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Research Article

Deformed Entropic and Information Inequalities for X-States of Two-Qubit and Single Qudit States

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The q-deformed entropies of quantum and classical systems are discussed. Standard and q-deformed entropic inequalities for X-states of the two-qubit system and the state of single qudit with $j = 3/2$ are presented.

1. Introduction

Quantum correlations of bipartite qudit systems are characterized, for example, by entropic inequalities [1–4] written for von Neumann entropies [5] of the system and its subsystems [6]. q-deformed entropies were introduced in [7, 8]. These entropies being functions of an extra parameter contain more detailed information on properties of density matrices of the qudit states and the qudit subsystem states. The Tsallis entropy of a bipartite qudit system was shown to satisfy the generalized subadditivity condition [9, 10]. This condition is the inequality available for Tsallis entropy of the bipartite system state and Tsallis entropies of two subsystem states. The Tsallis entropy is often used in finite-size or especially correlated systems. The properties and applications of the Tsallis entropy to describe the systems containing a large number of elements were discussed, for example, in [11]. In the approach of [12–16] it was shown that the relations for composite system state can be extended to be valid for noncomposite systems, for example, for the single qudit state. These inequalities reflect some quantum correlation properties of degrees of freedom of either of subsystems (in the case of bipartite system) or degrees of freedom of the single qudit in the case of noncomposite system states [17]. One of the important states of the two-qubit systems is X-states. The properties of these states were studied, for example, in [16, 18, 19]. The partial case of the X-state is the Werner state [20]. Entanglement properties of the Werner state were studied in detail, for example, in [21].

The aim of our work is to obtain a new deformed entropic inequality for X-state of composite (bipartite) and noncomposite (single qudit with $j = 3/2$) quantum systems. We consider both Rényi and Tsallis entropic inequalities.

The paper is constructed as follows. In Section 2 we review the notion of Rényi and Tsallis entropies for bipartite systems. In Section 3 we obtain the new Tsallis entropic inequalities for X-state of noncomposite quantum system. The latter entropic inequality is illustrated by an example of the Werner state of the single qudit.

2. Rényi and Tsallis Entropies

Let us introduce the quantum state in the Hilbert space $\mathscr{H}$ defined by the following density matrix:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \quad \operatorname{Tr}(\rho) = 1, \ \rho = \rho^{\dagger}, \ \rho \geq 0. \tag{1}$$

If we apply the invertible map of indices $1 \leftrightarrow 1/2\ 1/2$; $2 \leftrightarrow 1/2\ -1/2$; $3 \leftrightarrow -1/2\ 1/2$; and $4 \leftrightarrow -1/2\ -1/2$ then the latter matrix can describe the two-qubit state. This provides the possibility to construct reduced density operators $\rho_1 = \operatorname{Tr}_2 \rho(1,2)$ and $\rho_2 = \operatorname{Tr}_1 \rho(1,2)$ which describe the states of the subsystems 1 and 2, respectively. Applying another invertible map of indices $1 \leftrightarrow 3/2$, $2 \leftrightarrow 1/2$, $3 \leftrightarrow -1/2$, and $4 \leftrightarrow -3/2$, the density matrix (1) can be rewritten so that it can describe the noncomposite system of the single qudit with $j = 3/2$. Hence it is possible to use the density matrix in form (1) to describe both bipartite systems as well as systems without subsystems. This idea to use invertible map of integers $1, 2, 3, \ldots$ onto pairs (triples, etc.) of integers (i, k), $i, k = 1, 2, \ldots$, to formulate the quantum properties of systems without subsystems was applied in [12–16]. That gives us possibility to translate known properties of quantum correlations associated with the structure of the bipartite system like the entanglement to the system without subsystems, for example, single qudit.

An important measure of the entanglement is the entropy. The most known is the von Neumann entropy. It is obtained by

$$S_N = -\operatorname{Tr} \rho \ln \rho. \tag{2}$$

More flexible are Tsallis and Rényi entropies. The Rényi entropy generalizes the Shannon entropy, the Hartley entropy, the min-entropy, and the collision entropy. Both the Tsallis and Rényi entropies depend on extra parameter q; thus they are called q-entropies. The classical q-entropies of the probability vector, constructed from the diagonal elements of the density matrix (1) $\vec{p} = (p_1 = \rho_{11}, p_2 = \rho_{22}, p_3 = \rho_{33}, p_4 = \rho_{44})$, are

$$S_q^T = \frac{1}{1-q}\left(\sum_{i=1}^{4} p_i^q - 1\right), \qquad S_q^R = \frac{1}{1-q}\ln\left(\sum_{i=1}^{4} p_i^q\right). \tag{3}$$

