GLOBAL SOLUTIONS OF SEMILINEAR HEAT EQUATIONS IN HILBERT SPACES

G. MIHAI IANCU AND M. W. WONG*

Abstract. The existence, uniqueness, regularity and asymptotic behavior of global solutions of semilinear heat equations in Hilbert spaces are studied by developing new results in the theory of one-parameter strongly continuous semigroups of bounded linear operators. Applications to special semilinear heat equations in $L^2(\mathbb{R}^n)$ governed by pseudo-differential operators are given.

1. Introduction

Let X be a complex Hilbert space in which the norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively. Let $\mathcal{S}$ be a dense subspace of X. We assume that $\mathcal{S}$ is a topological vector space in which the topology is defined by a countable family of seminorms $\{|\cdot|_j : j = 1, 2, ...\}$. A sequence $\{\varphi_k\}$ in $\mathcal{S}$ is said to converge to an element φ in $\mathcal{S}$ if and only if $|\varphi_k - \varphi|_j \to 0$ as $k \to \infty$ for all $j = 1, 2, ...$ We let $\mathcal{S}'$ be the space of all continuous linear functionals on the space $\mathcal{S}$. We denote the value of a functional u in $\mathcal{S}'$ at an element φ in $\mathcal{S}$ by $< u, \varphi >$ and define $< \varphi, u >$ to be equal to $\overline{< u, \varphi >}$. We say that a functional u is continuous if and only if $< u, \varphi_k > \to 0$ as $k \to \infty$ for all sequences $\{\varphi_k\}$ converging to zero in $\mathcal{S}$ as $k \to \infty$. A sequence $\{u_k\}$ in $\mathcal{S}'$ is said to converge to an element u in $\mathcal{S}'$ if and only if $< u_k, \varphi > \to < u, \varphi >$ as $k \to \infty$ for all φ in $\mathcal{S}$.

We assume that the space X is continuously embedded in $\mathcal{S}'$. Furthermore, let us suppose that we can find a one-parameter family of Hilbert spaces X_s with norms denoted by $\|\cdot\|_s$, $-\infty < s < \infty$, and a one-parameter

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group of continuous linear mappings $J_s : \mathcal{S}' \to \mathcal{S}'$, $-\infty < s < \infty$, satisfying the following conditions:

(i) J_s maps $\mathcal{S}$ into $\mathcal{S}$, $-\infty < s < \infty$;
(ii) $X_s = \{u \in \mathcal{S}' : J_{-s}u \in X\}$, $-\infty < s < \infty$;
(iii) $\|u\|_s = \|J_{-s}u\|$, $u \in X_s$, $-\infty < s < \infty$;
(iv) Let $s \le t$. Then $X_t \subseteq X_s$, and $\|u\|_s \le \|u\|_t$, $u \in X_t$;
(v) $|(\varphi, \psi)| \le \|\varphi\|_s \|\psi\|_{-s}$, $\varphi, \psi \in \mathcal{S}$, $-\infty < s < \infty$;
(vi) X_s can be continuously embedded in $\mathcal{S}'$, $-\infty < s < \infty$;
(vii) $\mathcal{S}$ can be continuously embedded in X_s, $-\infty < s < \infty$;
(viii) $< u, \varphi >= (u, \varphi)$, $u \in X$, $\varphi \in \mathcal{S}$.

It can be proved that $\mathcal{S}$ is dense in X_s and $J_t : X_s \to X_{s+t}$ is a unitary operator for all s and t in $(-\infty, \infty)$. Furthermore, by (v) and the density of $\mathcal{S}$ in X_s, we can define (u, v) for any $u \in X_s$ and any $v \in X_{-s}$ by an obvious limiting argument and we have

$$|(u, v)| \le \|u\|_s \|v\|_{-s}, \quad u \in X_s, v \in X_{-s}.$$

Let A be a linear operator from X into X with domain $\mathcal{S}$. The formal adjoint A^* of the operator A, if it exists, is defined to be the restriction of the true adjoint A^t of the operator A to the space $\mathcal{S}$. It is clear from the definition that the formal adjoint exists if and only if $\mathcal{S}$ is contained in the domain of A^t. We assume in this paper that the formal adjoint A^* of the operator A exists. Furthermore, we assume that A maps $\mathcal{S}$ into $\mathcal{S}$ and A^* maps $\mathcal{S}$ into $\mathcal{S}$ continuously. In other words, if $\{\varphi_k\}$ is any sequence in $\mathcal{S}$ such that $\varphi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$, then $A\varphi_k \to 0$ and $A^*\varphi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$. We can now extend the linear operator A to the space $\mathcal{S}'$ as follows: For any u in $\mathcal{S}'$, we define Au to be the element in $\mathcal{S}'$ given by

$$< Au, \varphi >=< u, A^*\varphi >, \quad \varphi \in \mathcal{S}.$$

It is easy to prove that $A : \mathcal{S}' \to \mathcal{S}'$ is a continuous linear mapping.

