THE GENERALIZED CONLEY INDEX AND MULTIPLE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

E. N. DANCER AND YIHONG DU

ABSTRACT. We establish some framework so that the generalized Conley index can be easily used to study the multiple solution problem of semilinear elliptic boundary value problems. Both the parabolic flow and the gradient flow are used. Some examples are given to compare our approach here with other well-known methods. Our abstract results with parabolic flows may have applications to parabolic problems as well.

1. INTRODUCTION

In this paper, we continue our efforts to show how the generalized Conley index developed by Rybakowski can be applied to multiple solution problems of semilinear elliptic equations. Our main purpose here is to set up some framework so that the generalized Conley index can be easily used to superlinear problems. To that end, we use both the parabolic flows and the gradient flows of the elliptic problems. A simple example with superlinearity is given to illustrate some advantages of our approach compared with other well-established methods. We also study in some detail a more general version of a semilinear elliptic problem with a combined concave and convex nonlinearity, which was studied in [2] and [1] recently. This problem seems to serve as a good example to show when the generalized Conley index approach gives better results and when it does not. Here we also improve some results of A. Ambrosetti et al [1] and answer affirmatively a question asked therein.

In [26], various applications of the generalized Conley index to asymptotically linear parabolic and elliptic problems can be found. Recently, in [10] and [12], we used the generalized Conley index to study the multiplicity

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and related problems of some sublinear and asymptotically linear elliptic equations, where we made use of the Morse inequalities of the Morse decompositions together with the order-preserving property of the parabolic flow. The results obtained there seem difficult to get by using the standard topological and variational methods. In this paper, we extend some of the ideas in [10] and [12] to superlinear problems. As before, we use Rybakowski's version of the Conley index. We remark that Benci [4] developed a different version of the Conley index and he applied it also to some superlinear problems [4], [5]. He made use of his Morse index of critical point sets but did not use order structures and the full strength of Morse decompositions. Our ideas could probably also be carried out by using Benci's version of the Conley index.

We need to establish proper compactness settings in the generalized Conley index for superlinear nonlinearities. This is done by making use of the energy functionals, a priori estimates (for the parabolic flow) and P.S. condition (for the gradient flow). We find that while the parabolic flow has wider applications, for example, it can be used to study the existence of connecting orbits of the corresponding parabolic equations, the gradient flow seems to provide better compactness result (compare Theorems 2.1 and 3.2). One difficulty in using the gradient flow is that it does not have the strongly order preserving property and the solutions do not improve their regularity as time increases. This is overcome in our applications by the observation that certain entire orbits of the gradient flow are compact in $C^1(\overline{D})$, and the fact that the flow has a certain invariance property.

To be more precise, we consider the following semilinear elliptic problem:

$$-\Delta u = f(u) \quad \text{in } D, \quad Bu = 0 \quad \text{on } \partial D, \tag{1.1}$$

where D is a bounded domain in R^N with regular enough boundary ∂D, $Bu = u$ (Dirichlet boundary condition) or $Bu = \partial u/\partial n$ (Neumann boundary condition), where $n = n(x)$ is the unit outward normal to ∂D at $x \in \partial D$, and $f : R^1 \to R^1$ is a locally Lipschitz continuous function. Unless otherwise specified, we suppose that f has a subcritical growth rate, i.e., it satisfies

- (H_1) $|f(u)| \leq C(1 + |u|^\gamma)$ for all $u \in (-\infty, \infty)$, where C, γ are positive constants, and $\gamma < (N+2)/(N-2)$ if $N > 2$.

