

## Research Article

# Numerical Study for Time Delay Multistrain Tuberculosis Model of Fractional Order

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A novel mathematical fractional model of multistrain tuberculosis with time delay memory is presented. The proposed model is governed by a system of fractional delay differential equations, where the fractional derivative is defined in the sense of the Grünwald–Letnikov definition. Modified parameters are introduced to account for the fractional order. The stability of the equilibrium points is investigated for any time delay. Nonstandard finite difference method is proposed to solve the resulting system of fractional-order delay differential equations. Numerical simulations show that nonstandard finite difference method can be applied to solve such fractional delay differential equations simply and effectively.

## 1. Introduction

It is known that tuberculosis (TB) is one of the most important infectious diseases and is considered as the second largest cause of mortality by infectious diseases and a challenging disease to control [1]. Time delays required to treatment of active TB present a major obstacle to the control of a TB epidemic [2]; it worsens the disease, increases the risk of death, and enhances tuberculosis transmission to the community [3, 4]. Both patient and the health system may be responsible for the treatment delay [3]. On the other hand, mathematical models are quite important and efficient tool to describe and investigate TB diseases; see [5–9]. In [10], Silva et al. presented TB model with time-delay memory. Herein, we consider a general model of multistrain TB diseases with time-delay memory. A discrete time delay is incorporated, in the variables of active TB infection of two and three strains, to represent the required time to commencement of treatment and diagnosis [11].

The multistrain TB model incorporates three strains: *extensively drug-resistant* (XDR), emerging *multidrug resistant* (MDR), and drug sensitive, and has been developed by Arino and Soliman [12] in 2015. Several factors of spreading TB such as the exogenous reinfection, the fast infection, and secondary infection are included in this model. Sweilam et al. introduced some numerical studies for this model in [13–16].

Fractional differential equations have been the focus of many studies due to their frequent appearance in various sciences [13–20]. The general theory of differential equations with delays (DDEs) is widely developed and discussed in the literature [21–25]. *Delayed fractional differential equations* (DFDEs) are also used to describe dynamical systems [26–28]. Recently, DFDEs begin to raise the attention of many researchers [29–33]. Reliable and efficient numerical techniques for DFDEs are very necessary and important [34]. *Nonstandard finite difference method* (NSFDM) was firstly proposed by Mickens [35] in 1980s to solve numerically the *ordinary differential equations* (ODEs) and *partial differential*

TABLE 1: Interpretation of the variable states of system (1).

Variable	Interpretation
$S(t)$	Individuals have never encountered TB.
$L_s(t)$	The individuals infected with drug-sensitive TB but not infectious.
$L_m(t)$	Infected with MDR-TB but not infectious.
$L_x(t)$	Infected with XDR-TB but not infectious.
$I_s(t)$	Able to infect others with drug sensitive strain.
$I_m(t)$	Able to infect others with MDR strain.
$I_x(t)$	Able to infect others with XDR strain.
$R(t)$	Recovered by getting a successful treatment.
$N(t)$	The variable of population size. $N(t) = S(t) + L_s(t) + L_m(t) + L_x(t) + I_s(t) + I_m(t) + I_x(t) + R(t)$ .

TABLE 2: All adapted parameters and their interpretation of system (1).

Parameter	Interpretation
$d^\alpha$	Natural death rate
$b^\alpha$	Birth rate
$\lambda_j^\alpha$	Rate of infected individuals move to $L_j$ with strain $j \in \{s, m, x\}$
$1 - \lambda_j^\alpha$	Rate of newly infected individuals progressing to active TB with strain $j \in \{s, m, x\}$
$\beta_j^\alpha$	Transmission coefficient with strain $j \in \{s, m, x\}$
$\epsilon_j^\alpha$	Rate of endogenous reactivation of $L_j$
$\gamma_j^\alpha$	Rate of natural recovery to the latent stage $L_j$
$\delta_j^\alpha$	Rate of death due to TB of strain $j$
$\alpha_{j1}^\alpha, \alpha_{j2}^\alpha$	Rate of exogenous reinfection of $L_{j1}$ due to contact with $I_{j2}$
$1 - \sigma_j^\alpha$	Efficiency of treatment in preventing infection with strain $j$
$P_1^\alpha$	Probability of treatment success for $L_s$
$1 - P_1^\alpha$	Proportion of treated $L_s$ moved to $L_m$ due to incomplete treatment or lack of strict compliance in the use of drugs
$P_2^\alpha$	Probability of treatment success for $I_s$
$1 - P_2^\alpha$	Proportion of treated $I_s$ moved to $L_m$ due to incomplete treatment or lack of strict compliance in the use of drugs
$P_3^\alpha$	Probability of treatment success for $I_m$
$1 - P_3^\alpha$	proportion of treated $I_m$ moved to $L_x$ due to incomplete treatment or lack of strict compliance in the use of drugs
$t_{1s}^\alpha$	Rate of treatment for $L_s$
$t_{2j}^\alpha$	Rate of treatment for $I_j$ . Note that $t_{2x}$ is the rate of successful treatment of $I_x$ , $j \in \{x, m, s\}$

equations (PDEs) with more accuracy than *standard finite difference method* (SFDM). It is considered as a powerful numerical scheme that preserves properties of exact solutions of the differential equation [36].

The main aim of work is to study numerically the solutions of fractional-order model of multistrain TB with time delay memory. The presence of fractional-order and time delays in the model can lead to a notable increase in the complexity of the observed behavior, and the solution continuously depends on all the previous states. An efficient numerical method, NSFDM, is used to numerically solve the fractional-order delay model. The rest of the paper is organized as follows: In Section 2, we present a fractional order model with time delay for multistrain TB. Stability of equilibrium points is presented in Section 3. NSFDM for fractional-order delay differential equations is introduced in Section 4. Some numerical simulations are given in Section 5, and conclusion in Section 6. Some definitions on fractional calculus and some properties of nonstandard discretization are given in Appendix.

## 2. Fractional Multistrain TB Model with Time Delay

In this section, a multistrain TB model of fractional-order and time delay memory is presented. The population of interest is divided into eight compartments depending on their epidemiological stages as follows: susceptible ( $S$ ); latently infected with drug sensitive TB ( $L_s$ ); latently infected with MDR TB ( $L_m$ ); latently infected with XDR TB ( $L_x$ ); sensitive drug TB infectious ( $I_s$ ); MDR TB infectious ( $I_m$ ); XDR TB infectious ( $I_x$ ); recovered  $R$ . One biological meaning of the given parameters is given in Table 1. One of the main assumptions of this model is that the total population  $N(t)$ , with  $N(t) = S(t) + L_s(t) + L_m(t) + L_x(t) + I_s(t) + I_m(t) + I_x(t) + R(t)$ , is variable of the time. We introduce a discrete time delay in the state variables  $I_m$  and  $I_x$ , denoted by  $\tau$ , that represents the time required for diagnosis and commencement of treatment of active TB infection of two and three strains. The parameters in the modified the model are described in Table 2; see [37]. The modified system of multistrain TB model of fractional-order and time delay is