Let $T : \mathcal{S}' \to \mathcal{S}'$ be a continuous linear mapping. Suppose that there exists a real number m such that $T : X_s \to X_{s-m}$ is a bounded linear operator for all $s \in (-\infty, \infty)$. Then we call T an operator of order m if m is the least number for which $T : X_s \to X_{s-m}$ is a bounded linear operator. If the least number is equal to $-\infty$, then we call T an infinitely smoothing operator. If A is a linear operator from X into X with domain $\mathcal{S}$ such that A maps $\mathcal{S}$ into $\mathcal{S}$ and A^* maps $\mathcal{S}$ into $\mathcal{S}$ continuously, then we call A an operator of order m if the extended operator $A : \mathcal{S}' \to \mathcal{S}'$ is of order m.

The preceding theory and the proof of the following theorem can be found in the paper [9] by Wong.

Theorem 1.1. *Let A be a linear operator from X into X with domain $\mathcal{S}$ such that A maps $\mathcal{S}$ into $\mathcal{S}$ and its formal adjoint A^* maps $\mathcal{S}$ into $\mathcal{S}$ continuously. Suppose that A and A^* are of positive order $2m$ and there exists a linear operator B of order $-2m$ such that BA=I+R, where I is the identity operator and R is an infinitely smoothing operator. Then A has a unique* *closed extension A_0 such that $\mathcal{S}$ is contained in the domain of A_0^t, the domain of A_0 is equal to X_{2m}, and*

$$Au = A_0 u, \quad u \in X_{2m}.$$

The following theorem is the basis for the existence, uniqueness and regularity of global solutions of semilinear heat equations in Hilbert spaces. In fact, its hypotheses, used throughout the paper, give us valuable information about the asymptotic behavior of these global solutions.

Theorem 1.2. *In addition to the hypotheses of Theorem 1.1, we assume that there exist constants C and λ_0 ($C > 0$ and $\lambda_0 \geq 0$) such that*

$$Re(-A\varphi, \varphi) \geq C\|\varphi\|_m^2 - \lambda_0 \|\varphi\|^2, \quad \varphi \in \mathcal{S}. \tag{1.1}$$

Then A_0 is the infinitesimal generator of a C_0 semigroup of bounded linear operators on X.

We prove a spectral result in Section 2. Based on this result, the proof of Theorem 1.2 is given in Section 3. A well-known corollary on the existence, uniqueness and regularity of global solutions of semilinear heat equations in Hilbert spaces is stated in Remark 3.3. Theorems on the asymptotic stability of the equilibrium solutions and on the existence of absorbing sets for global solutions of semilinear heat equations are given in Sections 4 and 5. Applications to semilinear heat equations in $L^2(\mathbb{R}^n)$ governed by pseudo-differential operators are given in Section 6.

The impetus for the study of semilinear heat equations in Hilbert spaces carried out in this paper stems from our desire to obtain a better understanding and a more unified treatment of some of the technical results in the papers [10, 11, 12] of Wong by generalizing the semilinear evolution equations therein to similar equations in Hilbert spaces. The results obtained in this paper are illuminating and have applications to ordinary, partial and pseudo-differential equations arising in various disciplines in science and engineering.

Semilinear heat equations modelled by specific ordinary and partial differential operators have been studied extensively in, e.g., [1, 2] by Bellani-Morante and [7] by Tanabe.

2. A result in spectral theory

The following result in spectral theory will be used frequently in this paper.

Theorem 2.1. *In addition to the hypotheses of Theorem 1.1, we assume that there exists a positive constant C such that*

$$Re\,(A\varphi, \varphi) \geq C\|\varphi\|_m^2, \quad \varphi \in \mathcal{S}. \tag{2.1}$$

Then $0 \in \rho(A_0)$, where $\rho(A_0)$ is the resolvent set of A_0.

Proof. By (2.1) and the Schwarz inequality,

$$\|A\varphi\| \geq C\|\varphi\|, \quad \varphi \in \mathcal{S},$$

and hence, by a limiting argument, we get

$$\|A_0 u\| \geq C\|u\|, \quad u \in \mathcal{D}(A_0). \tag{2.2}$$

Next, let $a : X_m \times X_m \to \mathbb{C}$ be the bilinear form defined by

$$a(u, v) = (u, A^* v), \quad u, v \in X_m.$$

Since A^* is of order $2m$, it follows from (v) in Section 1 that there exists a positive constant C' such that

$$|a(u, v)| \leq C'\|u\|_m \|v\|_m, \quad u, v \in X_m.$$

Furthermore, by (2.1) and a limiting argument,

$$|a(u, u)| = |(u, A^* u)| \geq Re(u, A^* u) \geq C\|u\|_m^2, \quad u \in X_m.$$

So, for any $f \in X_m$, the Lax-Milgram theorem on page 26 of the book [7] by Tanabe can be used to ensure the existence of an element u in X_m such that

$$a(u, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}.$$

Thus,

$$(u, A^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}.$$

So, u is in the domain of the maximal operator A_1 of A and $A_1 u = f$. But, using the fact proved in the paper [9] by Wong that $A_0 = A_1$, we can conclude that $u \in \mathcal{D}(A_0)$ and $A_0 u = f$. Thus, the range of A_0 is dense in X, and hence, by (2.2), $0 \in \rho(A_0)$. This completes the proof of the theorem. ∎

3. Proof of theorem 1.2

In order to prove Theorem 1.2, we need some preparation. We begin by proving the following lemmas.

