41}$$

and introduce the following notations:

$$p_{1i}^j = p_{1i}\big|_{y=y_j(x)}, \quad p_{2i}^j = p_{2i}\big|_{y=y_j(x)}, \quad q_{1i}^j = q_{1i}\big|_{y=y_j(x)}, \quad q_{2i}^j = q_{2i}\big|_{y=y_j(x)}, \quad i,j = 1,2,3,4,5, \tag{42}$$

and $\tilde{f}_j = (dy/dx)|_{y=y_i(x)} = ((-2x+3x^3-x^5)/(2y-3y^3+y^5))|_{y=y_i(x)}$. By Lemma 7, we have

$$\begin{aligned}
c_{01}(\delta) &= \int_{L_{12}} Q_5 dx - P_5 dy\\
&= \int_1^{\sqrt{22}/2}\left(Q_5 - P_5\frac{dy}{dx}\right)\Big|_{y=y_1(x)} dx + \int_{\sqrt{22}/2}^{2}\left(Q_5 - P_5\frac{dy}{dx}\right)\Big|_{y=y_2(x)} dx\\
&= \sum_{i=0} a_i I_{1i}^1 + b_i I_{2i}^1 + a_i I_{1i}^2 + b_i I_{2i}^2,\\
c_{02}(\delta) &= \int_{L_{21}} Q_5 dx - P_5 dy\\
&= \int_2^1 \left(Q_5 - P_5\frac{dy}{dx}\right)\Big|_{y=y_3(x)} dx\\
&= \sum_{i=0} a_i I_{1i}^3 + b_i I_{2i}^3,\\
e_{01}(\delta) &= \int_{L_{81}} Q_5 dx - P_5 dy\\
&= \int_{-1}^{1}\left(Q_5 - P_5\frac{dy}{dx}\right)\Big|_{y=y_4(x)} dx\\
&= \sum_{i=0} a_i I_{1i}^4 + b_i I_{2i}^4,\\
e_{02}(\delta) &= \int_{L_{18}} Q_5 dx - P_5 dy\\
&= \int_1^{-1}\left(Q_5 - P_5\frac{dy}{dx}\right)\Big|_{y=y_5(x)} dx\\
&= \sum_{i=0} a_i I_{1i}^4 + b_i I_{2i}^4,
\end{aligned} \tag{43}$$



(a) The 2-polycycle L_1 (b) The 2-polycycle L_8

Figure 5

where, for $i = 0, 1, 2, 3, 4, 5$,

$$\begin{aligned}
I_{1i}^{1} &= \int_{1}^{\sqrt{22}/2} \left(p_{1i}^{1} + q_{1i}^{1}\widetilde{f}_1\right) dx, \\
I_{2i}^{1} &= \int_{1}^{\sqrt{22}/2} \left(p_{2i}^{1} + q_{2i}^{1}\widetilde{f}_1\right) dx, \\
I_{1i}^{2} &= \int_{\sqrt{22}/2}^{2} \left(p_{1i}^{2} + q_{1i}^{2}\widetilde{f}_2\right) dx, \\
I_{2i}^{2} &= \int_{\sqrt{22}/2}^{2} \left(p_{2i}^{2} + q_{2i}^{2}\widetilde{f}_2\right) dx, \\
I_{1i}^{3} &= \int_{2}^{1} \left(p_{1i}^{3} + q_{1i}^{3}\widetilde{f}_3\right) dx, \\
I_{2i}^{3} &= \int_{2}^{1} \left(p_{2i}^{3} + q_{2i}^{3}\widetilde{f}_3\right) dx, \\
I_{1i}^{4} &= \int_{-1}^{1} \left(p_{1i}^{4} + q_{1i}^{4}\widetilde{f}_4\right) dx, \\
I_{2i}^{4} &= \int_{-1}^{1} \left(p_{2i}^{4} + q_{2i}^{4}\widetilde{f}_4\right) dx, \\
I_{1i}^{5} &= \int_{1}^{-1} \left(p_{1i}^{5} + q_{1i}^{5}\widetilde{f}_5\right) dx, \\
I_{2i}^{5} &= \int_{1}^{-1} \left(p_{2i}^{5} + q_{2i}^{5}\widetilde{f}_5\right) dx.
\end{aligned} \tag{44}$$

Thus,

$$\begin{aligned}
c_0(\delta) &= c_{01}(\delta) + c_{02}(\delta) \\
&= \sum_{i=0}^{5} \left[\left(I_{1i}^{1} + I_{1i}^{2} + I_{1i}^{3}\right) a_i + \left(I_{2i}^{1} + I_{2i}^{2} + I_{2i}^{2}\right) b_i\right] \\
&\equiv l_0 a_0 + l_1 a_1 + l_2 a_2 + l_3 a_3 + l_4 a_4 + l_5 a_5, \\
e_0(\delta) &= e_{01}(\delta) + e_{02}(\delta) \\
&= \sum_{i=0}^{5} \left[\left(I_{1i}^{4} + I_{1i}^{5} + I_{1i}^{3}\right) a_i + \left(I_{2i}^{4} + I_{2i}^{5} + I_{2i}^{2}\right) b_i\right] \\
&\equiv m_0 a_0 + m_1 a_1 + m_2 a_2 + m_4 a_4 + m_5 a_5.
\end{aligned} \tag{45}$$