In the generalized Conley index approach, as in [26], [10] and [12], the most natural flow to work with is the corresponding parabolic flow of (1.1), that is the local semiflow induced by the following parabolic problem:

$$u_t - \Delta u = f(u),\ t > 0, x \in D, \quad Bu = 0,\ t > 0, x \in \partial D. \tag{1.2}$$

To be more precise, we choose $p > N, \theta \in (1/2 + N/(2p), 1)$ and let $X = L^p(D)$. By well known results (see, e.g., [20]), (1.2) defines a local semiflow $\pi(t) : X^\theta \to X^\theta$ by $\pi(t)u(0, \cdot) = u(t, \cdot)$, where X^θ is the fractional power space induced by (1.2) and $u(t, x)$ is a solution of (1.2). We will call π the parabolic flow for (1.1) and denote $\pi(t, u(0, \cdot)) = \pi(t)u(0, \cdot) = u(t, \cdot)$. By our choice of p and θ, X^θ is continuously imbedded into $C^1(\overline{D})$ (see [20]). Clearly, solutions of (1.1) are equilibria of π and vice versa.

If the nonlinearity f is superlinear, one difficulty in using the generalized Conley index theory with the above parabolic flow is that solutions of (1.2) may blow up in finite time. To overcome this, or more precisely, to meet the more demanding compactness requirements (i.e., the π-admissibility condition) in the generalized Conley index theory, we make use of its energy functional $J : W_B^{1,2}(D) \to R$,

$$J(u) = \int_D [|\nabla u|^2/2 - F(u)]dx,$$

where $F(u) = \int_0^u f(s)ds$, and a priori estimates for parabolic equations. Here and throughout this paper, we understand that $W_B^{1,2} = W_0^{1,2}$ if $Bu = u$ (i.e., the Dirichlet case), and $W_B^{1,2} = W^{1,2}$ if $Bu = \partial u/\partial n$ (i.e., the Neumann case). We find that the energy level sets are very convenient to work with for this purpose. We use ideas of F. Rothe [25] on a priori estimates for parabolic systems, and energy estimates for parabolic equations as in [7] and [19], and show in particular that, if f satisfies (H_1), and

- (H_2) For some $M > 0$ and $q > 1$, $0 < (1 + q)F(u) \le uf(u)$ for all $|u| \ge M$,

then the set of points between any two energy levels (i.e., the set $\{u \in X^\theta : a \le J(u) \le b\}$) is strongly π-admissible provided that a further condition $\gamma < 1+(4q+8)/(3N)$ is met . This compactness result enables us to use the generalized Conley index for π for superlinear problems. Note that condition (H_2) implies that $F(u) \ge C_1(|u|^{q+1}-1)$ for some $C_1 > 0$ and all u. Therefore we necessarily have $q \le \gamma$ if f satisfies both (H_1) and (H_2). It then follows from $\gamma < 1 + (4q + 8)/(3N)$ that $\gamma < (3N + 8)/(3N - 4)$ if $N > 1$.

It is well known that conditions (H_1) and (H_2) guarantee that the energy functional J satisfies the P.S. condition on $W_B^{1,2}$, i.e., $\{u_n\}$ has a convergent subsequence in $W_B^{1,2}$ whenever $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$. Therefore, standard variational methods are applicable to (1.1) under these conditions only, but our generalized Conley index setting is not. We suspect that our extra restriction $\gamma < 1 + (4q + 8)/(3N)$ is not necessary.

Another natural flow associated to (1.1) is a slight variant of the negative gradient flow of J, that is, the global flow $\eta : R \times W_B^{1,2}(D) \to W_B^{1,2}(D)$ given by $\eta(t, u) = \eta(t)$, where $\eta(t)$ is the unique solution of the problem

$$\eta_t = -\sigma(\eta)J'(\eta),\ \eta(0) = u, \tag{1.3}$$

where $\sigma(u) = 1/\max\{1, \|J'(u)\|\}$. Since critical points of J are solutions of (1.1), it is easy to see that equilibria of η are solutions of (1.1), and vice versa.