When $q \to 1$ holds, S_q^T reduces to the von Neumann entropy. Tsallis and Rényi entropies can be rewritten in the following forms:

$$S_q^T = -\operatorname{Tr} \rho \ln_q \rho, \qquad S_q^R = \frac{1}{1-q}\ln\left(\operatorname{Tr} \rho^q\right), \tag{4}$$

where

$$\ln_q \rho = \begin{cases} \dfrac{\rho^{q-1} - I}{q-1}, & \text{if } q \neq 1, \\ \ln \rho, & \text{if } q = 1, \end{cases} \tag{5}$$

for any real $q > 0$, where I is identity matrix. Logarithm (5) is called the q-logarithm or the deformed logarithm. Relations between Tsallis and Rényi entropies are given by the following formulas:

$$S_q^T = \frac{\exp\left(S_q^R (1-q)\right) - 1}{1-q}, \qquad S_q^R = \frac{\ln\left(1 + (1-q) S_q^T\right)}{1-q}. \tag{6}$$

If the density matrix (1) describes the bipartite state (the two-qubit system), then we can consider two subsystems on spaces $\mathscr{H}^1$ and $\mathscr{H}^2$ such that $\mathscr{H} = \mathscr{H}^1 \otimes \mathscr{H}^2$. Reduced density matrices ρ_1, ρ_2 are defined as partial traces of (1). Resulting matrices are density matrices of the density operators acting on spaces $\mathscr{H}^1$ and $\mathscr{H}^2$, respectively. Thus the reduced density matrices of the first and the second qubit are defined as

$$\rho_1 = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix}. \tag{7}$$

It is well known that the von Neumann entropy is subadditive. In [9] was proved the subadditivity of the Tsallis entropy for $q > 1$ of the composite system; namely,

$$S_q^T(\rho) \leq S_q^T(\rho_1) + S_q^T(\rho_2). \tag{8}$$

There are other entropic inequalities, for example, the strong subadditivity condition in [1], which holds for the von Neumann entropy of three-partite quantum system. The fact that the Tsallis entropy is not strong subadditive was recently proved in [10]. Let us define the q-information as

$$I_q^T = S_q^T(\rho_1) + S_q^T(\rho_2) - S_q^T(\rho) \geq 0. \tag{9}$$

The subadditivity condition for the Tsallis entropy provides the inequality for the Rényi entropy as follows:

$$\exp\left(S_q^R(\rho_1)(1-q)\right) + \exp\left(S_q^R(\rho_2)(1-q)\right) - \exp\left(S_q^R(\rho)(1-q)\right) < 1. \tag{10}$$

Since Rényi and Tsallis entropies tend to the von Neumann entropy for $q \to 1$, both inequalities (9) and (10) in this limit give the standard positivity condition of the von Neumann mutual information.

3. The Tsallis Entropy for the X-State

Using the invertible mapping $1 \leftrightarrow 3/2$, $2 \leftrightarrow 1/2$, $3 \leftrightarrow -1/2$, and $4 \leftrightarrow -3/2$, the density matrix (1) can be rewritten as

$$\rho_{3/2} = \begin{pmatrix} \rho_{3/2,3/2} & \rho_{3/2,1/2} & \rho_{3/2,-1/2} & \rho_{3/2,-3/2} \\ \rho_{1/2,3/2} & \rho_{1/2,1/2} & \rho_{1/2,-1/2} & \rho_{1/2,-3/2} \\ \rho_{-1/2,3/2} & \rho_{-1/2,1/2} & \rho_{-1/2,-1/2} & \rho_{-1/2,-3/2} \\ \rho_{-3/2,3/2} & \rho_{-3/2,1/2} & \rho_{-3/2,-1/2} & \rho_{-3/2,-3/2} \end{pmatrix}. \tag{11}$$

This matrix is a density matrix of the single qudit state with spin $j = 3/2$. Such system has no subsystems; thus it is impossible to write its reduced density matrices. But using form (1) we can successfully write them. If $\rho_{12} = \rho_{13} = \rho_{21} = \rho_{31} = \rho_{24} = \rho_{34} = \rho_{42} = \rho_{43} = 0$ holds, then the density matrix (1) has the view of the X-state density matrix. Consider

$$\rho^X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{12}$$

where ρ_{11}, ρ_{22}, ρ_{33}, and ρ_{44} are positive reals and ρ_{23}, ρ_{14} are complex quantities. The latter matrix has the unit trace and it is nonnegative if $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$, $\rho_{11}\rho_{44} \geq |\rho_{14}|^2$ hold. The reduced density matrices are defined as

$$\rho_1 = \begin{pmatrix} \rho_{11} + \rho_{22} & 0 \\ 0 & \rho_{33} + \rho_{44} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \rho_{11} + \rho_{33} & 0 \\ 0 & \rho_{22} + \rho_{44} \end{pmatrix}. \tag{13}$$

Hence, the q-information (9) for the X-state of the single qudit is

$$\begin{aligned} I_q^T = \frac{1}{1-q} \Big(& (\rho_{11} + \rho_{22}) \left((\rho_{11} + \rho_{22})^{q-1} - 1 \right) \\ & + (\rho_{11} + \rho_{33}) \left((\rho_{11} + \rho_{33})^{q-1} - 1 \right) \\ & + (\rho_{22} + \rho_{44}) \left((\rho_{22} + \rho_{44})^{q-1} - 1 \right) \\ & + (\rho_{33} + \rho_{44}) \left((\rho_{33} + \rho_{44})^{q-1} - 1 \right) \\ & - (\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44}) \left(\rho^{q-1} - 1 \right) \Big) \geq 0. \end{aligned} \tag{14}$$