By (4), the divergence of (1) at S_1 and S_2 is as follows:

$$\begin{aligned}
d_1(\delta) &= \left(\frac{dP_5}{dx} + \frac{dQ_5}{dy}\right)\Bigg|_{(1,2)} \\
&= 2a_0 + 20a_1 + 150a_2 - 14a_4 \\
&\quad - 70a_5 - 48b_4 + 240b_5, \\
d_2(\delta) &= \left(\frac{dP_5}{dx} + \frac{dQ_5}{dy}\right)\Bigg|_{(2,1)} \\
&= 2a_0 + 20a_1 + 150a_2 - 14a_4 \\
&\quad - 70a_5 + 48b_4 - 240b_5.
\end{aligned} \tag{46}$$

Note that

$$\begin{aligned}
\begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}\Bigg|_{S_1} &= \begin{pmatrix} 0 & 24 \\ 6 & 0 \end{pmatrix}, \\
\begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}\Bigg|_{S_2} &= \begin{pmatrix} 0 & -6 \\ -24 & 0 \end{pmatrix},
\end{aligned} \tag{47}$$

which yields

$$\lambda_1 = \lambda_2 = 12. \tag{48}$$

By Lemma 3, we have

$$\begin{aligned} c_1(\delta) = e_1(\delta) &= \sum_{i=1}^{2} -\frac{1}{|\lambda_i|}\left(\frac{dP_5}{dx}+\frac{dQ_5}{dy}\right)(S_i,\delta) \\ &= -\frac{d_1(\delta)}{|\lambda_1|} - \frac{d_2(\delta)}{|\lambda_1|} \\ &= -\frac{1}{3}a_0 - \frac{10}{3}a_1 - 25a_2 + \frac{7}{3}a_4 + \frac{35}{3}a_5. \end{aligned} \quad (49)$$

In the following by letting $b_4 = 5b_5$, then

$$d_1(\delta) = d_2(\delta), \qquad c_1(S_1,\delta) = c_1(S_2,\delta). \quad (50)$$

Letting $c_1(\delta) = 0$, then

$$a_0 = -10a_1 - 75a_2 + 7a_4 + 35a_5. \quad (51)$$

Under $c_1(\delta) = 0$, we can apply Lemma 7 to calculate the coefficients $c_2(\delta)$ and $e_2(\delta)$. For convenience, the following notations are introduced: $h = dx/dt = 2y - 3y^3 + y^5$. For $j = 1,2,3,4,5$, $h_j = (2y - 3y^3 + y^5)|_{y=y_j(x)}$, $(dP_5/dx + dQ_5/dy) \equiv \sum_{j=0}^{5} f_i a_i$, and

$$\begin{aligned} f_0^j(x) &= \left.\frac{f_0}{h}\right|_{y=y_j(x)} = \left.\frac{2}{2y-3y^3+y^5}\right|_{y=y_j(x)}, \\ f_1^j(x) &= \left.\frac{f_1}{h}\right|_{y=y_j(x)} = \left.\frac{4x^2+4y^2}{2y-3y^3+y^5}\right|_{y=y_j(x)}, \\ f_2^j(x) &= \left.\frac{f_2}{h}\right|_{y=y_j(x)} = \left.\frac{12y^2x^2+6x^4+6y^4}{2y-3y^3+y^5}\right|_{y=y_j(x)}, \\ f_3^j(x) &= \left.\frac{f_3}{h}\right|_{y=y_j(x)} = \left.\frac{0}{2y-3y^3+y^5}\right|_{y=y_j(x)} = 0, \\ f_4^j(x) &= \left.\frac{f_4}{h}\right|_{y=y_j(x)} = \left.\frac{-12y^2x^2+2x^4+2y^4}{2y-3y^3+y^5}\right|_{y=y_j(x)}, \\ f_5^j(x) &= \left.\frac{f_5}{h}\right|_{y=y_j(x)} = \left.\frac{-60y^2x^2+10x^4+10y^4}{2y-3y^3+y^5}\right|_{y=y_j(x)}. \end{aligned} \quad (52)$$

By Lemma 7, we have

$$\begin{aligned} c_{21}(\delta) &= \int_{L_{12}} \left(\frac{dP_5}{dx}+\frac{dQ_5}{dy}\right)dt = \sum_{i=0}^{5}\int_{L_{12}} a_i\frac{f_i}{h}dx, \\ c_{22}(\delta) &= \int_{L_{21}} \left(\frac{dP_5}{dx}+\frac{dQ_5}{dy}\right)dt = \sum_{i=0}^{5}\int_{L_{21}} a_i\frac{f_i}{h}dx, \\ e_{21}(\delta) &= \int_{L_{81}} \left(\frac{dP_5}{dx}+\frac{dQ_5}{dy}\right)dt = \sum_{i=0}^{5}\int_{L_{81}} a_i\frac{f_i}{h}dx, \\ e_{22}(\delta) &= \int_{L_{18}} \left(\frac{dP_5}{dx}+\frac{dQ_5}{dy}\right)dt = \sum_{i=0}^{5}\int_{L_{18}} a_i\frac{f_i}{h}dx. \end{aligned} \quad (53)$$