$$\begin{aligned}
D_t^\alpha S &= b^\alpha - d^\alpha S - \beta_s^\alpha \frac{SI_s}{N} - \beta_m^\alpha \frac{SI_m}{N} - \beta_x^\alpha \frac{SI_x}{N}, \\
D_t^\alpha L_s &= \lambda_s^\alpha \beta_s^\alpha \frac{SI_s}{N} + \sigma_s^\alpha \lambda_s^\alpha \beta_s^\alpha \frac{RI_s}{N} + \gamma_s^\alpha I_s - \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} \\
&\quad - \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_s I_m}{N} - \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_s I_x}{N} - (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha) L_s, \\
D_t^\alpha L_m &= \lambda_m^\alpha \beta_m^\alpha \frac{SI_m}{N} + \sigma_m^\alpha \lambda_m^\alpha \beta_m^\alpha \frac{RI_m}{N} + \gamma_m^\alpha I_m \\
&\quad + \alpha_{sm}^\alpha \beta_m^\alpha \lambda_m^\alpha \frac{L_s I_m}{N} + (1 - P_1^\alpha) t_{1s}^\alpha L_s + (1 - P_2^\alpha) t_{2s}^\alpha I_s \\
&\quad - \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m I_m}{N} - \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_m I_x}{N} - (d^\alpha + \varepsilon_m^\alpha) L_m, \\
D_t^\alpha L_x &= \lambda_x^\alpha \beta_x^\alpha \frac{SI_x}{N} + \sigma_x^\alpha \lambda_x^\alpha \beta_x^\alpha \frac{RI_x}{N} + \gamma_x^\alpha I_x + \alpha_{sx}^\alpha \beta_x^\alpha \lambda_x^\alpha \\
&\quad \cdot \frac{L_s I_x}{N} + \alpha_{mx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_m I_x}{N} + (1 - P_3^\alpha) t_{2m}^\alpha I_m - \alpha_{xx}^\alpha \beta_x^\alpha \\
&\quad \cdot \frac{L_x I_x}{N} - (d^\alpha + \varepsilon_x^\alpha) L_x, \\
D_t^\alpha I_s &= \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} + (1 - \lambda_s^\alpha) \beta_s^\alpha \left( \frac{SI_s}{N} + \sigma_s^\alpha \frac{RI_s}{N} \right) \\
&\quad + \varepsilon_s^\alpha L_s - (d^\alpha + \delta_s^\alpha + t_{2s}^\alpha + \gamma_s^\alpha) I_s, \\
D_t^\alpha I_m &= \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m I_m}{N} + (1 - \lambda_m^\alpha) \\
&\quad \cdot \beta_m^\alpha \left( \frac{SI_m}{N} + \sigma_m^\alpha \frac{RI_m}{N} + \alpha_{sm}^\alpha \frac{L_s I_m}{N} \right) + \varepsilon_m^\alpha L_m \\
&\quad - (d^\alpha + \delta_m^\alpha + \gamma_m^\alpha) I_m - t_{2m}^\alpha I_m (t - \tau), \\
D_t^\alpha I_x &= \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x I_x}{N} + (1 - \lambda_x^\alpha) \\
&\quad \cdot \beta_x^\alpha \left( \frac{SI_x}{N} + \sigma_x^\alpha \frac{RI_x}{N} + \alpha_{sx}^\alpha \frac{L_s I_x}{N} + \alpha_{mx}^\alpha \frac{L_m I_x}{N} \right) + \varepsilon_x^\alpha L_x \\
&\quad - (d^\alpha + \delta_x^\alpha + \gamma_x^\alpha) I_x - t_{2x}^\alpha I_x (t - \tau), \\
D_t^\alpha R &= P_1^\alpha t_{1s}^\alpha L_s + P_2^\alpha t_{2s}^\alpha I_s + P_3^\alpha t_{2m}^\alpha I_m + t_{2x}^\alpha I_x (t - \tau) \\
&\quad - \sigma_s^\alpha \beta_s^\alpha \frac{RI_s}{N} - \sigma_m^\alpha \beta_m^\alpha \frac{RI_m}{N} - \sigma_x^\alpha \beta_x^\alpha \frac{RI_x}{N} - d^\alpha R.
\end{aligned} \tag{1}$$

The initial conditions for system (1) are  $S(\xi) = \theta_1(\xi)$ ,  $L_s(\xi) = \theta_2(\xi)$ ,  $L_m(\xi) = \theta_3(\xi)$ ,  $L_x(\xi) = \theta_4(\xi)$ ,  $I_s(\xi) = \theta_5(\xi)$ ,  $I_m(\xi) = \theta_6(\xi)$ ,  $I_x(\xi) = \theta_7(\xi)$ ,  $R(\xi) = \theta_8(\xi)$ ,  $\xi \in [-\tau, 0]$ , where

$\theta = (\theta_1, \theta_2, \dots, \theta_8)^T \in C$ , where  $C$  is the Banach space  $C([0, \tau], \mathbb{R}^8)$ . From biological meaning, we further assume that  $\theta_i > 0$  for  $i = 1, \dots, 8$ . Throughout this paper, we focus on the dynamics of the solutions of (1) in the restricted region,  $\Omega = \{(S, L_s, L_m, L_x, I_s, I_m, I_x, R) \in \mathbb{R}^8 \mid S + L_s + L_m + L_x + I_s + I_m + I_x + R \leq b^\alpha/d^\alpha\}$ . We refer here to [24, 31], for local existence, uniqueness, and continuation results.

The unique solution  $(S(t), L_s(t), L_m(t), L_x(t), I_s(t), I_m(t), I_x(t), R(t))$  of (1) with initial condition exists for all time  $t \geq 0$ . Consider the solutions of (1), for  $(S, L_s, L_m, L_x, I_s, I_m, I_x, R) \in \Omega'$ , where  $\Omega'$  is the interior of  $\Omega$ , for all  $\xi \in [-\tau, 0]$ . Then the solutions stay in the interior of the region for all time  $t \geq 0$ ; that is, the region is positively invariant with respect to system (1) (see, e.g., [31]). Model (1) has a disease-free equilibrium given by  $E_0 = (b^\alpha/d^\alpha, 0, 0, 0, 0, 0, 0, 0)$ ; see [32].

**2.1. Basic Reproduction Number.** The basic reproduction number,  $R_0$ , is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual [32]. Herein, we apply the method in [32] to drive  $R_0$ . The order of the infected variables is

$$\mathfrak{F} := (L_s, L_m, L_x, I_s, I_m, I_x)^T. \tag{2}$$

The vector representing new infections into the infected classes  $F$  is given by

$$F := \begin{pmatrix} \lambda_s^\alpha \beta_s^\alpha \frac{SI_s}{N} + \sigma_s^\alpha \lambda_s^\alpha \beta_s^\alpha \frac{RI_s}{N} \\ \lambda_m^\alpha \beta_m^\alpha \frac{SI_m}{N} + \sigma_m^\alpha \lambda_m^\alpha \beta_m^\alpha \frac{RI_m}{N} \\ \lambda_x^\alpha \beta_x^\alpha \frac{SI_x}{N} + \sigma_x^\alpha \lambda_x^\alpha \beta_x^\alpha \frac{RI_x}{N} \\ (1 - \lambda_s^\alpha) \beta_s^\alpha \left( \frac{SI_s}{N} + \sigma_s^\alpha \frac{RI_s}{N} \right) \\ (1 - \lambda_m^\alpha) \beta_m^\alpha \left( \frac{SI_m}{N} + \sigma_m^\alpha \frac{RI_m}{N} \right) \\ (1 - \lambda_x^\alpha) \beta_x^\alpha \left( \frac{SI_x}{N} + \sigma_x^\alpha \frac{RI_x}{N} \right) \end{pmatrix}. \tag{3}$$

The vector  $V$  representing other flows within and out of the infected classes  $\mathfrak{F}$  is given by

$$V := \begin{pmatrix} \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} - \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_s I_m}{N} - \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_s I_x}{N} + \gamma_s^\alpha I_s - (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha) L_s, \\ + \gamma_m^\alpha I_m + \alpha_{sm}^\alpha \beta_m^\alpha \lambda_m^\alpha \frac{L_s I_m}{N} + (1 - P_1^\alpha) t_{1s}^\alpha L_s + (1 - P_2^\alpha) t_{2s}^\alpha I_s, \\ - \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m I_m}{N} - \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_m I_x}{N} - (d^\alpha + \varepsilon_m^\alpha) L_m, \\ + \gamma_x^\alpha I_x + \alpha_{sx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_s I_x}{N} + \alpha_{mx}^\alpha \beta_x^\alpha \lambda_x^\alpha \frac{L_m I_x}{N} + (1 - P_3^\alpha) t_{2m}^\alpha I_m - \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x I_x}{N} - (d^\alpha + \varepsilon_x^\alpha) L_x, \\ \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s I_s}{N} + \varepsilon_s^\alpha L_s - (d^\alpha + \delta_s^\alpha + t_{2s}^\alpha + \gamma_s^\alpha) I_s, \\ \left( \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m I_m}{N} + \alpha_{sm}^\alpha \frac{L_s I_m}{N} \right) + \varepsilon_m^\alpha L_m - (d^\alpha + \delta_m^\alpha + \gamma_m^\alpha + t_{2m}^\alpha) I_m, \\ \left( \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x I_x}{N} + (1 - \lambda_x^\alpha) \beta_x^\alpha + \alpha_{sx}^\alpha \frac{L_s I_x}{N} + \alpha_{mx}^\alpha \frac{L_m I_x}{N} \right) + \varepsilon_x^\alpha L_x - (d^\alpha + \delta_x^\alpha + \gamma_x^\alpha + t_{2x}^\alpha) I_x, \end{pmatrix}. \tag{4}$$

The matrix of new infections  $F$  and the matrix of transfers between compartments  $V$  are the Jacobian matrices obtained by differentiating  $F$  and  $V$  with respect to the infected variables  $\mathfrak{I}$  and evaluating at the disease-free equilibrium. They take the form

$$\begin{aligned} F &:= \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}, \\ V &:= \begin{pmatrix} C & D \\ E & F_2 \end{pmatrix}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} A &= \begin{pmatrix} \lambda_s^\alpha \beta_s^\alpha & 0 & 0 \\ 0 & \lambda_m^\alpha \beta_m^\alpha & 0 \\ 0 & 0 & \lambda_x^\alpha \beta_x^\alpha \end{pmatrix}, \\ B &= \begin{pmatrix} (1 - \lambda_s^\alpha) \beta_s^\alpha & 0 & 0 \\ 0 & (1 - \lambda_m^\alpha) \beta_m^\alpha & 0 \\ 0 & 0 & (1 - \lambda_x^\alpha) \beta_x^\alpha \end{pmatrix}, \\ C &= \begin{pmatrix} (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha) & 0 & 0 \\ (-1 + P_1^\alpha) t_{1s}^\alpha & (d^\alpha + \varepsilon_m^\alpha) & 0 \\ 0 & 0 & (d^\alpha + \varepsilon_x^\alpha) \end{pmatrix}, \\ D &= \begin{pmatrix} -\gamma_s^\alpha & 0 & 0 \\ (-1 + P_2^\alpha) t_{2s}^\alpha & -\gamma_m^\alpha & 0 \\ 0 & (-1 + P_3^\alpha) t_{2m}^\alpha & -\gamma_x^\alpha \end{pmatrix}, \\ F_2 &= \begin{pmatrix} (d^\alpha + \delta_s^\alpha + \gamma_s^\alpha + t_{2s}^\alpha) & 0 & 0 \\ 0 & (d^\alpha + \delta_m^\alpha + \gamma_m^\alpha + t_{2m}^\alpha) & 0 \\ 0 & 0 & (d^\alpha + \delta_x^\alpha + \gamma_x^\alpha + t_{2x}^\alpha) \end{pmatrix}, \\ E &= \begin{pmatrix} -\varepsilon_s^\alpha & 0 & 0 \\ 0 & -\varepsilon_m^\alpha & 0 \\ 0 & 0 & -\varepsilon_x^\alpha \end{pmatrix}. \end{aligned} \quad (6)$$