Here we do not have blow-ups, but we still need η-admissibility. We find that η possesses this compactness property under (H_1) and (H_2) only. In fact, we will show that, under condition (H_1), J satisfies the P.S. condition if and only if $\{u \in W_B^{1,2}(D) : a \le J(u) \le b\}$ is strongly η-admissible for any finite numbers $a < b$. Hence, in almost all the cases where the standard variational method can be used to (1.1), the generalized Conley index theory with the gradient flow works. One disadvantage of η compared with π is that it does not have the order preserving property that the parabolic flow enjoys. Moreover, unlike the parabolic flow, solutions along η do not improve their regularity when time goes from zero to positive. These poses some difficulties in applications. We overcome these by using an invariance property of η used by Hofer [21], and an observation that any entire orbit lying between two energy levels is precompact in $C^1(\overline{D})$. This makes it possible to use upper and lower solution arguments in our generalized Conley index setting. These properties of η come from the fact that $J'(u) = u - Au$ where A is compact, order preserving, and improves regularity.

As a simple example to show that the generalized Conley index approach may give better results than the other well established methods, we apply our general results on the parabolic and gradient flows to a special case of (1.1), i.e., the following Neumann problem:

$$-\Delta u = f(u), \quad \frac{\partial u}{\partial n}|_{\partial D} = 0, \tag{1.4}$$

where f is C^1 and satisfies (H_1), (H_2) and

- (H_3) There exist real numbers α and β with $\alpha < \beta$ such that $f(\alpha) = f(\beta) = 0$, $f'(\alpha) > 0, f'(\beta) > 0$, $f(x) < 0$ for $x < \alpha$, $f(x) > 0$ for $x > \beta$.

Recall that condition (H_2) implies that $f(u)$ is superlinear near infinity. It follows from condition (H_3) that $u = \alpha$ and $u = \beta$ are the only constant solutions of (1.4) outside the interval (α, β). We are interested in establishing a relationship between the position of the point $(f'(\alpha), f'(\beta))$ in R^2 and the number of nonconstant solutions of (1.4) which are outside (α, β). Here we say $u = u(x)$ is outside (α, β) if for some $x_0 \in D$, $u(x_0) \notin (\alpha, \beta)$, that is, in the sense that (α, β) is regarded as an order interval in a proper function space. Note that it follows from the continuity of f that (1.4) has at least one constant solution in (α, β). Moreover, since we have no restriction on f for $u \in (\alpha, \beta)$ except requiring it to be C^1, by varying f in (α, β) one can produce many constant and nonconstant solutions of (1.4) which lie inside this interval. The point of our method is that it does not depend on how f varies in (α, β).

Let $0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le ... \le \lambda_k \le \lambda_{k+1} \le ...$ be all the eigenvalues of $-\Delta u = \lambda u$, $Bu|_{\partial D} = 0$, counting multiplicity. Note that $\{\lambda_k\}$ with Dirichlet boundary conditions is different from that with Neumann boundary conditions. In concrete problems, we always understand that $\{\lambda_k\}$ refers to the one with the boundary condition of the problem under consideration.

Define the sets G_1, G_2 and G_3 in R^2 as follows:

$G_1 = [(\lambda_1, \lambda_2) \times (\lambda_1, \lambda_2)] \cup [(\lambda_1, \lambda_2) \times (\lambda_2, \lambda_3)] \cup [(\lambda_2, \lambda_3) \times (\lambda_1, \lambda_2)]$.

$G_2 = [(\lambda_1, \lambda_2) \times (\lambda_3, \infty)] \cup [(\lambda_3, \infty) \times (\lambda_1, \lambda_2)]$.

$G_3 = [(\lambda_2, \lambda_3) \times (\lambda_2, \lambda_3)] \cup [(\lambda_3, \infty) \times (\lambda_3, \infty)] \cup \{(\lambda_2, \lambda_3) \times [(\lambda_3, \infty) \setminus \{\lambda_k : k > 3\}]\}$

$$\cup \{[(\lambda_3, \infty) \setminus \{\lambda_k : k > 3\}] \times (\lambda_2, \lambda_3)\}.$$

Note that the union of the closure of G_1, G_2 and G_3 is the entire first quadrant in R^2 (where the Neumann boundary condition is used).