Substituting (52) into (53), with $f_0 = 2$ being considered, we have

$$\begin{aligned} c_{21}(\delta) &= \sum_{i=1}^{5}\int_{L_{12}} a_i\frac{f_i}{h}dx + \int_{L_{12}} \frac{-20a_1-150a_2+14a_4+70a_5}{h}dx \\ &= \sum_{i=1} a_iJ_i^1 + \sum_{i=1} a_iJ_i^2, \\ c_{22}(\delta) &= \sum_{i=1}^{5}\int_{L_{21}} a_i\frac{f_i}{h}dx + \int_{L_{21}} \frac{-20a_1-150a_2+14a_4+70a_5}{h}dx \\ &= \sum_{i=1} a_iJ_i^3, \\ e_{21}(\delta) &= \sum_{i=1}^{5}\int_{L_{81}} a_i\frac{f_i}{h}dx + \int_{L_{81}} \frac{-20a_1-150a_2+14a_4+70a_5}{h}dx \\ &= \sum_{i=1} a_iJ_i^4, \\ e_{22}(\delta) &= \sum_{i=1}^{5}\int_{L_{18}} a_i\frac{f_i}{h}dx + \int_{L_{18}} \frac{-20a_1-150a_2+14a_4+70a_5}{h}dx \\ &= \sum_{i=1} a_iJ_i^5, \end{aligned} \quad (54)$$

where $J_3^1 = J_3^2 = J_3^3 = J_3^4 = J_3^5 = 0$ and

$$\begin{aligned} J_1^1 &= \int_1^{\sqrt{22}/2} f_1^1(x) + \frac{-20}{h_1}dx, \\ J_1^2 &= \int_{\sqrt{22}/2}^{2} f_1^2(x) + \frac{-20}{h_2}dx, \\ J_1^3 &= \int_2^{1} f_1^3(x) + \frac{-20}{h_3}dx, \\ J_2^1 &= \int_1^{\sqrt{22}/2} f_2^1(x) + \frac{-150}{h_1}dx, \end{aligned}$$

$$J_2^2 = \int_{\sqrt{22}/2}^{2} f_2^2(x) + \frac{-150}{h_2} dx,$$

$$J_2^3 = \int_{2}^{1} f_2^3(x) + \frac{-150}{h_3} dx,$$

$$J_4^1 = \int_{1}^{\sqrt{22}/2} f_4^1(x) + \frac{14}{h_1} dx,$$

$$J_4^2 = \int_{\sqrt{22}/2}^{2} f_4^2(x) + \frac{14}{h_2} dx,$$

$$J_4^3 = \int_{2}^{1} f_4^3(x) + \frac{14}{h_3} dx,$$

$$J_5^1 = \int_{1}^{\sqrt{22}/2} f_5^1(x) + \frac{70}{h_1} dx,$$

$$J_5^2 = \int_{\sqrt{22}/2}^{2} f_5^2(x) + \frac{70}{h_2} dx,$$

$$J_5^3 = \int_{2}^{1} f_5^3(x) + \frac{70}{h_3} dx,$$

$$J_1^4 = \int_{-1}^{1} f_1^1(x) + \frac{-20}{h_4} dx,$$

$$J_1^5 = \int_{1}^{-1} f_1^2(x) + \frac{-20}{h_5} dx,$$

$$J_2^4 = \int_{-1}^{1} f_2^1(x) + \frac{-150}{h_4} dx,$$

$$J_2^5 = \int_{1}^{-1} f_2^2(x) + \frac{-150}{h_5} dx,$$

$$J_4^4 = \int_{-1}^{1} f_4^1(x) + \frac{14}{h_4} dx,$$

$$J_4^5 = \int_{1}^{-1} f_4^2(x) + \frac{14}{h_5} dx,$$

$$J_5^4 = \int_{-1}^{1} f_5^1(x) + \frac{70}{h_4} dx,$$

$$J_5^5 = \int_{1}^{-1} f_5^2(x) + \frac{70}{h_5} dx. \quad (55)$$

Applying Lemma 7, we have

$$\begin{aligned} c_2(\delta) &= c_{21}(\delta) + c_{22}(\delta) \\ &= \sum_{i=0} \left(J_i^1 + J_i^2 + J_i^3\right) a_i \\ &\equiv J_1 a_1 + J_2 a_2 + J_4 a_4 + J_5 a_5, \end{aligned}$$

$$\begin{aligned} e_2(\delta) &= e_{21}(\delta) + e_{22}(\delta) \\ &= \sum_{i=0} \left(J_i^4 + J_i^5 + J_i^3\right) a_i \\ &\equiv R_1 a_1 + R_2 a_2 + R_4 a_4 + R_5 a_5. \end{aligned} \quad (56)$$

The integrals in (44) and (3) and the coefficients in (45) and (56) can be obtained by numeral computation on Maple 13; see Appendix.