Lemma 3.1. *Let A be as in Theorem 1.1. Let $f \in X_s, -\infty < s < \infty$. Then any solution u in $\bigcup_{t \in \mathbb{R}} X_t$ of the equation $Au = f$ is in X_{s+2m}.*

Proof. Since $BA = I + R$, it follows that

$$u = BAu - Ru = Bf - Ru. \tag{3.1}$$

Using the fact that B is of order $-2m$ and f is in X_s, we conclude that Bf is in X_{s+2m}. Since R is infinitely smoothing, it follows that Ru lies in X_t for any $t \in (-\infty, \infty)$ and, in particular, in X_{s+2m}. Hence, by (3.1), u is in X_{s+2m}. ∎

Lemma 3.2. *Let $\lambda > \lambda_0$. Then, for every $f \in X$, there exists a unique solution u in X_{2m} such that $(\lambda I - A)u = f$. Moreover,*

$$\|(\lambda I - A_0)u\| \geq (\lambda - \lambda_0)\|u\|, \;\; u \in X_{2m}. \tag{3.2}$$

Proof. By (1.1), for $\lambda > \lambda_0$,

$$\begin{aligned} Re\,((\lambda I - A)\varphi, \varphi) &= Re\,((\lambda_0 I - A)\varphi, \varphi) + (\lambda - \lambda_0)\|\varphi\|^2 \\ &\geq C\|\varphi\|_m^2 + (\lambda - \lambda_0)\|\varphi\|^2 \\ &\geq (\lambda - \lambda_0)\|\varphi\|^2, \quad \varphi \in \mathcal{S}. \end{aligned}$$

So, by a limiting argument, we get

$$Re\,((\lambda I - A_0)u, u) \geq (\lambda - \lambda_0)\|u\|^2, \quad u \in X_{2m}.$$

Thus, by Theorem 2.1, we have, for any f in X, a unique u in X_{2m} for which $(\lambda I - A_0)u = f$. Moreover,

$$\begin{aligned}\|(\lambda I - A_0)\varphi\|^2 &= \|(\lambda_0 I - A_0)\varphi + (\lambda - \lambda_0)\varphi\|^2 \\ &= (\lambda - \lambda_0)^2\|\varphi\|^2 + 2(\lambda - \lambda_0)Re\,((\lambda_0 I - A_0)\varphi, \varphi) \\ &\quad + \|(\lambda_0 I - A_0)\varphi\|^2 \\ &\geq (\lambda - \lambda_0)^2\|\varphi\|^2, \quad \varphi \in \mathcal{S}.\end{aligned}$$

Thus, by a standard limiting argument again, we obtain

$$\|(\lambda I - A_0)u\| \geq (\lambda - \lambda_0)\|u\|, \quad \lambda > \lambda_0, u \in X_{2m},$$

and the proof is complete. ∎

Proof of Theorem 1.2. A_0 is a closed and densely defined linear operator from X into X. By Lemma 3.2, $(\lambda I - A_0)^{-1}$ exists for $\lambda > \lambda_0$, and for such values of λ, (3.2) implies that the operator norm of $(\lambda I - A_0)^{-1}$ is at most $(\lambda - \lambda_0)^{-1}$. Hence, by the Hille-Yosida-Phillips theorem, the proof is complete. ∎

Remark 3.3. *It is well-known from the general theory of semilinear evolution equations that the nonhomogeneous initial value problem for the semilinear heat equation*

$$\begin{cases} u'(t) = A\{u(t)\} + F(u(t)), & t > 0, \\ u(0) = f, \end{cases}$$

has a unique solution in the intersection of $C([0,\infty), X_{2m})$ *and* $C^1([0,\infty), X)$ *for every initial value* $f \in \mathcal{D}(A_0)$ *if* F *is Lipschitz continuous and continuously Fréchet differentiable. For details, see Chapter 6 of the book* [6] *by Pazy. In this paper, we assume the existence and uniqueness of global solutions under the conditions on* F *stipulated above.*

In Remark 3.3 and in the sequel, the derivative $u'(t)$, at any time t, is understood to be the strong limit in X (if it exists) of the difference quotient

$$\frac{u(t+h) - u(t)}{h}$$

as $h \to 0$.

4. Asymptotic stability

The main result of this section is the following theorem.