Then the basic reproduction number  $R_0$  for system (1) is the spectral radius of the next generation matrix and is given by

$$R_0 = \rho(FV^{-1}) = \max(R_{0s}, R_{0m}, R_{0x}), \quad (7)$$

where

$$\begin{aligned} R_{0s} &= \frac{\beta_s^\alpha (\varepsilon_s^\alpha + (1 - \lambda_s^\alpha) (d^\alpha + t_{1s}^\alpha))}{(\varepsilon_s^\alpha + d^\alpha + t_{1s}^\alpha) (t_{2s}^\alpha + \delta_s^\alpha + d^\alpha) + \gamma_s^\alpha (t_{1s}^\alpha + d^\alpha)}, \\ R_{0m} &= \frac{\beta_m^\alpha (\varepsilon_m^\alpha + (1 - \lambda_m^\alpha) d^\alpha)}{(\varepsilon_m^\alpha + d^\alpha) (t_{2m}^\alpha + \delta_m^\alpha + d^\alpha) + d^\alpha \gamma_m^\alpha}, \\ R_{0x} &= \frac{\beta_x^\alpha (\varepsilon_x^\alpha + (1 - \lambda_x^\alpha) d^\alpha)}{(\varepsilon_x^\alpha + d^\alpha) (t_{2x}^\alpha + \delta_x^\alpha + d^\alpha) + d^\alpha \gamma_x^\alpha}. \end{aligned} \quad (8)$$

### 3. Equilibrium Points and Their Asymptotic Stability

To discuss the local asymptotic stability for evaluating the equilibrium points, let us consider the following [38]:

$$\begin{aligned} D_t^\alpha S &= D_t^\alpha L_s = D_t^\alpha L_m = D_t^\alpha L_x = D_t^\alpha I_s = D_t^\alpha I_m \\ &= D_t^\alpha I_x = D_t^\alpha R = 0. \end{aligned} \quad (9)$$

Then, from (A.1)

$$g_i(\bar{S}, \bar{L}_s, \bar{L}_m, \bar{L}_x, \bar{I}_s, \bar{I}_m, \bar{I}_x, \bar{R}) = 0, \quad i = 1, 2, 3, \dots, 8, \quad (10)$$

where  $(\bar{S}, \bar{L}_s, \bar{L}_m, \bar{L}_x, \bar{I}_s, \bar{I}_m, \bar{I}_x, \bar{R})$  denotes any equilibrium point.

**3.1. Stability of the Disease-Free Equilibrium.** If  $I_s(t) = I_m(t) = I_x(t) = 0$  then  $L_s(t) = L_m(t) = L_x(t) = 0$ ,  $R(t) = 0$ , and

$S(t) = b^\alpha/d^\alpha$ . Then the disease-free equilibrium (DFE) is  $E_0 = \{(b^\alpha/d^\alpha, 0, 0, 0, 0, 0, 0, 0)\}$ .

Let us consider the coordinate transformation:  $s(t) = S(t) - \bar{S}$ ,  $l_s(t) = L_s(t) - \bar{L}_s$ ,  $l_m(t) = L_m(t) - \bar{L}_m$ ,  $l_x(t) =$

$L_x(t) - \bar{L}_x$ ,  $i_s(t) = I_s(t) - \bar{I}_s$ ,  $i_m(t) = I_m(t) - \bar{I}_m$ ,  $i_x(t) = I_x(t) - \bar{I}_x$ ,  $r(t) = R(t) - \bar{R}$ . The corresponding characteristic equation for DFE is given as follows:

$$(J(E_0) - \lambda I) = \begin{pmatrix} \lambda - a & 0 & 0 & 0 & b & c & d_1 & 0 \\ 0 & \lambda - e & 0 & 0 & f & 0 & 0 & 0 \\ 0 & g & \lambda - h & 0 & p & q & 0 & 0 \\ 0 & 0 & 0 & \lambda - r & 0 & s & t & 0 \\ 0 & u & 0 & 0 & \lambda - v & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & \lambda - x + t_{2m}^\alpha e^{(-\lambda\tau)} & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 & \lambda - z + t_{2x}^\alpha e^{(-\lambda\tau)} & 0 \\ 0 & m & 0 & 0 & n & j & t_{2x}^\alpha e^{(-\lambda\tau)} & \lambda - a \end{pmatrix}, \quad (11)$$

where  $a = -d^\alpha$ ,  $b = -\beta_s^\alpha$ ,  $c = -\beta_m^\alpha$ ,  $d_1 = -\beta_x^\alpha$ ,  $e = -(d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha)$ ,  $f = \gamma_s^\alpha + \lambda_s^\alpha \beta_s^\alpha$ ,  $g = (1 - p_1^\alpha)t_{1s}^\alpha$ ,  $h = -(d^\alpha + \varepsilon_m^\alpha)$ ,  $p = (1 - p_2^\alpha)t_{2s}^\alpha$ ,  $q = \gamma_m^\alpha + \lambda_m^\alpha \beta_m^\alpha$ ,  $r = -(d^\alpha + \varepsilon_x^\alpha)$ ,  $s = (1 - p_3^\alpha)t_{2m}^\alpha$ ,  $t = \gamma_x^\alpha + \lambda_x^\alpha \beta_x^\alpha$ ,  $u = \varepsilon_s^\alpha$ ,  $v = -(d^\alpha + \delta_s^\alpha + t_{2s}^\alpha + \gamma_s^\alpha)$ ,  $w = \varepsilon_m^\alpha$ ,  $x = -(d^\alpha + \delta_m^\alpha + \gamma_m^\alpha)$ ,  $y = \varepsilon_x^\alpha$ ,  $z = -(d^\alpha + \delta_x^\alpha + \gamma_x^\alpha)$ ,  $m = p_1^\alpha t_{1s}^\alpha$ ,  $n = p_2^\alpha t_{2s}^\alpha$ ,  $j = p_3^\alpha t_{2m}^\alpha$ .

The characteristic equation associated with above matrix is [38]

$$\begin{aligned} \Delta(\lambda) &= |J(E_0) - \lambda I| = 0, \\ (a - \lambda)^2 (\lambda^2 - (r + z + t_{2x}^\alpha e^{(-\lambda\tau)})\lambda - yt \\ &+ (z - t_{2x}^\alpha e^{(-\lambda\tau)})r) (-\lambda^2 + (h + x - t_{2m}^\alpha e^{(-\lambda\tau)})\lambda \\ &- (x + t_{2m}^\alpha e^{(-\lambda\tau)})h + wq) (-\lambda^2 + (e + v)\lambda + uf \\ &- ve) = 0. \end{aligned} \quad (12)$$

**Lemma 1.** *If  $R_0 < 1$ , then the disease-free equilibrium  $E_0$  is locally asymptotically stable for  $\tau = 0$ .*

*Proof.* When  $\tau = 0$ , the associated transcendental characteristic equation  $\Delta(\lambda)$  of system (1) at  $E_0$  becomes  $\Delta(\lambda) = P(\lambda) = 0$ , and then the eigenvalues of the Jacobian matrix are

$$\lambda_{1,2} = -d,$$

$$\lambda_{3,4}$$

$$= \frac{r + (z - t_{2x}^\alpha) \pm \sqrt{(r^2 - 2(z - t_{2x}^\alpha)r + (z - t_{2x}^\alpha)^2 + 4yt)}}{2},$$

$$\begin{aligned} \lambda_{5,6} &= \frac{x - t_{2m}^\alpha + h \pm \sqrt{(x - t_{2m}^\alpha)^2 - 2(x - t_{2m}^\alpha)h + h^2 + 4wq}}{2}, \\ \lambda_{7,8} &= \frac{v + e \pm \sqrt{(v^2 - 2ve + e^2 + 4uf)}}{2}, \end{aligned} \quad (13)$$

and by using Routh Hurwitz Theorem [28], these roots are negative or have negative real parts and all eigenvalues satisfy Matignon's conditions [39], given by ( $|\arg \lambda_i| > \alpha\pi/2$ ) so the disease-free equilibrium  $E_0$  is locally asymptotically stable.  $\square$