In order to find the local coefficient $c_3(S_1, \delta)$ at the saddle $S_1(1, 2)$ we make a change of variables of the form $u = (\sqrt{2}/2)(x-1)$, $v = \sqrt{2}(y-2)$ and time rescaling $\tau = kt$, $k = 1$ so that the system (1) becomes

$$\frac{du}{d\tau} = \widetilde{H}_v + \varepsilon \widetilde{p}(u, v, \delta), \qquad \frac{dv}{d\tau} = -\widetilde{H}_u + \varepsilon \widetilde{q}(u, v, \delta), \quad (57)$$

where

$$\begin{aligned} \widetilde{H}(u, v) &= H(x, y)\big|_{\{x=\sqrt{2}u+1, y=(\sqrt{2}/2)v+2\}} \\ &= 6v^2 - 6u^2 - \frac{10\sqrt{2}}{3}u^3 + \frac{25\sqrt{2}}{6}v^3 \\ &\quad + 5u^4 + \frac{35}{16}v^4 + 4\sqrt{2}u^5 + \frac{\sqrt{2}}{4}v^5 \\ &\quad + \frac{4}{3}u^6 + \frac{1}{48}v^6 - \frac{5}{12}, \end{aligned} \quad (58)$$

$$\widetilde{p}(u, v) = \frac{\sqrt{2}}{2} P_5(x, y)\big|_{\{x=\sqrt{2}u+1, y=(\sqrt{2}/2)v+2\}},$$

$$\widetilde{q}(u, v) = \sqrt{2} Q_5(x, y)\big|_{\{x=\sqrt{2}u+1, y=(\sqrt{2}/2)v+2\}}.$$

Writing functions $\widetilde{p}(u, v)$ and $\widetilde{q}(u, v)$ as the form

$$\begin{aligned} \widetilde{p}(u, v) &= \sum_{i+j=0}^{5} \widetilde{a}_{ij} u^i v^j, \\ \widetilde{q}(u, v) &= \sum_{i+j=0}^{5} \widetilde{b}_{ij} u^i v^j, \end{aligned} \quad (59)$$

then the formula for the second local coefficient at the saddle in Lemma 2 can be applied directly; we have

$$c_3(S_1, \delta) = \frac{7}{216}a_1 + \frac{35}{72}a_2 + \frac{55}{216}a_4 + \frac{275}{216}a_5. \quad (60)$$

Similarly, we have

$$c_3(S_2, \delta) = \frac{7}{216}a_1 + \frac{35}{72}a_2 + \frac{55}{216}a_4 + \frac{275}{216}a_5. \quad (61)$$

Applying Lemma 7, we have

$$\begin{aligned} c_3(\delta) &= e_3(\delta) = c_3(S_1, \delta) + c_3(S_2, \delta) \\ &= \frac{7}{108}a_1 + \frac{35}{36}a_2 + \frac{55}{108}a_4 + \frac{275}{108}a_5. \end{aligned} \quad (62)$$

In order to find $B_i(\delta)$, $i = 0, 1, 2$, we move the center $C_1 = (2, 2)$ into the origin by letting $u = 2\sqrt{6}(x-2)$ and $v = 2\sqrt{6}(y-2)$; that is, $x = (\sqrt{6}/12)u+2$ and $y = (\sqrt{6}/12)v+2$ and make the time rescaling $d\tau = 24dt$ so that the system (1) becomes

$$\frac{du}{d\tau} = \frac{dH_1^c}{dv} + \varepsilon p_1(u,v), \qquad \frac{dv}{d\tau} = -\frac{dH_1^c}{du} + \varepsilon q_1(u,v), \tag{63}$$

where

$$\begin{aligned} H_1^c(u,v) &= H(x,y)\big|_{\{x=(\sqrt{6}/12)u+2,\, y=(\sqrt{6}/12)v+2\}} \\ &= -\frac{8}{3} + \frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{25\sqrt{6}}{432}u^3 \\ &\quad + \frac{25\sqrt{6}}{432}v^3 + \frac{35}{2304}u^4 + \frac{35}{2304}v^4 + \frac{\sqrt{6}}{3456}u^5 \\ &\quad + \frac{\sqrt{6}}{3456}v^5 + \frac{1}{82944}u^6 + \frac{1}{82944}v^6, \\ p_1(u,v) &= \frac{\sqrt{6}}{12}p(x,y)\Big|_{\{x=(\sqrt{6}/12)u+2,\, y=(\sqrt{6}/12)v+2\}}, \\ q_1(u,v) &= \frac{\sqrt{6}}{12}q(x,y)\Big|_{\{x=(\sqrt{6}/12)u+2,\, y=(\sqrt{6}/12)v+2\}}. \end{aligned} \tag{64}$$

Let

$$\begin{aligned} \widetilde{M}_1^c(h,\delta) &= \oint_{H_1^c(u,v)=h} q_1 du - p_1 dv \\ &= \sum_{k\geq 0} \widetilde{b}_k\left(h + \frac{8}{3}\right)^{k+1} \end{aligned} \tag{65}$$

which is the Melnikov function of the new system (63). Applying the formula for the Hopf coefficients $\widetilde{b}_0(\delta)$, $\widetilde{b}_1(\delta)$, and $\widetilde{b}_2(\delta)$ directly in [20], we have

$$\begin{aligned} \widetilde{b}_0(\delta) &= \frac{1}{6}\pi a_0 + \frac{8}{3}\pi a_1 + 32\pi a_2 - \frac{32}{3}\pi a_4 - \frac{160}{3}\pi a_5, \\ \widetilde{b}_1(\delta)\big|_{\widetilde{b}_0=0} &= -\frac{11}{108}\pi a_1 - \frac{19}{9}\pi a_2 \\ &\quad + \frac{25}{27}\pi a_4 + \frac{125}{27}\pi a_5, \\ \widetilde{b}_2(\delta)\big|_{\widetilde{b}_0=0} &= -\frac{17545}{839808}\pi a_1 - \frac{28379}{69984}\pi a_2 \\ &\quad + \frac{37445}{209952}\pi a_4 + \frac{187225}{209952}\pi a_5. \end{aligned} \tag{66}$$