Theorem 4.1. *Under the hypotheses of Theorem 1.2, we denote by* $s(A_0)$ *the supremum of the real part of the spectrum* $\Sigma(A_0)$ *of* A_0*, i.e.,* $s(A_0) = sup\, Re\Sigma(A_0)$*, and let* F *be a continuous mapping from* X *into* X *such that*

$$\lim_{\|u\|\to 0} \frac{\|F(u)\|}{\|u\|} = 0.$$

Then, for $\lambda > s(A_0)$, the equilibrium solution $u(t) \equiv 0$ of

$$(*) \qquad u'(t) + (\lambda I - A)\{u(t)\} = F\{u(t)\}, \quad t > 0,$$

is asymptotically stable.

Proof. We have already seen in Section 3 that A_0 is the infinitesimal generator of a C_0 semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of bounded linear operators on X. We will show in the following that the C_0 semigroup is in fact an analytic semigroup. But the analyticity of $\{\mathcal{T}(t)\}_{t\geq 0}$ implies that we have the spectral mapping theorem for the semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, i.e.,

$$\Sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\Sigma(A_0)}, \quad t \geq 0.$$

Thus, by Theorem 1.22 on page 15 of the book [3] by Davies,

$$\lim_{t\to\infty} \frac{1}{t} \ln \|e^{A_0 t}\| \leq s(A_0).$$

Thus, for any number ε in $(0, \lambda - s(A_0))$, we can find a positive number t_ε such that

$$\frac{1}{t} \ln \|e^{A_0 t}\| < s(A_0) + \varepsilon, \quad t > t_\varepsilon.$$

Hence

$$\|e^{A_0 t}\| < e^{(s(A_0)+\varepsilon)t}, \quad t > t_\varepsilon.$$

Thus,

$$\|e^{(-\lambda I + A_0)t}\| < e^{-(\lambda - s(A_0) - \varepsilon)t}, \quad t > t_\varepsilon.$$

Therefore we can find a positive constant M such that

$$\|e^{(-\lambda I + A_0)t}\| < M\, e^{-(\lambda - s(A_0) - \varepsilon)t}, \quad t \geq 0.$$

Hence the proof of Theorem 2.1 in the paper [11] by Wong can be used to conclude that the equilibrium solution $u(t) \equiv 0$ of $(*)$ is asymptotically stable, and the proof is complete modulo the proof that $\{\mathcal{T}(t)\}_{t\geq 0}$ is analytic. To complete the proof of the theorem, let $\tilde{A}_0 = A_0 - \lambda_0 I$. Using the hypotheses of Theorem 1.2, we have

$$(4.1) \qquad Re\,(-\tilde{A}_0\varphi, \varphi) \geq C\|\varphi\|_m^2, \quad \varphi \in \mathcal{S}.$$

Also,

$$(4.2) \qquad |Im(-\tilde{A}_0\varphi, \varphi)| \leq |(-\tilde{A}_0\varphi, \varphi)| \leq K\|\varphi\|_m^2, \quad \varphi \in \mathcal{S},$$

for some constant $K > 0$. By a standard limiting argument, we conclude from (4.1) and (4.2) that the numerical range $S(-\tilde{A}_0)$ of $-\tilde{A}_0$ is contained in the sector S_{θ_1} given by

$$S_{\theta_1} = \{\lambda \in \mathbb{C} \,:\, -\theta_1 < arg\,\lambda < \theta_1\},$$

where $\theta_1 = tan^{-1}(\frac{K}{C}) < \frac{\pi}{2}$. We choose θ such that $\theta_1 < \theta < \frac{\pi}{2}$ and define

$$\Sigma'_\theta = \{\lambda \in \mathbb{C} \,:\, |arg\,\lambda| > \theta\}.$$

Then, for all $\lambda \in \Sigma'_\theta$, there exists a constant C_θ such that

$$d(\lambda, S(-\tilde{A}_0)) \geq C_\theta |\lambda|,$$

where $d(\lambda, S(-\tilde{A}_0))$ is the distance between λ and the set $S(-\tilde{A}_0)$. Since, by Lemma 3.2, all positive numbers μ are in the resolvent set of $\tilde{A}_0$, the set $\{\mu \in \mathbb{R} \ : \ \mu < 0\}$ is in the resolvent set of $-\tilde{A}_0$. Thus, Σ'_θ is contained in a component of the complement of the closure of S_{θ_1} which has a nonempty intersection with $\rho(-\tilde{A}_0)$.Then, by Theorem 3.9 on page 12 of the book [6] by Pazy, $\rho(-\tilde{A}_0) \supset \Sigma'_\theta$. Now, for all $\lambda \in \Sigma'_\theta$,

$$\|R(\lambda; -\tilde{A}_0)\| \leq d\left(\lambda\ ,\ \overline{S(-\tilde{A}_0)}\right)^{-1} \leq \frac{1}{C_\theta|\lambda|},$$

where $\|R(\lambda; -\tilde{A}_0)\|$ is the operator norm of the resolvent of $-\tilde{A}_0$ at λ. Therefore $\tilde{A}_0$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on X. But $\tilde{A}_0 = A_0 - \lambda_0 I$, and $\lambda_0 I$ is a bounded linear operator, which finally implies that A_0 is the infinitesimal generator of an analytic semigroup of bounded linear operators on X. ∎

5. Existence of an absorbing set

The existence of an absorbing set for a semilinear heat equation in a Hilbert space is guaranteed by the following theorem.