**Lemma 2.** *Let  $R_0 < 1$ , and then the disease-free equilibrium  $E_0$  is locally asymptotically stable for  $\tau > 0$ .*

*Proof.* Let us consider  $\tau > 0$ , and we noted that second and third factor of the characteristic equation (12), which are  $(\lambda^2 - (r + z + t_{2x}^\alpha e^{(-\lambda\tau)})\lambda - yt + (z - t_{2x}^\alpha e^{(-\lambda\tau)})r)$  and  $(-\lambda^2 + (h + x - t_{2m}^\alpha e^{(-\lambda\tau)})\lambda - (x + t_{2m}^\alpha e^{(-\lambda\tau)})h + wq)$ , have no pure imaginary roots for any value of the delay  $\tau$ , if  $R_0 < 1$ . Hence all the roots of the characteristic equation have negative real parts and we get that DFE is locally asymptotically stable regardless of the value of the delay and all eigenvalues satisfy Matignon's conditions [39], given by ( $|\arg \lambda_i| > \alpha\pi/2$ ) so the disease-free equilibrium  $E_0$  is locally asymptotically stable.  $\square$

**3.2. Stability of the Endemic Equilibrium.** System (1) has an endemic equilibrium if at least one of the infected variables is not zero. The expression "analytic" is complexity and not useful for our purposes. Consider the values of parameters from Table 3. Then the basic reproduction number is  $R_0 > 1$ . The endemic equilibrium  $S = 338.2$ ,  $L_s = 0$ ,  $L_m = 0$ ,  $L_x = 2233.8$ ,  $I_s = 0$ ,  $I_m = 0$ ,  $I_x = 4820.6$ ,  $R = 62.0$ . The matrices

$A_1$  and  $A_2$  associated with the linearized system at the endemic equilibrium are computed as

$$A_1 = \begin{pmatrix} -9.0226 & 0.4107 & 0.4107 & 0.4107 & -0.2244 & -0.2244 & -0.2244 & 0.4107 \\ 0 & -2.2127 & 0 & 0 & 0.6321 & 0 & 0 & 0 \\ 0 & 0.8800 & -1.3327 & 0 & 0.800 & 0.6321 & 0 & 0 \\ 4.4475 & 0.1472 & 0.1472 & -1.4118 & -0.0791 & -0.0750 & 0.3432 & 1.0525 \\ 0 & 0.5000 & 0 & 0 & -1.2729 & 0 & 0 & 0 \\ 0 & 0.0146 & 0.5 & 0 & 0 & -0.4415 & 0 & 0 \\ 4.1762 & -0.1241 & -0.3181 & 0.6023 & -0.3504 & -0.3504 & -0.5675 & 0.7813 \\ 0.0188 & 0.0188 & 0.0188 & 0.0188 & -0.0103 & 0.0196 & 0.0237 & -2.6245 \end{pmatrix}, \quad (14)$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t_{2m}^\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -t_{2x}^\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{2x}^\alpha & 0 \end{pmatrix}.$$

The transcendental characteristic equation  $\Delta\lambda = (\lambda I - A_1 - e^{-\tau\lambda} A_2)$  is given by

$$\begin{aligned} & \lambda^8 + (18.8862 - 0.0680e^{-\tau\lambda}) \lambda^7 \\ & + (1.2702 \times 10^2 - 1.2255e^{-\tau\lambda} + 0.0012e^{-2\tau\lambda}) \lambda^6 \\ & + (4.2273 \times 10^2 - 7.5306e^{-\tau\lambda} + 0.0198e^{-2\tau\lambda}) \lambda^5 \\ & + (7.7354 \times 10^2 - 22.1163e^{-\tau\lambda} + 0.10986e^{-2\tau\lambda}) \lambda^4 \\ & + (7.9327 \times 10^2 - 34.3423e^{-\tau\lambda} + 0.2751e^{-2\tau\lambda}) \lambda^3 \\ & + (4.3522 \times 10^2 - 28.4367e^{-\tau\lambda} + 0.3354e^{-2\tau\lambda}) \lambda^2 \\ & + (1.1156 \times 10^2 - 11.6693e^{-\tau\lambda} + 0.18817e^{-2\tau\lambda}) \\ & \cdot \lambda (9.6913 - 1.8329e^{-\tau\lambda} + 0.0367e^{-2\tau\lambda}) = 0, \end{aligned} \quad (15)$$

when  $\tau = 0$ , and we have the following characteristic equation:

$$\begin{aligned} & \lambda^8 + 18.8182\lambda^7 + 125.7957\lambda^6 + 415.2192\lambda^5 \\ & + 751.5336\lambda^4 + 759.2028\lambda^3 + 407.1187\lambda^2 \\ & + 100.0789\lambda + 7.8951 = 0. \end{aligned} \quad (16)$$

The roots of (16) are  $-0.14208$ ,  $-0.41505$ ,  $-1.57465$ ,  $-1.40036$ ,  $-1.01044$ ,  $-9.15198$ ,  $-2.56181 + 0.05148i$ , and  $-2.56181 + 0.05148i$ , and these roots are negative or have negative real parts and all eigenvalues satisfy Matignon's conditions [39], given by  $(|\arg \lambda_i| > \alpha\pi/2)$   $i = 1, 2, \dots, 8$  so the endemic equilibrium is locally asymptotically stable.

Consider now the case  $\tau > 0$ ; we noted that the roots of the characteristic equation (15) have no pure imaginary roots for any value of the delay  $\tau$ , if  $R_0 > 1$ . Hence all the roots of the characteristic equation have negative real parts and all eigenvalues satisfy Matignon's conditions [39]. Therefore, the endemic equilibrium is locally asymptotically.

#### 4. NSFD for Fractional Delay Differential Equations

In this section, we apply NSFD method with GLFDs to obtain the discretization of the delay fractional multistrain TB model (1), which will yield the following equations:

$$\begin{aligned} \sum_{j=0}^{n+1} \omega_j^\alpha S^{n+1-j} &= b^\alpha - d^\alpha S^{n+1} - \beta_s^\alpha \frac{S^{n+1} I_s^n}{N^n} - \beta_m^\alpha \frac{S^{n+1} I_m^n}{N^n} \\ &- \beta_x^\alpha \frac{S^{n+1} I_x^n}{N^n}, \end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{n+1} \omega_j^\alpha L_s^{n+1-j} &= \lambda_s^\alpha \beta_s^\alpha \frac{S^{n+1} I_s^n}{N^n} + \sigma_s^\alpha \lambda_s^\alpha \beta_s^\alpha \frac{R^{n+1} I_s^n}{N^n} + \gamma_s^\alpha I_s^n \\
&\quad - \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s^{n+1} I_s^n}{N^n} - \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_s^{n+1} I_x^n}{N^n} - (d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha) \\
&\quad \cdot L_s^{n+1} - \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_s^{n+1} I_m^n}{N^n}, \\
\sum_{j=0}^{n+1} \omega_j^\alpha L_m^{n+1-j} &= \lambda_m^\alpha \beta_m^\alpha \frac{S^{n+1} I_m^n}{N^n} + \sigma_m^\alpha \lambda_m^\alpha \beta_m^\alpha \frac{R^{n+1} I_m^n}{N^n} \\
&\quad + \lambda_m^\alpha \alpha_{sm}^\alpha \beta_m^\alpha \frac{L_s^{n+1} I_m^n}{N^n} + \gamma_m^\alpha I_m^n + t_{1s}^\alpha L_s^{n+1} - P_1^\alpha t_{1s}^\alpha L_s^{n+1} \\
&\quad + t_{2s}^\alpha I_s^n - P_2^\alpha t_{2s}^\alpha I_s^n - \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m^{n+1} I_m^n}{N^n} - \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_m^{n+1} I_x^n}{N^n} \\
&\quad - (d^\alpha + \varepsilon_m^\alpha) L_m^{n+1}, \\
\sum_{j=0}^{n+1} \omega_j^\alpha L_x^{n+1-j} &= \lambda_x^\alpha \beta_x^\alpha \frac{S^{n+1} I_x^n}{N^n} + \sigma_x^\alpha \lambda_x^\alpha \beta_x^\alpha \frac{R^{n+1} I_x^n}{N^n} \\
&\quad + \lambda_x^\alpha \alpha_{sx}^\alpha \beta_x^\alpha \frac{L_s^{n+1} I_x^n}{N^n} + \gamma_x^\alpha I_x^n + \lambda_x^\alpha \alpha_{mx}^\alpha \beta_x^\alpha \frac{L_m^{n+1} I_x^n}{N^n} \\
&\quad + t_{2m}^\alpha I_m^n - P_3^\alpha t_{2m}^\alpha I_m^n - \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x^{n+1} I_x^n}{N^n} - (d^\alpha + \varepsilon_x^\alpha) \\
&\quad \cdot L_x^{n+1}, \\
\sum_{j=0}^{n+1} \omega_j^\alpha I_s^{n+1-j} &= \alpha_{ss}^\alpha \beta_s^\alpha \frac{L_s^{n+1} I_s^n}{N^n} + (1 - \lambda_s^\alpha) \\
&\quad \cdot \beta_s^\alpha \left( \frac{S^{n+1} I_s^n}{N^n} + \sigma_s^\alpha \frac{R^{n+1} I_s^n}{N^n} \right) + \varepsilon_s^\alpha L_s^{n+1} - (d^\alpha + \delta_s^\alpha) \\
&\quad \cdot I_s^{n+1} - (\gamma_s^\alpha + t_{2s}^\alpha) I_s^n, \\
\sum_{j=0}^{n+1} \omega_j^\alpha I_m^{n+1-j} &= \alpha_{mm}^\alpha \beta_m^\alpha \frac{L_m^{n+1} I_m^n}{N^n} + (1 - \lambda_m^\alpha) \\
&\quad \cdot \beta_m^\alpha \left( \frac{S^{n+1} I_m^n}{N^n} + \sigma_m^\alpha \frac{R^{n+1} I_m^n}{N^n} + \alpha_{sm}^\alpha \frac{L_s^{n+1} I_m^n}{N^n} \right) \\
&\quad + \varepsilon_m^\alpha L_m^{n+1} - (d^\alpha + \delta_m^\alpha) I_m^{n+1} - \gamma_m^\alpha I_m^n - t_{2m}^\alpha I_m^{n-\kappa}, \\
\sum_{j=0}^{n+1} \omega_j^\alpha I_x^{n+1-j} &= \alpha_{xx}^\alpha \beta_x^\alpha \frac{L_x^{n+1} I_x^n}{N^n} + (1 - \lambda_x^\alpha)
\end{aligned}$$