By Lemma 6,

$$M_1(h,\delta) = \widetilde{M}_1^c(h,\delta). \tag{67}$$

Therefore

$$\begin{aligned} B_0(\delta) &= \widetilde{b}_0(\delta) = \frac{1}{6}\pi a_0 + \frac{8}{3}\pi a_1 \\ &\quad + 32\pi a_2 - \frac{32}{3}\pi a_4 - \frac{160}{3}\pi a_5, \\ B_1(\delta)\big|_{B_0=0} &= \widetilde{b}_1(\delta)\big|_{\widetilde{b}_0=0} = -\frac{11}{108}\pi a_1 - \frac{19}{9}\pi a_2 \\ &\quad + \frac{25}{27}\pi a_4 + \frac{125}{27}\pi a_5, \\ B_2(\delta)\big|_{B_0=0} &= \widetilde{b}_2(\delta)\big|_{\widetilde{b}_0=0} = -\frac{17545}{839808}\pi a_1 - \frac{28379}{69984}\pi a_2 \\ &\quad + \frac{37445}{209952}\pi a_4 + \frac{187225}{209952}\pi a_5. \end{aligned} \tag{68}$$

Similarly, the expressions of $d_k(\delta)$ are obtained as follows:

$$\begin{aligned} d_0(\delta) &= \frac{\sqrt{6}}{6}\pi a_0 + \frac{4\sqrt{6}}{3}\pi a_1 + 8\sqrt{6}\pi a_2 \\ &\quad + \frac{8\sqrt{6}}{3}\pi a_4 + \frac{40\sqrt{6}}{3}\pi a_5, \\ d_1(\delta)\big|_{d_0=0} &= -\frac{\sqrt{6}}{108}\pi a_1 + \frac{\sqrt{6}}{18}\pi a_2 \\ &\quad - \frac{35\sqrt{6}}{54}\pi a_4 - \frac{175\sqrt{6}}{54}\pi a_5, \\ d_2(\delta)\big|_{d_0=0} &= \frac{535}{104976}\pi\sqrt{6}a_1 + \frac{1949}{17496}\pi\sqrt{6}a_2 \\ &\quad - \frac{12055}{52488}\pi\sqrt{6}a_4 - \frac{60275}{52488}\pi\sqrt{6}a_5. \end{aligned} \tag{69}$$

We will use the coefficients $c_0(\delta)$, $c_1(\delta)$, $c_2(\delta)$, $c_3(\delta)$, $B_0(\delta)$, $B_1(\delta)$, $B_2(\delta)$, $e_0(\delta)$, $e_1(\delta)$, $e_2(\delta)$, $e_3(\delta)$, $d_0(\delta)$, $d_1(\delta)$, and $d_2(\delta)$ obtained above to study the limit cycle bifurcation.

(1) Solving the equations

$$c_0(\delta) = c_1(\delta) = d_0(\delta) = d_1(\delta) = 0 \tag{70}$$

gives

$$\begin{aligned} a_0 &= \mu_1 a_1, \qquad a_2 = \mu_2 a_1, \\ a_4 &= \mu_3 a_1, \qquad a_5 = \mu_4 a_1, \end{aligned} \tag{71}$$

where

$$\begin{aligned} \mu_1 &= -\frac{232}{73}, \qquad \mu_2 = -\frac{163}{1752}, \\ \mu_3 &= \frac{1}{1752}\,\frac{39l_5 + 27840l_0 - 8760l_1 + 815l_2}{-l_5 + 5l_4}, \\ \mu_4 &= -\frac{1}{1752}\,\frac{5568l_0 - 1752l_1 + 163l_2 + 39l_4}{-l_5 + 5l_4}. \end{aligned} \tag{72}$$

Approximate computation using Maple.13 gives

$$\begin{aligned} a_0 &= \mu_1 a_1 = -3.1780821917808219178 a_1 \equiv a_0^*, \\ a_2 &= \mu_2 a_1 \\ &= -0.093036529680365296804 a_1 \equiv a_2^*, \\ a_4 &= \mu_3 a_1 \\ &= 1.3418001549779299848 \times 10^8 a_1 \equiv a_4^*, \\ a_5 &= \mu_4 a_1 \\ &= -2.6836003104010654490 \times 10^7 a_1 \equiv a_5^*. \end{aligned} \tag{73}$$

We can easily find that

$$\operatorname{rank} \frac{\partial (c_0, c_1, d_0, d_1)}{\partial (a_0, a_2, a_4, a_5)} = 4. \tag{74}$$

We fix $a_1 > 0$, $a_3 \neq 0$, and take $\delta = (a_0, a_2, a_4, a_5)$, $\delta_0 = (a_0^*, a_2^*, a_4^*, a_5^*)$. Inserting δ_0 into $c_2(\delta)$, $d_2(\delta)$, $e_0(\delta)$ and $B_0(\delta)$ we have

$$\begin{aligned} c_2(\delta_0) &= -1.6154178146383561644 a_1, \\ d_2(\delta_0) &= -0.00015502565057430061 0\pi\sqrt{6} a_1, \\ e_0(\delta_0) &= 0.001040156096575342 a_1, \\ B_0(\delta_0) &= -0.6027397260050228310 \pi a_1. \end{aligned} \tag{75}$$