Theorem 5.1. *Under the hypotheses of Theorem 1.2, we consider the initial value problem*

$$(**) \qquad \begin{cases} u'(t) = A\{u(t)\} + F(u(t)), & t > 0, \\ u(0) = f, \end{cases}$$

where $f \in \mathcal{D}(A_0)$, and $F : X \to X$ is a mapping from X into X such that there exists a strictly positive constant M for which

$$\|F(u)\| \leq M\|u\|, \quad u \in X.$$

Assume that $C > \lambda_0 + M$, where C and λ_0 are the constants in formula (1.1). *Then, for all initial values f belonging to a bounded subset Ω of X and for all $\rho > 0$, there exists a positive number $t_0 = t_0(r, b, \rho)$ such that any global solution $u(t)$ of $(**)$ has the property that $u(t) \in B(0, \rho)$, $t > t_0$, where $b = C - \lambda_0 - M$, $B(0, \rho)$ is the open ball with centre at* 0 *and radius ρ, and r is the radius of the smallest ball with centre at* 0 *containing Ω.*

Proof. Let u be any global solution of $(**)$. Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= Re\,(u'(t), u(t)) \leq -(C - \lambda_0 - M)\|u(t)\|^2 \\ &= -b\|u(t)\|^2, \quad t \geq 0. \end{aligned}$$

Thus,

$$\frac{d}{dt}\|u(t)\|^2 + 2b\|u(t)\|^2 \leq 0, \quad t \geq 0.$$

Let $g(t) = \|u(t)\|^2 e^{2bt}$, $t \geq 0$. Then we obtain

$$g'(t) \leq 0, \quad t \geq 0.$$

So, integrating from 0 to t, we obtain

$$g(t) - g(0) \leq 0, \quad t \geq 0,$$

which is equivalent to

$$\|u(t)\| \leq \|f\| e^{-bt}, \quad t \geq 0.$$

Let $f \in \Omega$ and let $t_0 = \max\{0, \frac{1}{b}\ln(\frac{r}{\rho})\}$. Then, for $t > t_0$, $u(t)$ belongs to $B(0,\rho)$. To see this, note the following cases:

• If $r > \rho$, then $t_0 = \frac{1}{b}\ln(\frac{r}{\rho})$. So,

$$\|u(t)\|^2 \leq \|f\|^2 e^{-2bt} < r^2 e^{-2bt_0} = r^2 e^{-2\ln(\frac{r}{\rho})} = \rho^2, \quad t > t_0.$$

• If $r \leq \rho$, then $t_0 = 0$. So,

$$\|u(t)\|^2 \leq \|f\|^2 e^{-2bt} < r^2 \leq \rho^2, \quad t > t_0.$$

This concludes the proof of the theorem. ▪

Theorem 5.1 is in fact another result on the asymptotic stability of equilibrium solutions of semilinear heat equations in Hilbert spaces. It has an advantage over Theorem 4.1 in applications because the latter requires an explicit knowledge of the supremum of the real part of the spectrum of A_0 and this is usually very difficult to determine.

Next, we impose another set of conditions on F to get a result on the existence of a "universal" absorbing set for all global solutions of a semilinear heat equation in a Hilbert space.

Theorem 5.2. *Under the hypotheses of Theorem 1.2, we consider the initial value problem*

$$(**) \qquad \begin{cases} u'(t) = A\{u(t)\} + F(u(t)), & t > 0, \\ u(0) = f, \end{cases}$$

where $f \in \mathcal{D}(A_0)$, and $F : X \to X$ is a Lipschitz continuous mapping from X into X with Lipschitz constant M. Assume that $C > \lambda_0 + M$, where C and λ_0 are the constants in formula (1.1)*. Then, for all initial values f belonging to a bounded subset Ω of X and for all $\varepsilon > 0$, there exists a positive number $t_0 = t_0(r, b, \varepsilon)$ such that any global solution of* $(**)$ *has the property that $u(t) \in B(0,\rho)$, $t > t_0$, where b and r are as in Theorem 5.1, and $\rho = \frac{N}{b} + \varepsilon$, with $N = \inf_{u_0 \in X}\{M\|u_0\| + \|F(u_0)\|\}$.*

Proof. Since F is Lipschitz continuous with Lipschitz constant M, it follows that

$$\begin{aligned} Re\ (F(u), u) &= Re\ (F(u) - F(u_0) + F(u_0), u) \\ &= Re\ (F(u) - F(u_0), u) + Re\ (F(u_0), u) \\ &\leq \|F(u) - F(u_0)\|\|u\| + \|F(u_0)\|\|u\| \\ &\leq M\|u - u_0\|\|u\| + \|F(u_0)\|\|u\| \\ &\leq M\|u\|^2 + (M\|u_0\| + \|F(u_0)\|)\,\|u\|, \quad u, u_0 \in X. \end{aligned}$$