$$\begin{aligned}
&\quad \cdot \beta_m^\alpha \left( \frac{S^{n+1} I_x^n}{N^n} + \sigma_x \frac{R^{n+1} I_x^n}{N^n} + \alpha_{mx} \frac{L_x^{n+1} I_m^n}{N^n} \right) \\
&\quad + \varepsilon_x^\alpha L_x^{n+1} - (d^\alpha + \delta_x^\alpha) I_x^{n+1} - \gamma_x^\alpha I_x^n - t_{2x}^\alpha I_x^{n-\kappa}, \\
\sum_{j=0}^{n+1} \omega_j^\alpha R^{n+1-j} &= P_1^\alpha t_{1s}^\alpha L_s^{n+1} + P_2^\alpha t_{2s}^\alpha I_s^n + P_3^\alpha t_{2m}^\alpha I_m^n \\
&\quad + t_{2x}^\alpha I_x^{n-\kappa} - d^\alpha R^{n+1} - \sigma_s^\alpha \beta_s^\alpha \frac{R^{n+1} I_s^n}{N^n} - \sigma_m^\alpha \beta_m^\alpha \frac{R^{n+1} I_m^n}{N^n} \\
&\quad - \sigma_x^\alpha \beta_x^\alpha \frac{R^{n+1} I_x^n}{N^n},
\end{aligned} \tag{17}$$

where

$$N^n = S^n + L_s^n + L_m^n + L_x^n + I_s^n + I_m^n + I_x^n + R^n, \tag{18}$$

and  $\omega_0^\alpha = (\varphi_i(h))^{-\alpha}$ ,  $i = 1, 2, \dots, 8$ ,  $n = -\kappa, -\kappa + 1, \dots, 0, 1$ , where the nonlocal approximations are used for the nonlinear terms and the following functions of denominator:

$$\begin{aligned}
\varphi_1(h) &= \frac{e^{d^\alpha h} - 1}{d^\alpha}, \\
\varphi_2(h) &= \frac{e^{(d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha)h} - 1}{(d^\alpha + \varepsilon_s^\alpha + t_{1s}^\alpha)}, \\
\varphi_3(h) &= \frac{e^{(d^\alpha + \varepsilon_m^\alpha)h} - 1}{(d^\alpha + \varepsilon_m^\alpha)}, \\
\varphi_4(h) &= \frac{e^{(d^\alpha + \varepsilon_x^\alpha)h} - 1}{(d^\alpha + \varepsilon_x^\alpha)}, \\
\varphi_5(h) &= \frac{1 - e^{-(d^\alpha + \delta_s^\alpha)h}}{(\gamma_s^\alpha + t_{2s}^\alpha)}, \\
\varphi_6(h) &= \frac{1 - e^{-(d^\alpha + \delta_m^\alpha)h}}{(\gamma_m^\alpha + t_{2m}^\alpha)}, \\
\varphi_7(h) &= \frac{1 - e^{-(d^\alpha + \delta_x^\alpha)h}}{(\gamma_x^\alpha + t_{2x}^\alpha)}, \\
\varphi_8(h) &= \frac{e^{d^\alpha h} - 1}{d^\alpha}.
\end{aligned} \tag{19}$$

Then we obtain

$$\begin{aligned}
S^{n+1} &= \frac{b^\alpha - \sum_{j=1}^{n+1} \omega_j^\alpha S^{n+1-j}}{(\varphi_1(h))^{-\alpha} + d^\alpha + (\beta_s^\alpha I_s^n + \beta_m^\alpha I_m^n + \beta_x^\alpha I_x^n) / N^n}, \\
L_s^{n+1} &= \frac{(\beta_s^\alpha I_s^n / N^n) \lambda_s^\alpha (S^{n+1} + \sigma_s^\alpha R^{n+1}) + \gamma_s^\alpha I_s^n - \sum_{j=1}^{n+1} \omega_j^\alpha L_s^{n+1-j}}{(\varphi_2(h))^{-\alpha} + (d^\alpha + t_{1s}^\alpha + \varepsilon_s^\alpha) + (1/N^n) (\alpha_{ss}^\alpha \beta_s^\alpha I_s^n + \alpha_{sm}^\alpha \beta_m^\alpha I_m^n + \alpha_{sx}^\alpha \beta_x^\alpha I_x^n)},
\end{aligned}$$

$$\begin{aligned}
L_m^{n+1} &= \frac{(\beta_m^\alpha \lambda_m^\alpha I_m^n / N^n) (S^{n+1} + \sigma_m^\alpha R^{n+1} + \alpha_{sm}^\alpha L_s^{n+1}) + \gamma_m^\alpha I_m^n + t_{1s}^\alpha L_s^{n+1} (1 - P_1^\alpha)}{(\varphi_3(h))^{-\alpha} + (d^\alpha + \varepsilon_m^\alpha) + (1/N^n) (\alpha_{mm}^\alpha \beta_m^\alpha I_m^n + \alpha_{mx}^\alpha \beta_x^\alpha I_x^n)} \\
&\quad + \frac{t_{2s}^\alpha I_s^n (1 - P_2^\alpha) - \sum_{j=1}^{n+1} \omega_j^\alpha L_m^{n+1-j}}{(\varphi_3(h))^{-\alpha} + (d^\alpha + \varepsilon_m^\alpha) + (1/N^n) (\alpha_{mm}^\alpha \beta_m^\alpha I_m^n + \alpha_{mx}^\alpha \beta_x^\alpha I_x^n)}, \\
L_x^{n+1} &= \frac{(\beta_x^\alpha \lambda_x^\alpha I_x^n / N^n) (S^{n+1} + \sigma_x^\alpha R^{n+1} + \alpha_{sx}^\alpha L_s^{n+1} + \alpha_{mx}^\alpha L_m^{n+1}) + t_{2m}^\alpha I_m^n (1 - P_3^\alpha)}{(\varphi_4(h))^{-\alpha} + (d^\alpha + \varepsilon_x^\alpha) + (1/N^n) (\alpha_{xx}^\alpha \beta_x^\alpha I_x^n)} \\
&\quad + \frac{\gamma_x^\alpha I_x^n - \sum_{j=1}^{n+1} \omega_j^\alpha L_x^{n+1-j}}{(\varphi_4(h))^{-\alpha} + (d^\alpha + \varepsilon_x^\alpha) + (1/N^n) (\alpha_{xx}^\alpha \beta_x^\alpha I_x^n)}, \\
I_s^{n+1} &= \frac{\varphi_5(h) \beta_s^\alpha (I_s^n / N^n) (\alpha_{ss}^\alpha L_s^{n+1} + (1 - \lambda_s^\alpha) (S^{n+1} + \sigma_s^\alpha R^{n+1}))}{(\varphi_5(h))^{-\alpha} + (d^\alpha + \delta_s^\alpha)} - \frac{(\gamma_s^\alpha + (t_{2s}^\alpha)) I_s^n + \varepsilon_s^\alpha L_s^{n+1} - \sum_{j=1}^{n+1} \omega_j^\alpha I_s^{n+1-j}}{(\varphi_5(h))^{-\alpha} + (d^\alpha + \delta_s^\alpha)}, \\
I_m^{n+1} &= \frac{\beta_m^\alpha (I_m^n / N^n) (\alpha_{mm}^\alpha L_m^{n+1} + (1 - \lambda_m^\alpha) (S^{n+1} + \sigma_m^\alpha R^{n+1} + \alpha_{sm}^\alpha L_s^{n+1}))}{(\varphi_6(h))^{-\alpha} + (d^\alpha + \delta_m^\alpha)} - \frac{\gamma_m I_m^n - t_{2m}^\alpha I_m^{n-\kappa} + \varepsilon_m^\alpha L_m^{n+1} - \sum_{j=1}^{n+1} \omega_j^\alpha I_m^{n+1-j}}{(\varphi_6(h))^{-\alpha} + (d^\alpha + \delta_m^\alpha)}, \\
I_x^{n+1} &= \frac{\beta_x^\alpha (I_x^n / N^n) (\alpha_{xx}^\alpha L_x^{n+1} + (1 - \lambda_x^\alpha) (S^{n+1} + \sigma_x^\alpha R^{n+1} + \alpha_{sx}^\alpha L_s^{n+1} + \alpha_{mx}^\alpha L_m^{n+1}))}{(\varphi_7(h))^{-\alpha} + (d^\alpha + \delta_x^\alpha)} \\
&\quad - \frac{\gamma_x^\alpha I_x^n - t_{2x}^\alpha I_x^{n-\kappa} + \varepsilon_x^\alpha L_x^{n+1} - \sum_{j=1}^{n+1} \omega_j^\alpha I_x^{n+1-j}}{(\varphi_7(h))^{-\alpha} + (d^\alpha + \delta_x^\alpha)}, \\
R^{n+1} &= \frac{t_{1s}^\alpha P_1^\alpha L_s^{n+1} + P_2^\alpha t_{2s}^\alpha I_s^n + t_{2m}^\alpha P_3^\alpha I_m^n + t_{2x}^\alpha I_x^{n-\kappa} - \sum_{j=1}^{n+1} \omega_j^\alpha R^{n+1-j}}{(\varphi_8(h))^{-\alpha} + d^\alpha + (1/N^n) (\sigma_s^\alpha \beta_s^\alpha I_s^n + \sigma_m^\alpha \beta_m^\alpha I_m^n + \sigma_x^\alpha \beta_x^\alpha I_x^n)}.
\end{aligned} \tag{20}$$