Taking $h_1 = -5/12 - \varepsilon_1$, $h_2 = -8/3 + \varepsilon_1$, $h_3 = -5/12 - \varepsilon_1$, and $h_4 = -4/3 + \varepsilon_1$ with $\varepsilon_1 > 0$ small enough, we have

$$\begin{aligned} M_1(h_1, \delta_0) &= c_2(\delta_0)(-\varepsilon_1) + O\left((-\varepsilon_1)^2 \ln|\varepsilon_1|\right) > 0, \\ M_1(h_2, \delta_0) &= B_0(\delta_0)(\varepsilon_1) + O\left(\varepsilon_1^2\right) < 0, \\ M_8(h_3, \delta_0) &= e_0(\delta_0) + O\left((\varepsilon_1)^2 \ln|\varepsilon_1|\right) > 0, \\ M_8(h_4, \delta_0) &= d_2(\delta_0)(\varepsilon_1)^3 + O\left((\varepsilon_1)^4\right) < 0, \end{aligned} \tag{76}$$

which yield $(1 - \operatorname{sgn}(M_1(h_1, \delta_0) M_1(h_2, \delta_0)))/2 = 1$ and $(1 - \operatorname{sgn}(M_8(h_3, \delta_0) M_8(h_4, \delta_0)))/2 = 1$. Hence, by Theorem 8, (1) can have 6 limit cycles inside the 2-polycycles L_1 and L_8, 2 limit cycles near the 2-polycycle L_1, 2 limit cycles near the center C_8, 1 limit cycle between the center C_1 and the polycycle L_1, and 1 limit cycle between the center C_8 and the polycycle L_8. Considering that the system (1) is Z_4-equivariant, the system (1) can have at least 24 limit cycles. See Figure 2(a) for their distribution.

(2) Further let

$$c_0(\delta) = c_1(\delta) = B_0(\delta) = e_0(\delta) = 0, \tag{77}$$

to obtain

$$\begin{aligned} a_0 &= w_1 a_1, \qquad a_2 = w_2 a_1, \\ a_4 &= w_3 a_1, \qquad a_5 = w_4 a_1, \end{aligned} \tag{78}$$

where $w_1 = -16(55m_2l_4 - 11m_2l_5 - 55m_4l_2 + 15m_5l_4 + 11m_5l_2 - 15m_4l_5 + 360m_4l_1 - 72m_5l_1 - 360m_1l_4 + 72m_1l_5)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$, $w_2 = -(-2m_4l_5 + 880m_4l_0 - 95m_4l_1 - 176m_5l_0 + 2m_5l_4 + 19m_5l_1 - 880m_0l_4 + 176m_0l_5 + 95m_1l_4 - 19m_1l_5)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$, $w_3 = (240m_5l_0 - 2m_2l_5 + 2m_5l_2 - 240m_0l_5 - 39m_5l_1 + 39m_1l_5 - 5760l_0m_1 - 880l_2m_0 + 95l_2m_1 + 880l_0m_2 + 5760l_1m_0 - 95l_1m_2)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$, and $w_4 = (240m_0l_4 - 39m_1l_4 + 2m_2l_4 - 19l_2m_1 - 240m_4l_0 + 176l_2m_0 + 1152l_0m_1 - 2m_4l_2 - 176l_0m_2 + 39m_4l_1 + 19l_1m_2 - 1152l_1m_0)/(5760m_4l_0 - 95m_4l_2 - 1152m_5l_0 + 39m_5l_4 + 19m_5l_2 - 5760m_0l_4 + 1152m_0l_5 - 39m_4l_5 + 95m_2l_4 - 19m_2l_5)$. Approximate computation using Maple.13 gives

$$\begin{aligned} a_0 &= -9.4485863716681452021 a_1 \equiv a_0^*, \\ a_2 &= 0.0030582821716100337153 a_1 \equiv a_2^*, \\ a_4 &= 8655624.6835808229498 a_1 \equiv a_4^*, \\ a_5 &= -1731124.9144080276985 a_1 \equiv a_5^*. \end{aligned} \tag{79}$$

As before, we have

$$\operatorname{rank} \frac{\partial (c_0, c_1, B_0, e_0)}{\partial (a_0, a_2, a_4, a_5)} = 4, \tag{80}$$

fixing $a_1 > 0$, $a_3 \neq 0$, and taking $\delta = (a_0, a_2, a_4, a_5)$ and $\delta_0 = (a_0^*, a_2^*, a_4^*, a_5^*)$. Noting that

$$\begin{aligned} c_2(\delta_0) &= -0.45572159725036392404 a_1, \\ B_1(\delta_0) &= -0.0050298137906989600656 \pi a_1, \\ e_2(\delta_0) &= -0.85071588097 a_1, \\ d_0(\delta_0) &= 0.080477020648213603 0\pi\sqrt{6} a_1, \end{aligned} \tag{81}$$