We have the following result.

Proposition 1.1. *Suppose that f satisfies $(H_1) - (H_3)$. Then* (1.4) *has at least i nonconstant solutions outside* (α, β) *if* $(f'(\alpha), f'(\beta)) \in G_i,\ i = 1, 2, 3$.

This result is proved by using the generalized Conley index theory for the gradient flow of (1.4). It seems difficult to obtain by using the standard topological and variational methods. We will also study a slightly more general version of a problem studied by [1] and [2] recently, i.e., the following Dirichlet problem:

$$-\Delta u = \lambda |u|^{r-1} u + g(u), \quad u|_{\partial D} = 0, \tag{1.5}$$

where $\lambda > 0$ and $r \in (0, 1)$ are constants, $g : R^1 \to R^1$ is C^1 and

- (H_4) $g(u)u \geq 0,\ \ \forall u;\quad \lim_{u\to 0} g(u)/u = 0.$

We will show that if g is asymptotically linear, then our generalized Conley index approach as developed in [12] gives better results than other well-known methods; however, if g is superlinear, the degree method used in [11] seems to give the best result.

We prove in particular the following.

Proposition 1.2. *Suppose that g satisfies (H_4) and either*

- (H_5) $\lim_{|u|\to\infty} g(u)/u = a \in (\lambda_k, +\infty) \setminus \{\lambda_i\},\quad k \geq 2$, *or*
- (H_6) $\lim_{|u|\to\infty} g(u)/|u|^{\gamma-1} u = b > 0,$

where $1 < \gamma < (N + 2)/(N - 2)$ *if* $N > 2$, $1 < \gamma < \infty$ *if* $N = 1, 2$. *Then there exists* $\Lambda^+, \Lambda^- > 0$ *such that*

(i) *for* $\lambda > \Lambda^+$ *(resp.* Λ^-*),* (1.5) *has no positive (resp. negative) solution;*
(ii) *for* $0 < \lambda < \Lambda^+$ *(resp.* Λ^-*),* (1.5) *has at least two positive (resp. negative) solutions;*
(iii) *for* $0 < \lambda < \max\{\Lambda^+, \Lambda^-\}$*,* (1.5) *has at least two sign-changing solutions.*

Remarks. 1. The above result (essentially part (iii)) improves a result in [1] where they proved that under conditions similar to but slightly more restrictive than (H_4) and (H_6) for g, for all *small* positive λ , (1.5) has at least 6 nontrivial solutions. Parts (i) and (ii) above follow essentially from [2].

2. In fact we will prove more than Proposition 1.2 in section 4 later. In particular, we will answer affirmatively a question in [1] (see Remark 4.1 for details).

3. If $g(u) = |u|^{p-1}u$, where $p > 1$ and $p < (N + 2)/(N - 2)$ when $N > 2$, then the above result follows directly from [2]. In fact, in this case, using the oddness of the nonlinearity, $\Lambda^+ = \Lambda^-$ and (iv) can be improved significantly, i.e., there are infinitely many nontrivial solutions for any real λ, see [2] and the note added in proof at the end of [2].

4. Conditions (H_5) and (H_6) can be relaxed considerably. For example, (H_5) can be replaced by

$$\lim_{u\to\pm\infty} g(u)/u = a^{\pm} \in (\lambda_k, +\infty) \setminus \{\lambda_i\}, k \geq 2,$$

and (H_6) can be replaced by

$$\lim_{u\to\pm\infty} g(u)/|u|^{\gamma^\pm-1}u = b^\pm > 0,$$

where $\gamma^\pm$ satisfies the same condition as γ. See also Remark 4.4 at the end of section 4.