Thus,

$$Re\ (F(u(t)), u(t)) \leq M\|u(t)\|^2 + N\|u(t)\|, \quad t \geq 0.$$

Therefore

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= Re\ (u'(t), u(t)) = Re\ (A\{u(t)\} + F(u(t)), u(t)) \\ &= -Re\ (-A\{u(t)\}, u(t)) + Re\ (F(u(t)), u(t)) \\ &\leq -(C - \lambda_0)\|u(t)\|^2 + M\|u(t)\|^2 + N\|u(t)\| \\ &= -b\|u(t)\|^2 + N\|u(t)\|, \quad t \geq 0. \end{aligned} \tag{5.1}$$

Let $x(t) = \|u(t)\|^2$, $t \geq 0$. Then (5.1) becomes

$$x'(t) \leq -2b\, x(t) + 2N\sqrt{x(t)}, \quad t \geq 0. \tag{5.2}$$

So, if we let $g(t) = \sqrt{x(t)}e^{bt}$, $t \geq 0$, then, by (5.2), we get

$$g'(t) \leq Ne^{bt}, \quad t \geq 0.$$

Integration from 0 to t yields

$$g(t) - g(0) \leq \frac{1}{b}N\left(e^{bt} - 1\right), \quad t \geq 0.$$

Thus,

$$\|u(t)\| \leq \|f\|e^{-bt} + \frac{N}{b}\left(1 - e^{-bt}\right), \quad t \geq 0.$$

Now, let $t_0 = \max\{0, \frac{1}{b}\ln\frac{r}{\varepsilon}\}$. Then, for $t > t_0$, $u(t)$ belongs to $B(0, \rho)$. To see this, note the following cases:

- If $r > \varepsilon$, then $t_0 = \frac{1}{b}\ \ln\frac{r}{\varepsilon}$, and, for $t > t_0$,

$$\begin{aligned} \|u(t)\| &\leq \|f\|e^{-bt} + \frac{N}{b}\left(1 - e^{-bt}\right) < re^{-bt} + \frac{N}{b} \\ &< re^{-bt_0} + \frac{N}{b} = re^{-\ln\frac{r}{\varepsilon}} + \frac{N}{b} \\ &= \varepsilon + \frac{N}{b} = \rho. \end{aligned}$$

- If $r \leq \varepsilon$, then $t_0 = 0$, and, for $t > 0$,

$$\begin{aligned} \|u(t)\| &\leq \|f\|e^{-bt} + \frac{N}{b}\left(1 - e^{-bt}\right) < \|f\| + \frac{N}{b} \\ &\leq r + \frac{N}{b} \leq \varepsilon + \frac{N}{b} = \rho. \end{aligned}$$

This completes the proof of the theorem. ∎

6. An Application to pseudo-differential operators

The existence, uniqueness and regularity of the dynamics of semilinear systems modelled by pseudo-differential operators are established. The asymptotic stability of the zero equilibrium solutions and the existence of absorbing sets for global solutions of semilinear heat equations governed by pseudo-differential operators are also formulated in this section.

Let $m > 0$. We define S^m to be the set of all C^∞ functions σ on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \quad x,\xi \in \mathbb{R}^n.$$

We call any function in S^m a symbol of order m. Let $\sigma \in S^m$. Then the pseudo-differential operator T_σ is defined on the Schwartz space $\mathcal{S}$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)d\xi, \quad x \in \mathbb{R}^n,$$

for all functions φ in $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n.$$

It is easy to prove that T_σ maps $\mathcal{S}$ into $\mathcal{S}$. Hence we can consider T_σ as a linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with dense domain $\mathcal{S}$. It is well-known that the formal adjoint T_σ^* of T_σ exists and is also a pseudo-differential operator with symbol in S^m. Using the formal adjoint, we can extend $T_\sigma : \mathcal{S} \to \mathcal{S}$ to a continuous linear mapping from $\mathcal{S}'$ into $\mathcal{S}'$, where $\mathcal{S}'$ is the space of all tempered distributions. It is also well-known that $T_\sigma : H^{s,2} \to H^{s-m,2}$ is a bounded linear operator for $-\infty < s < \infty$, where

$$H^{s,2} = \left\{ u \in \mathcal{S}' : (1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}^n) \right\}$$

and $H^{s,2}$ is a Hilbert space with norm $\| \|_{s,2}$ given by

$$\|u\|_{s,2} = \left\{ \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right\}^{\frac{1}{2}}, \quad u \in H^{s,2}.$$