we have

$$\begin{aligned} M_1(h_1, \delta_0) &= c_2(\delta_0)(-\varepsilon_3) + O\left(-\varepsilon_3 \ln|\varepsilon_3|\right) > 0, \\ M_1(h_2, \delta_0) &= B_1(\delta_0)\left(\varepsilon_3^2\right) + O\left(\varepsilon_3^3\right) < 0, \\ M_8(h_3, \delta_0) &= e_2(\delta_0)(-\varepsilon_3) + O\left((-\varepsilon_3)\ln|\varepsilon_3|\right) > 0, \\ M_8(h_4, \delta_0) &= d_0(\delta_0)\varepsilon_3 + O\left((-\varepsilon_3)^2\right) > 0, \end{aligned} \tag{82}$$

where $h_1 = -5/12 - \varepsilon_3$, $h_2 = -8/3 + \varepsilon_3$, $h_3 = -5/12 - \varepsilon_3$, and $h_4 = -4/3 + \varepsilon_3$ with $\varepsilon_1 > 0$being small. Hence, noting that $(1 - \operatorname{sgn}(M_1(h_1, \delta_0) M_1(h_2, \delta_0)))/2 = 1$, $(1 - \operatorname{sgn}(M_8(h_3, \delta_0) M_8(h_4, \delta_0)))/2 = 0$, and $(1 + \operatorname{sgn}(c_2(\delta_0) e_2(\delta_0)))/2 = 1$, by Theorem 8 again, we can obtain 6 limit cycles inside the 2-polycycles L_1 and L_8. By Z_4-equivariance, the system (1) can have 24 limit cycles. See Figure 2(b). Then Theorem 1 is proved.

4. Conclusion

In this paper, we proved that a Z_4-equivalent quintic near-Hamiltonian system can also have 24 limit cycles compared to a z_6 and z_3-equivalent quintic near-Hamiltonian system having 24 limit cycles. Certainly, the distributions of 24 limit cycles obtained in this paper are new. The method we use is the expansions of the corresponding Melnikov functions, which is different from the methods of detect function and normal form, which are the main methods of the previous work on z_q-equivalent quintic near-Hamiltonian system.

Appendix

In this section, by numeral computation using Maple.13, we give the approximate values of the integrals in (44) and (3) and the coefficients in (45) and (56) as

$$I_{10}^1 = 3.309003029,$$
$$I_{10}^2 = 1.730303665,$$
$$I_{10}^3 = -3.217505541,$$
$$I_{11}^1 = 28.51638338,$$
$$I_{11}^2 = 12.79094042,$$
$$I_{11}^3 = -16.08752770,$$
$$I_{12}^1 = 254.3849935,$$
$$I_{12}^2 = 97.6750154,$$
$$I_{12}^3 = -80.43763849,$$
$$I_{13}^1 = -13.03562364,$$
$$I_{13}^2 = 1.03562364, \qquad I_{13}^3 = 12,$$
$$I_{14}^1 = -141.0435372,$$
$$I_{14}^2 = -0.5116118346,$$
$$I_{14}^3 = 60, \qquad I_{15}^1 = -317.8639878,$$
$$I_{15}^2 = -149.9117575,$$
$$I_{15}^3 = 60, \qquad I_{10}^4 = 4.636476089,$$
$$I_{10}^5 = -3.114725692,$$
$$I_{11}^4 = 23.18238044,$$
$$I_{11}^5 = -10.86431923,$$
$$I_{12}^4 = 115.9119022,$$
$$I_{12}^5 = -39.42284767,$$
$$I_{13}^4 = 12, \qquad I_{13}^5 = -12.00000000,$$
$$I_{14}^4 = 60, \qquad I_{14}^5 = -43.62567211,$$
$$I_{15}^4 = 60, \qquad I_{15}^5 = 21.87163945,$$
$$I_{20}^1 = \frac{9}{4}, \qquad I_{20}^2 = -\frac{9}{4}, \qquad I_{20}^3 = 0,$$
$$I_{20}^4 = 0, \qquad I_{20}^5 = 0, \qquad I_{21}^1 = \frac{261}{16},$$
$$I_{21}^2 = -\frac{261}{16}, \qquad I_{21}^3 = 0,$$
$$I_{21}^4 = 0, \qquad I_{21}^5 = 0,$$
$$I_{22}^1 = \frac{1953}{16}, \qquad I_{22}^2 = -\frac{1953}{16}, \qquad I_{22}^3 = 0,$$
$$I_{22}^4 = 0, \qquad I_{22}^5 = 0, \qquad I_{23}^1 = -\frac{315}{16},$$
$$I_{23}^2 = \frac{315}{16}, \qquad I_{23}^3 = 0, \qquad I_{23}^4 = 0,$$
$$I_{23}^5 = 0, \qquad I_{24}^1 = -\frac{2385}{16}, \qquad I_{24}^2 = \frac{2385}{16},$$
$$I_{24}^3 = 0, \qquad I_{24}^4 = 0, \qquad I_{24}^5 = 0,$$
$$I_{25}^1 = -\frac{549}{16}, \qquad I_{25}^2 = \frac{549}{16}, \qquad I_{25}^3 = 0,$$
$$I_{25}^4 = 0, \qquad I_{25}^5 = 0,$$
$$J_1^1 = 1.831700353,$$
$$J_1^2 = 1.100027834, \qquad J_1^3 = 0,$$
$$J_2^1 = 36.46355425,$$
$$J_2^2 = 20.04837845,$$
$$J_2^3 = 0, \qquad J_4^1 = -14.15186229,$$
$$J_4^2 = -7.471228579,$$
$$J_4^3 = -10.29601773,$$
$$J_5^1 = -70.75931146,$$
$$J_5^2 = -37.35614289,$$
$$J_5^3 = -51.48008865,$$
$$J_1^4 = 0, \qquad J_1^5 = -3.964132938,$$
$$J_2^4 = 0, \qquad J_2^5 = -49.77854225,$$
$$J_4^4 = 14.83672348,$$
$$J_4^5 = 14.44096924,$$
$$J_5^4 = 74.18361742,$$