## 5. Numerical Results and Simulations

In this section, we show the effectiveness of the numerical technique for delay fractional differential equations. Throughout this section, all simulations are performed with initial conditions  $(S(0), L_s(0), L_m(0), L_x(0), I_s(0), I_m(0), I_x(0), R(0)) = (5000, 50, 50, 50, 30, 30, 30, 60)$ , with the parameters in Table 3. The approximate solutions of the proposed system are given in Figures 1–12 at different values of  $\tau$  and  $\alpha$ . Figure 1 shows the behavior of the approximate solutions of  $R(t)$  in two cases with and without delay using dde23 at  $\alpha = 1$  and  $\tau = 0.3$ . In Figure 2, we use the same data in Figure 1 and use NSFDM; we noted that the number of individuals  $R(t)$  increases in the case of nondelay, that is, the delays in diagnosis and commencement of treatment to the individuals of  $I_m$  and  $I_x$  causing a shortage in the number of individuals of  $R(t)$ . Figure 3 shows the relationship between  $I_m(t)$  and  $I_m(t - \tau)$  and chaotic attractors at  $\tau = 0.1$  and  $\alpha = 1$ . Figure 4, shows the relationship between  $I_x(t)$  and  $I_x(t - \tau)$  at  $\tau = 0.1$  in case of integer order. Figures 5 and 6 show the relationship between  $I_m(t)$  and  $I_m(t - \tau)$  and  $I_x(t)$  with  $I_x(t - \tau)$ , respectively, in case of fraction order where  $\alpha = 0.95$ .

Figures 7 and 8 show the approximate solutions for  $I_m(t - \tau)$  and  $I_x(t - \tau)$  at  $\tau = 2$ ,  $\alpha = 0.98$  by using NSFDM. Figures 9 and 10 show the approximate solutions of different  $\tau$  in both fraction and integer cases; we noted that increasing the value of  $\tau$  causes decreasing the values of  $R(t)$ . Figures 11 and 12 show the behavior of the approximate solutions with different value of  $\alpha$ , which are given to demonstrate how the fractional model is a generalization of the integer order model.

## 6. Conclusion

Fractional models have the potential to describe more complex dynamics than the integer models and can include easily the memory effect present in many real world phenomena. In this paper, multistrain TB model of fractional order derivatives with time delay memory is presented. A nonstandard numerical scheme is introduced to numerically study the approximate solution of proposed model problem. The obtained results show that the delays in diagnosis and commencement of treatment to the individuals of  $I_m$  and  $I_x$  cause a shortage in the number of individuals of  $R(t)$ . The approximate solution of proposed model changes when  $\tau$  and



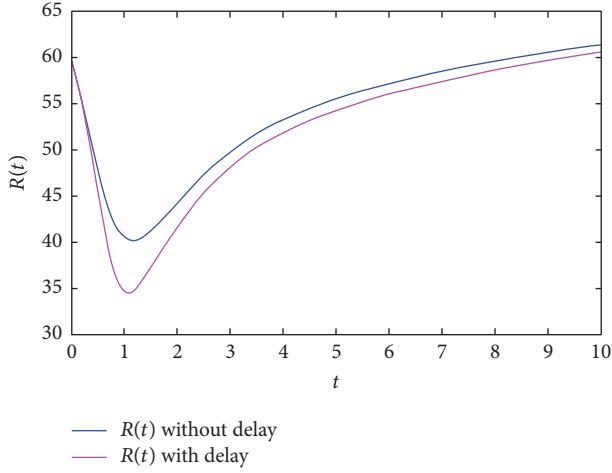


FIGURE 1: The approximate solution of  $R(t)$  with  $\tau = 0.3$ , using dde23.

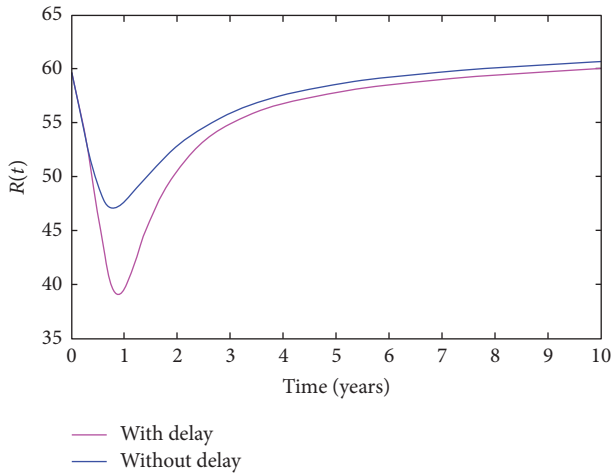


FIGURE 2: The approximate solution of  $R(t)$  with  $\tau = 0.3$ , using NSFDM,  $\alpha = 1$ .

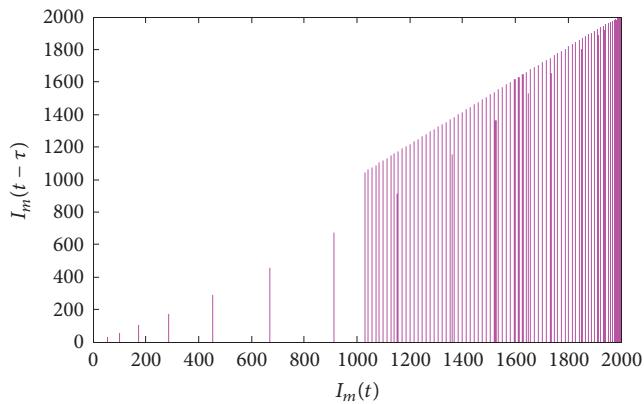


FIGURE 3: The relationship between  $I_m(t)$ ,  $I_m(t - \tau)$  with  $\tau = 0.1$ ,  $\alpha = 1$  using NSFDM.

TABLE 3: Parameter values of system (1).

Parameter	Value
$b^\alpha$	$3190^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$d^\alpha$	$0.38^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\beta_s^\alpha = \beta_m^\alpha = \beta_x^\alpha$	$14^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\lambda_s^\alpha = \lambda_m^\alpha = \lambda_x^\alpha$	$0.5^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\epsilon_s^\alpha = \epsilon_m^\alpha = \epsilon_x^\alpha$	$0.5^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\alpha_{r1,r2}^\alpha$	$0.05^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\gamma_s^\alpha = \gamma_m^\alpha = \gamma_x^\alpha$	$0.3^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$t_{1s}^\alpha$	$0.88^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$t_{2r}^\alpha : r \in (s, m, x)$	$t_{2s}^\alpha = 0.88^\alpha;$ $t_{2m}^\alpha = t_{2x}^\alpha = 0.034^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\sigma_r^\alpha$	$0.25^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$p_r^\alpha$	$0.88^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$
$\delta_r^\alpha$	$0.045^\alpha \left(\frac{1}{\text{year}}\right)^\alpha$

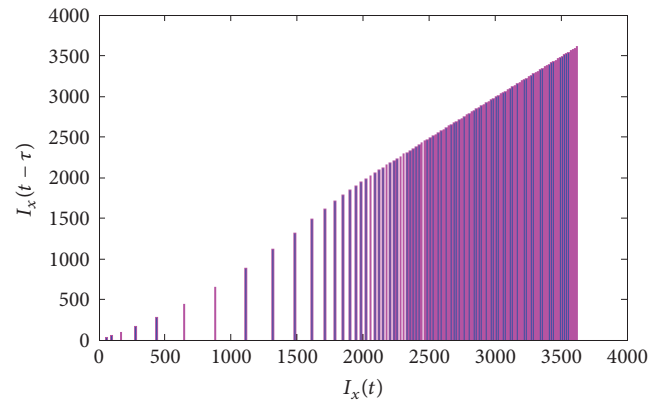


FIGURE 4: The relationship between  $I_x(t)$ ,  $I_x(t - \tau)$  with  $\tau = 0.1$ ,  $\alpha = 1$  using NSFDM.