The one-parameter family of spaces $H^{s,2}$, indexed by s, $-\infty < s < \infty$, satisfies the conditions $(i)-(viii)$ in Section 1 provided that J_s is chosen to be the pseudo-differential operator of which the symbol is given by

$$\sigma_s(\xi) = (1+|\xi|^2)^{-\frac{s}{2}}, \quad \xi \in \mathbb{R}^n.$$

Let $\sigma \in S^m$, $m > 0$. Then we call σ a strongly elliptic symbol of order m if there exist positive constants C and R such that

$$Re\, \sigma(x,\xi) \ge C(1+|\xi|)^m, \quad |\xi| \ge R.$$

We call σ an elliptic symbol of order m if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \ge C(1+|\xi|)^m, \quad |\xi| \ge R.$$

It is obvious that strong ellipticity implies ellipticity. It is also a well-known fact that for any elliptic symbol $\sigma \in \mathcal{S}^m$, $m > 0$, we can find a symbol τ in S^{-m} such that

$$T_\sigma T_\tau = I + R$$

and

$$T_\tau T_\sigma = I + S,$$

where I is the identity operator, and R and S are pseudo-differential operators with symbols in $\bigcap_{k\in\mathbb{R}} S^k$. The abovementioned results concerning pseudo-differential operators can be found in the book [8] by Wong.

For any symbol σ in S^m, $m > 0$, we can define another linear operator W_σ associated with σ on the Schwartz space $\mathcal{S}$ by

$$(W_\sigma\varphi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma\left(\frac{x+y}{2},\xi\right)\varphi(y)dyd\xi, \quad x \in \mathbb{R}^n,$$

for all functions φ in $\mathcal{S}$, where the integral is an oscillatory integral. See Section 6 in Chapter 1 of the book [5] by Kumano-go for a discussion of oscillatory integrals. We call W_σ the pseudo-differential operator of the Weyl type with symbol σ.

Let $\sigma \in S^m$, $m > 0$, be strongly elliptic. Then it is a well-known fact that for any positive numbers a and ε, there exists a positive constant $C_{a\varepsilon}$ such that

$$Re\ (W_\sigma\varphi,\varphi) \geq (C-\varepsilon)\|\varphi\|^2_{\frac{m}{2}} - C_{a\varepsilon}\|\varphi\|^2_{\frac{m-a}{2}}, \quad \varphi \in \mathcal{S}, \tag{6.1}$$

where C is the constant in the strong ellipticity condition for σ. The proof of (6.1), also known as the Garding inequality, can be found in Chapter 2 of the book [4] by Folland. It can be proved that there is a one to one correspondence between the pseudo-differential operators of the Weyl type and the pseudo-differential operators. Moreover, if T_σ is a pseudo-differential operator with symbol σ, then $T_\sigma - W_\sigma$ is a pseudo-differential operator of the Weyl type of which the symbol is in S^{m-1}. See the book [4] by Folland for a discussion of the connection between pseudo-differential operators T_σ and W_σ.

The following theorem is a direct consequence of Theorem 1.1.

Theorem 6.1. *Let $\sigma \in S^m$, $m > 0$, be elliptic. Then the pseudo-differential operator T_σ from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with dense domain $\mathcal{S}$ has a unique closed extension $T_{\sigma 0}$ such that $\mathcal{S}$ is contained in the domain of $(T_{\sigma 0})^t$, the domain of $T_{\sigma 0}$ is equal to $H^{m,2}$, and*

$$T_\sigma u = T_{\sigma 0}u, \quad u \in H^{m,2}.$$

Let us first consider the initial value problem for the semilinear heat equation corresponding to the pseudo-differential operator T_σ, i.e.,

$$\begin{cases} u'(t) = -T_\sigma\{u(t)\} + F(u(t)), & t > 0, \\ u(0) = f, \end{cases} \tag{\#}$$

where $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Lipschitz continuous and continuously Fréchet differentiable, $f \in H^{2m,2}$, and σ is a strongly elliptic symbol of order $2m$. Then, by Theorems 1.2, 6.1 and Remark 3.3, we can conclude that the initial value problem (#) has a unique solution u in the intersection of $C([0,\infty), H^{2m,2})$ and $C^1([0,\infty), H^{0,2})$. To show that the hypotheses of Theorem 1.2 are satisfied, we only have to check that inequality (1.1) is satisfied. Since σ is a strongly elliptic symbol of order $2m$, it follows from the previous observation that $T_\sigma - W_\sigma = W_\kappa$, where κ is a symbol of order $2m-1$, the Garding inequality (6.1) and the Rellich inequality that a positive constant C_1 and a nonnegative constant λ_0 can be found such that

$$Re\ (W_\sigma\varphi,\varphi) \geq C_1\|\varphi\|_m^2 - \lambda_0\|\varphi\|^2, \quad \varphi \in \mathcal{S}. \tag{6.2}$$

It can be proved easily that there is a positive number C_2 such that

$$Re\ (W_\kappa\varphi,\varphi) \leq \|W_\kappa\varphi\|_{-a}\|\varphi\|_a \leq C_2\|\varphi\|_{2m-1-a}\|\varphi\|_a, \quad \varphi \in \mathcal{S}. \tag{6.3}$$