$$J_5^5 = 72.20484619, \tag{A.1}$$

and $l_3 = m_3 = 0$,

$$\begin{aligned} l_0 &= 1.821801153, \\ l_1 &= 25.21979610, \\ l_2 &= 271.6223704, \\ l_4 &= -81.55514903, \\ l_5 &= -407.7757453, \\ m_0 &= 1.521750397, \\ m_1 &= 12.31806121, \\ m_2 &= 76.48905453, \\ m_4 &= 16.37432789, \\ m_5 &= 81.87163945, \\ J_1 &= 2.931728187, \\ J_2 &= 56.51193270, \\ J_3 = 0, \quad J_4 &= -31.91910860, \\ J_5 &= -159.5955430, \\ R_1 &= -3.964132938, \\ R_2 &= -49.77854225, \\ R_3 = 0, \quad R_4 &= 29.27769272, \\ R_5 &= 146.3884636. \end{aligned} \tag{A.2}$$

Conflict of Interests

The authors declare that there is no conflict of interests for any of the authors in this paper.

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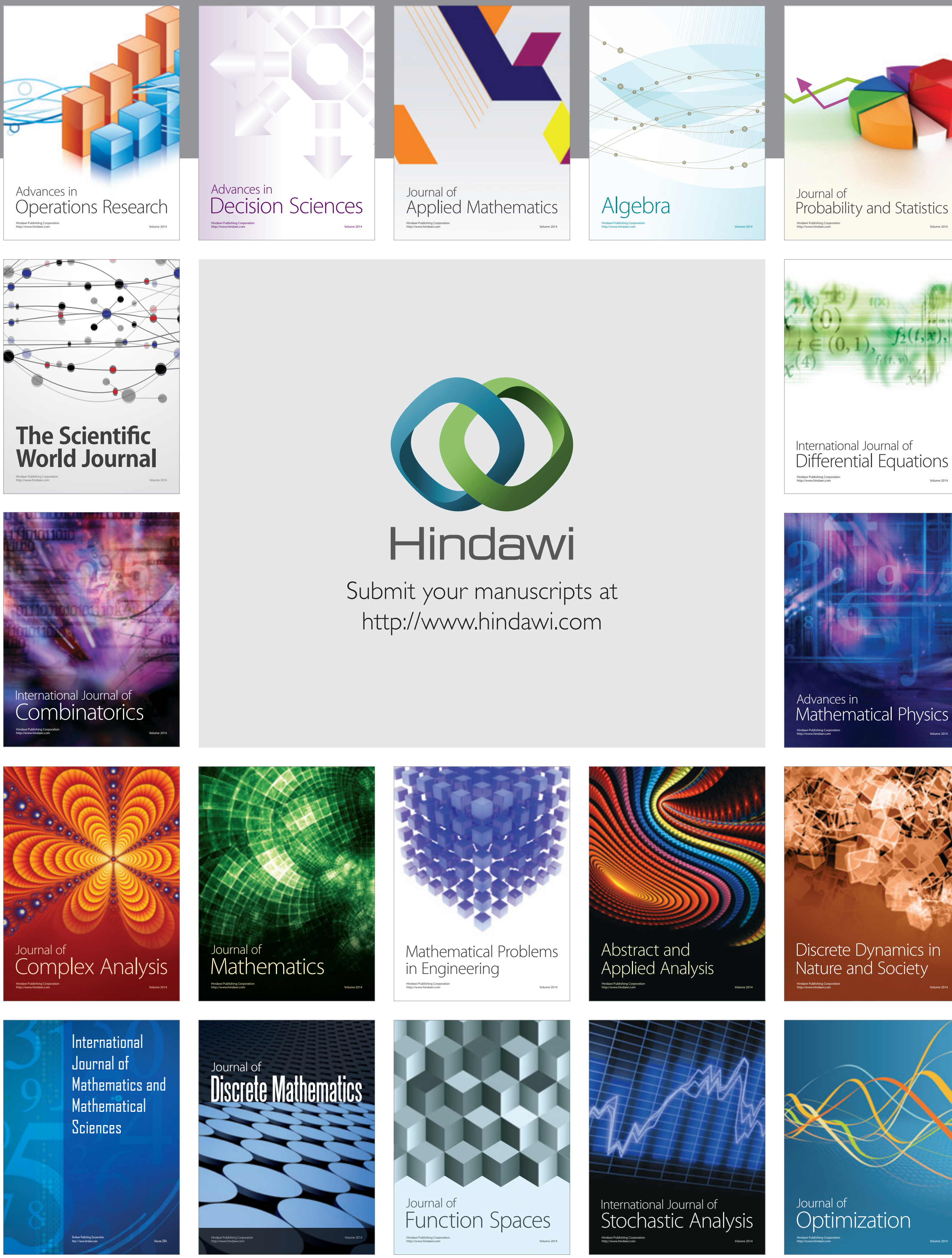
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