As an example of the X-state density matrix of the qudit state with spin $j = 3/2$ the Werner state matrix can be taken. Consider

$$\rho^W = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}, \tag{15}$$

where the parameter p satisfies the inequality $-1/3 \leq p \leq 1$. The parameter domain $1/3 < p \leq 1$ corresponds to the entangled state. The information (14) of the latter state can be seen in Figure 1. The dashed line in the point $p = 1/3$ marks the border between the separable and the entangled Werner states. One can see general behavior of the q-information against parameter p for different values of the deformation parameter q. In the domain of the entangled states the q-information increases with increasing the degree of the state entanglement. The sensitivity of the q-information to the degree of entanglement depends on the deformation parameter q.

Let us show, in the example, how the variation of the mapping impacts the correlations in the quantum system. To this end, we use the invertible mapping described by the unitary transformation $\rho \rightarrow U\rho U^\dagger$, for example, $1 \leftrightarrow 2$; $2 \leftrightarrow 3$; $3 \leftrightarrow 4$; and $4 \leftrightarrow 1$. Then the density matrix (1) can be rewritten as

$$\rho_U = \begin{pmatrix} \rho_{22} & \rho_{23} & \rho_{24} & \rho_{21} \\ \rho_{32} & \rho_{33} & \rho_{34} & \rho_{31} \\ \rho_{32} & \rho_{43} & \rho_{44} & \rho_{41} \\ \rho_{12} & \rho_{13} & \rho_{14} & \rho_{11} \end{pmatrix}. \tag{16}$$

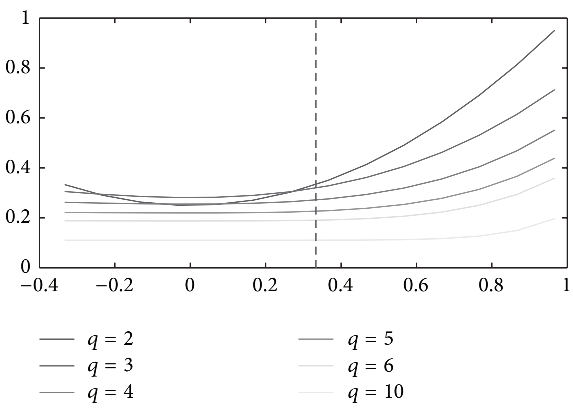

Figure 1: The q-information I_q^T of the Werner state (17) of the qudit with spin $j = 3/2$ against the parameter p for different values of the deformation parameter q.

Obviously, the full entropy of the system does not change. However, the entropies $S_q^T(\rho_1)$, $S_q^T(\rho_2)$ change due to the modification of the reduced density matrices. Hence, it is possible to find such mapping that $S_q^T(\rho_1) + S_q^T(\rho_2)$ takes minimum or maximum values.

Let us select the matrix of the Werner state with two parameters. Consider

$$\rho^W(p,b) = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{4} & b & 0 \\ 0 & b & \frac{1-p}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix} \tag{17}$$

as an example of the density matrix (1). To provide the positivity of (17) the parameters must be in the domain $-1/3 \leq p \leq 1$, $(1-p)/4 \geq |b|$. Let us select $b = (1-p)/5$. For this state the sum of the reduced entropies is

$$S_q^T(\rho_1) + S_q^T(\rho_2) = 1. \tag{18}$$

Using the introduced mapping, the matrix (17) can be rewritten as

$$\rho^W(p,b) = \begin{pmatrix} \frac{1+p}{4} & b & 0 & 0 \\ b & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & \frac{p}{2} \\ 0 & 0 & \frac{p}{2} & \frac{1+p}{4} \end{pmatrix} \tag{19}$$

and the sum of the reduced entropies is

$$S_q^T(\rho_1) + S_q^T(\rho_2) = -\frac{(p-1)(17p+23)}{25}. \tag{20}$$

The latter sum is less than (18). This example shows that the choice of the mapping may impact on the values of the reduced entropies.

4. Summary

To conclude, we point out the main results of the work. We applied the q-deformed entropies of Rényi and Tsallis as a measure of the entanglement for the systems without subsystems. New deformed entropic inequality of the X-state of the noncomposite quantum state (the qudit with spin $j = 3/2$) was obtained. As an example of the X-state, the Werner state with one parameter was taken.

Despite the fact that there are no subsystems in such systems it is possible to introduce analogs of partial traces like, for example, composite systems using special mapping illustrated in the text. Certainly, it is necessary to understand that these partial traces for the noncomposite systems do not imply the same meaning as quantum state of the composite system. The physical and probabilistic background of the correlations inside the system without subsystems will be developed in the future article of the authors.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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