So, if we let $a = m - \frac{1}{2}$ in (6.3), then we obtain

$$Re\ (W_\kappa\varphi,\varphi) \leq C_2\|\varphi\|_{m-\frac{1}{2}}^2, \quad \varphi \in \mathcal{S}. \tag{6.4}$$

Therefore, by (6.2) and (6.4), we have

$$\begin{aligned} Re\ (T_\sigma\varphi,\varphi) &= Re\ (W_\sigma\varphi,\varphi) + Re\ (W_\kappa\varphi,\varphi) \\ &\geq C_1\|\varphi\|_m^2 - \lambda_0\|\varphi\|^2 - C_2\|\varphi\|_{m-\frac{1}{2}}^2, \quad \varphi \in \mathcal{S}. \end{aligned} \tag{6.5}$$

Now, for any positive number ε, the Rellich inequality gives a positive constant C_ε such that

$$\|\varphi\|_{m-\frac{1}{2}}^2 \leq \varepsilon\|\varphi\|_m^2 + C_\varepsilon\|\varphi\|^2, \quad \varphi \in \mathcal{S}. \tag{6.6}$$

So, by (6.5) and (6.6),

$$Re\ (T_\sigma\varphi,\varphi) \geq (C_1 - \varepsilon C_2)\|\varphi\|_m^2 - (C_2C_\varepsilon + \lambda_0)\|\varphi\|^2, \quad \varphi \in \mathcal{S}. \tag{6.7}$$

Thus, by (6.7), the inequality (1.1) is satisfied if we choose ε to be such that $C_1 - \varepsilon C_2 > 0$. Thus, we have proved the following theorem.

Theorem 6.2. *Let σ be a strongly elliptic symbol in S^{2m}, $m > 0$. If $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Lipschitz continuous and continuously Fréchet differentiable, then, for any f in $H^{2m,2}$, the initial value problem for the semilinear heat equation (#) has a unique solution u in the intersection of $C([0,\infty), H^{2m,2})$ and $C^1([0,\infty), H^{0,2})$.*

The following theorems follow from Theorems 4.1, 5.1 and 5.2 respectively.

Theorem 6.3. *Let σ be a strongly elliptic symbol in S^{2m}, $m > 0$, and F be a continuous mapping from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ such that*

$$\lim_{\|u\|\to 0} \frac{\|F(u)\|}{\|u\|} = 0.$$

Then, for $\lambda > s(-T_{\sigma 0})$, the equilibrium solution $u(t) \equiv 0$ of

$$u'(t) + (\lambda I + T_\sigma)\{u(t)\} = F\{u(t)\}, \quad t > 0,$$

is asymptotically stable.

Theorem 6.4. *Let σ be a strongly elliptic symbol in S^{2m}, $m > 0$, $f \in H^{2m,2}$ and $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a mapping from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ such that there exists a strictly positive constant M for which*

$$\|F(u)\| \leq M\|u\|, \quad u \in L^2(\mathbb{R}^n).$$

Assume that $C > \lambda_0 + M$, where C and λ_0 are the constants in the Garding inequality (6.1). *Then, for all initial values f belonging to a bounded subset Ω of $L^2(\mathbb{R}^n)$ and for all $\rho > 0$, there exists a positive number $t_0 = t_0(r, b, \rho)$ such that any global solution $u(t)$ of* (#) *has the property that $u(t) \in B(0, \rho)$, $t > t_0$, where $b = C - \lambda_0 - M$, $B(0, \rho)$ is the open ball with centre at* 0 *and radius ρ, and r is the radius of the smallest ball with centre at* 0 *containing Ω.*

Theorem 6.5. *Let σ be a strongly elliptic symbol in S^{2m}, $m > 0$, $f \in H^{2m,2}$ and $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a Lipschitz continuous mapping from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ with Lipschitz constant M. Assume that $C > \lambda_0 + M$, where C and λ_0 are the constants in the Garding inequality* (6.1). *Then, for all initial values f belonging to a bounded subset Ω of $L^2(\mathbb{R}^n)$ and for all $\varepsilon > 0$, there exists a positive number $t_0 = t_0(r, b, \varepsilon)$ such that any global solution of* (#) *has the property that $u(t) \in B(0, \rho)$, $t > t_0$, where b and r are as in Theorem 6.4, and $\rho = \frac{N}{b} + \varepsilon$, with $N = \inf_{u_0 \in L^2(\mathbb{R}^n)}\{M\|u_0\| + \|F(u_0)\|\}$.*

Remark 6.6. *The results in this section can be generalized to pseudo-differential operators T_σ and W_σ, where σ is in the class $S^m_{\rho,\delta}$, $m > 0$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ studied in the books* [4] *and* [5] *by Folland and Kumano-go respectively.*

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Department of Mathematics and Statistics
York Univeristy
4700 Keele Street
North York
Ontario, M3J 1P3 CANADA

E-mail addresses: ysma9303@mathstat.yorku.ca, mwwong@mathstat.yorku.ca