$\alpha$  take different values. Some figures are given to demonstrate how the fractional delay model is a generalization of the integer order model. It is concluded that NSFDM can be applied to solve such fractional delay differential equations simply and effectively. All results are obtained by using MATLAB programming.

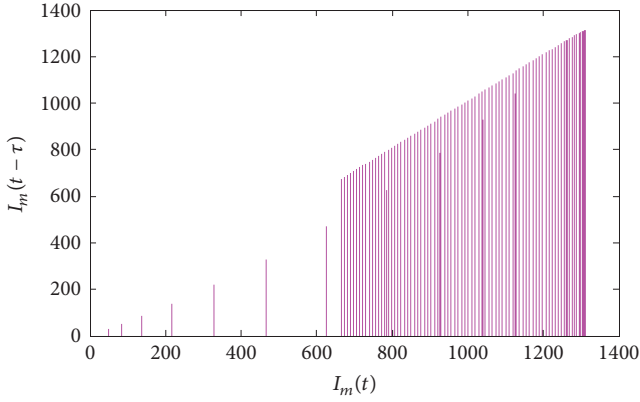


FIGURE 5: The relationship between  $I_m(t)$ ,  $I_m(t - \tau)$  with  $\tau = 0.1$ ,  $\alpha = 0.95$  using NSFDM.

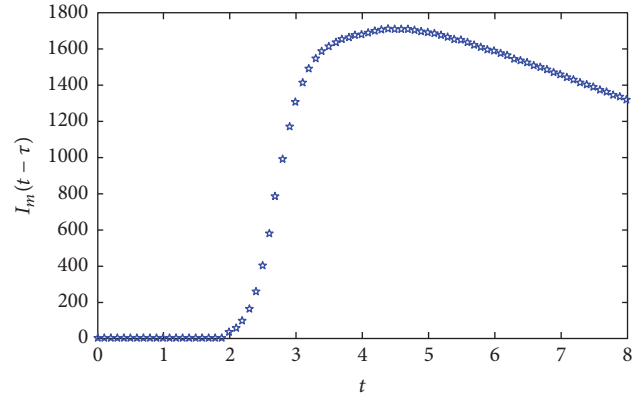


FIGURE 8: The approximate solutions  $I_m(t - \tau)$  with  $\tau = 2$ ,  $\alpha = 0.98$  using NSFDM.

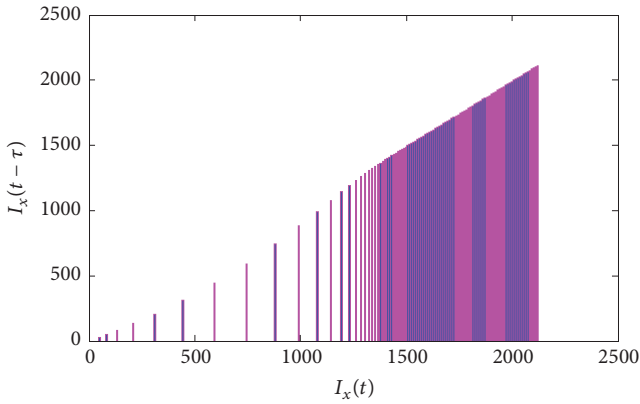


FIGURE 6: The relationship between  $I_x(t)$ ,  $I_x(t - \tau)$  with  $\tau = 0.1$ ,  $\alpha = 0.95$  using NSFDM.

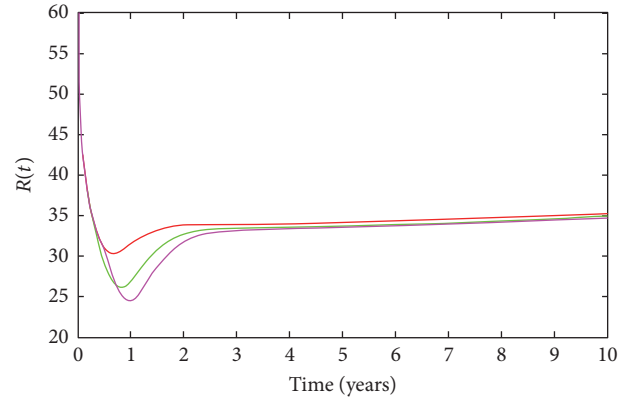


FIGURE 9: The approximate solutions with different  $\tau$  and  $\alpha = 0.9$  using NSFDM.

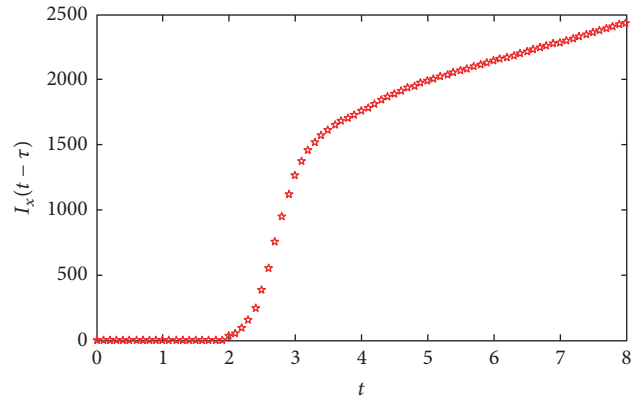


FIGURE 7: The approximate solutions  $I_x(t - \tau)$  with  $\tau = 2$ ,  $\alpha = 0.98$  using NSFDM.

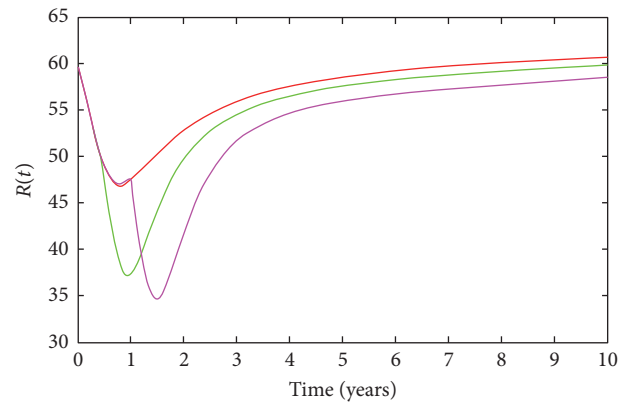


FIGURE 10: The approximate solution of  $R(t)$  with different  $\tau$ , using NSFDM,  $\alpha = 1$ .

## Appendix

### A. Preliminaries and Notations

In this section, some basic definitions and properties in the theory of the fractional calculus are presented. Moreover,

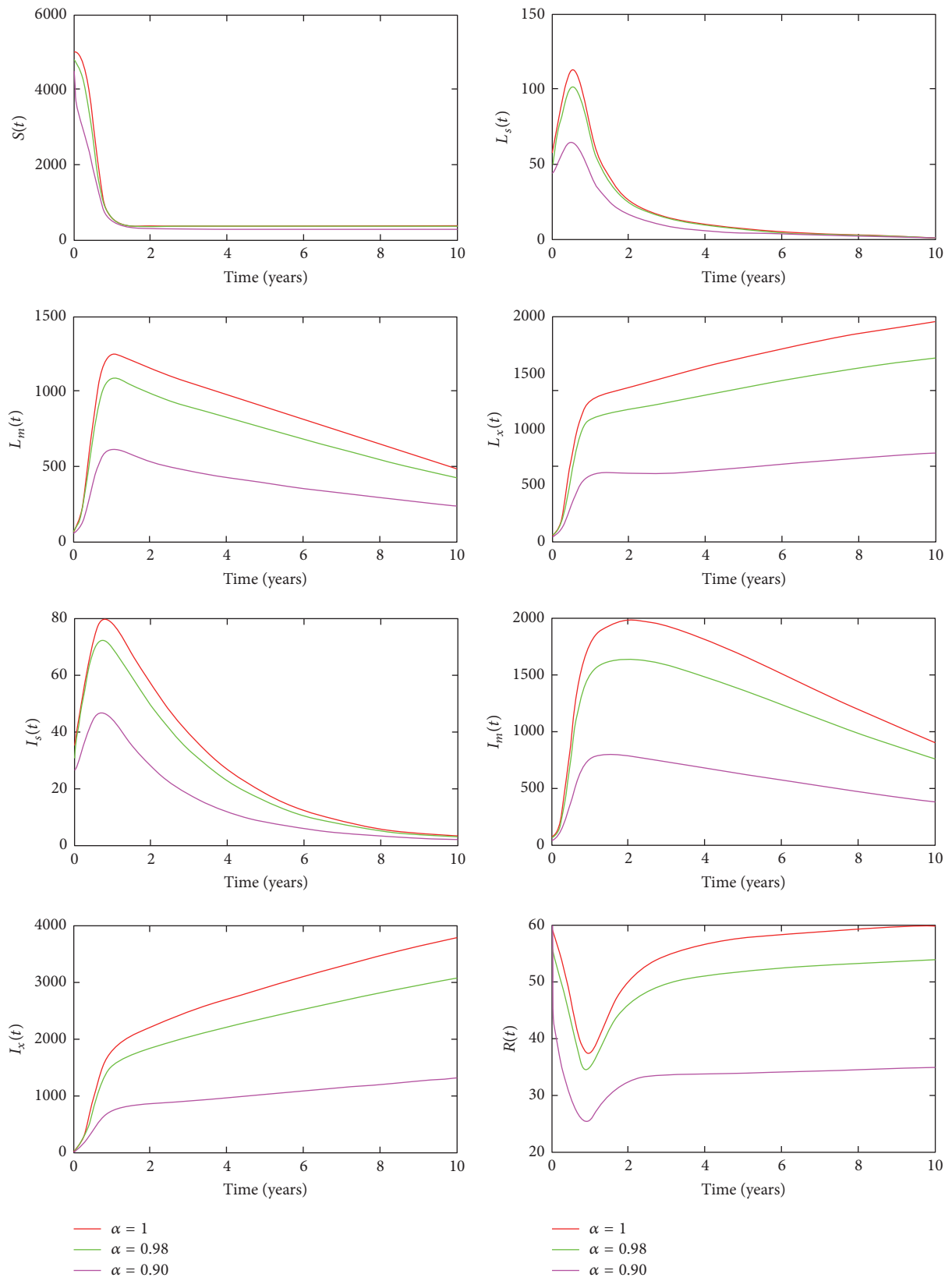


FIGURE 11: The approximate solutions with  $\tau = 0.4$  and different  $\alpha$  using NSFDM.

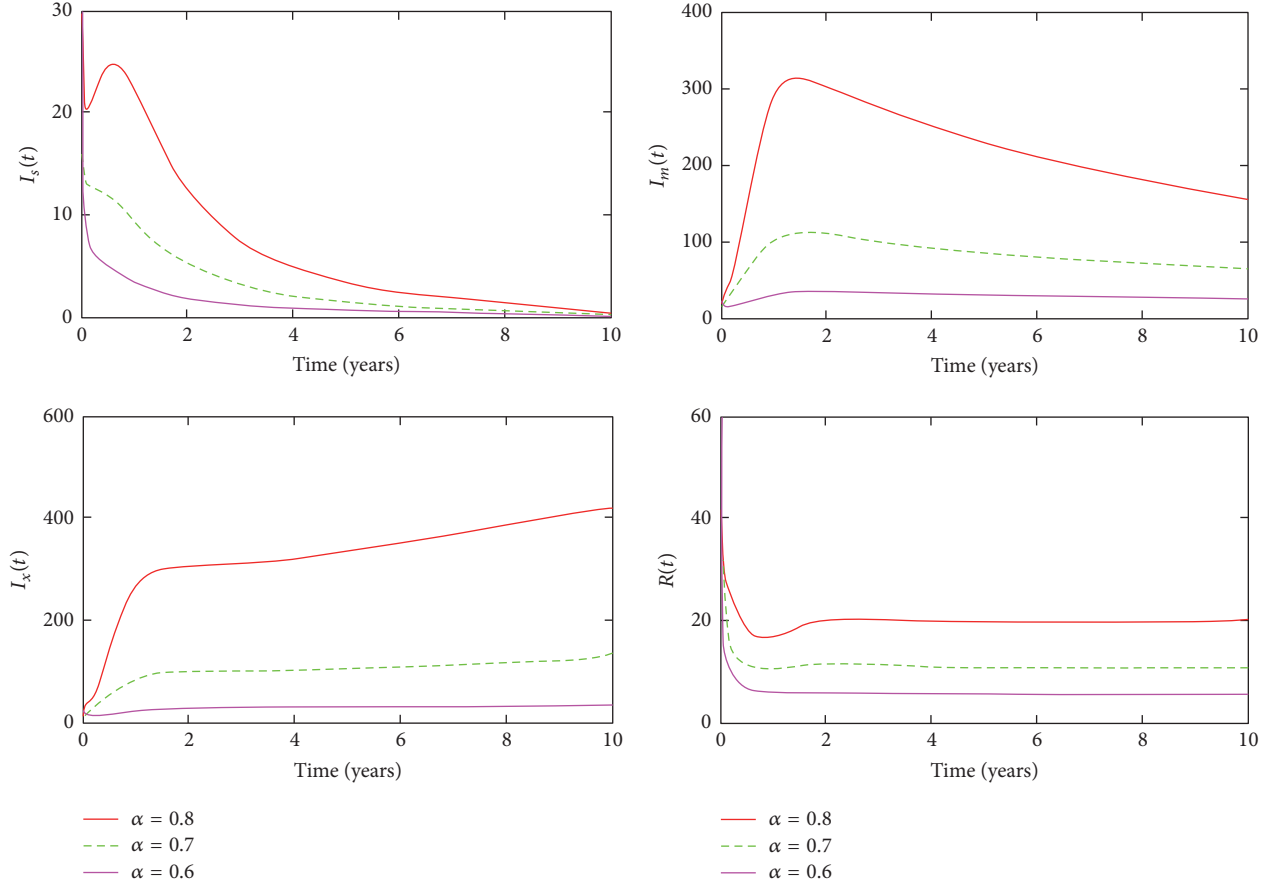


FIGURE 12: The approximate solutions with  $\tau = 0.4$  and different  $\alpha$  using NSFDM.

we introduce the main aspects concerning nonstandard discretization methods.

*A.1. Grünwald–Letnikov Fractional Derivatives (GLFDs).* We will begin with the signal fractional differential equation (see [17, 40, 41])

$$\begin{aligned} D_x^\alpha z(x) &= g(x, z(x)), \quad T \geq x \geq 0, \\ z(x_0) &= 0, \end{aligned} \quad (\text{A.1})$$

where  $\alpha > 0$ .  $T$  is the final time and  $D^\alpha$  denotes the fractional derivative, where  $n - 1 < \alpha < n$ , defined by

$$D_x^\alpha z(x) = J^{n-\alpha} D_x^n z(x), \quad (\text{A.2})$$

$\forall n \in \mathbb{N}$  and  $J^n$  is the  $n$ th-order Riemann–Liouville integral operator:

$$J^n z(x) = \frac{1}{\Gamma(x)} \int_0^x (x-\tau)^{n-1} z(\tau) d\tau, \quad (\text{A.3})$$

with  $\Gamma(\cdot)$  being the gamma function,

and  $x > 0$ . The Grünwald–Letnikov approximation of the fractional derivative is defined as follows [42]:

$$D_x^\alpha z(x) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{r=0}^m (-1)^r \binom{\alpha}{r} z(x-rh), \quad (\text{A.4})$$

where  $m = [x/h]$  denotes the integer part of  $x/h$  and  $h$  is the step-size. Equation (A.4) can be discretized as follows:

$$\sum_{r=0}^m \omega_r^\alpha z(x_{n-r}) = g(x_n, z(x_n)) \quad n = 1, 2, 3, \dots, \quad (\text{A.5})$$

where  $x_n = nh$  and  $\omega_r^\alpha$  are the Grünwald–Letnikov coefficients defined as

$$\omega_r^\alpha = \left(1 - \frac{1+\alpha}{r}\right) \omega_{r-1}^\alpha, \quad \omega_0^\alpha = h^{-\alpha}, \quad r = 1, 2, 3, \dots \quad (\text{A.6})$$

*A.2. NSFDM Discretization.* It is known that the numerical scheme is called nonstandard method if at least one of the following conditions is satisfied [36]:

- (1) the discretization of derivatives is not traditional and uses a nonnegative function [35, 43],
- (2) nonlocal approximations are used.

To construct the numerical scheme for system (1) using NSFDM, the approximations of temporal derivatives are made based on generalized forward scheme of first order. Hence, if  $g(t) \in C^1(\mathbb{R})$ , we define its derivative as follows:

$$\frac{dg(t)}{dt} = \frac{g(t + \Delta t) - g(t)}{\varphi(\Delta t)} + O(\varphi(\Delta t)), \quad (\text{A.7})$$

as  $\Delta t \rightarrow 0$ ,

where  $\varphi(\Delta t)$  is a real-valued function on  $\mathbb{R}$  and  $\Delta t = h$ . In the following, the denominator functions are little complex functions of the step-size of time than the classical one [44].

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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