Our results on the parabolic and gradient flows, especially that on the parabolic flow, may have many other applications. For example, many equation systems have certain single equations as limiting problems, and solutions of the single equation problems often induce solutions to the original systems (some such examples can be found in [13], [15] and [16]). Since the generalized Conley index possesses continuation stability, it may be useful to consider the flow π for a single limiting equation as a limit of the flow π' generated by the original system and use results on π to study π'. In this case, the flow η is difficult to use as there is in general no gradient flow for systems due to the fact that most natural reaction-diffusion *systems* lack variational structures. Another use of the flow π is to study the existence of connecting orbits (see Remark 2.3). We should point out that our results on the gradient flow are sufficient for the applications in this paper. The main point to include the results on the parabolic flow here is for comparison with the gradient flow (which justifies our use of the less natural gradient flow) and for possible future applications.

Most of our results on the flows π and η hold true for much more general flows, where they are induced by more general uniformly elliptic operators, and the nonlinearity f can also be x dependent.

The rest of this paper is organized as follows. In section 2, we show how a framework for π can be established so that the generalized Conley index works. A weaker version of Proposition 1.1 is proved using the flow π. In section 3, we study the gradient flow η and prove Proposition 1.1. Section 4 is devoted to the study of (1.5). Section 5 is an appendix, where we give proofs for the a priori estimates used in section 2.

2. The Parabolic Flow

In this section, we study (1.1) by making use of its parabolic flow and the generalized Conley index. We consider only the superlinear case since the sublinear and asymptotically linear cases are relatively well understood (see, e.g., [26], [10] and [12]).

Let $J_1 = J|_{X^\theta}$ and for any interval Λ in R, define

$$J_1^{-1}\Lambda = \{u \in X^\theta : J_1(u) \in \Lambda\}, \quad J^{-1}\Lambda = \{u \in W_B^{1,2} : J(u) \in \Lambda\}.$$

The following result plays an important role in this section.

Theorem 2.1. *Let $a < b$ be real numbers, and $(H_1), (H_2)$ be satisfied. If further $\gamma < 1 + (4q+8)/(3N)$, then $J_1^{-1}[a,b]$ is strongly π-admissible.*

Recall that a set $N \in X^\theta$ is called strongly π-admissible ([26]) if (i) N is closed; (ii) π does not explode in N, i.e., $u \in X^\theta$, $\{t_n\} \subset R^+$ bounded and $\{\pi(t,u) : t \in [0,t_n]\} \subset N$ imply $\{\pi(t_n,u)\}$ is bounded; (iii) for any $u_n \in X^\theta$,

$\{t_n\} \subset R^+$ with $t_n \to \infty$ and $\{\pi(t, u_n) : t \in [0, t_n]\} \subset N$, $\{\pi(t_n, u_n)\}$ has a convergent subsequence.

The proof of Theorem 2.1 relies on some a priori estimates for solutions of (1.2).

Lemma 2.1. *Let the conditions of Theorem 2.1 be satisfied and $u = u(t,x)$ be a solution of (1.2) satisfying $u(t,\cdot) \in J_1^{-1}[a,b]$ for $t \in [T_1, T_2]$. Then for any $r \in (1, (2q+4)/3)$, there exists $M = M_r$ independent of u and T_1, T_2 such that*

$$\sup_{t\in[T_1,T_2]} \|u(t,\cdot)\|_{L^r} \le M.$$

Lemma 2.1 follows from some well known energy estimates and interpolation inequalities, which were used, for example, in [7] and [19]. For the convenience of the readers, we will give the proof of Lemma 2.1 in the Appendix.

Lemma 2.2. *Under the conditions of Lemma 2.1, there exist positive constants M and δ independent of u and T_1, T_2 such that*

$$\|u(t,\cdot)\|_{L^\infty} \le [m(t - T_1)]^{-\delta} M, \forall t \in (T_1, T_2].$$

Here $m(s) = \min\{s, 1\}$.

The proof of Lemma 2.2 is based on some well known methods of F. Rothe [25] and is rather technical. Since the ideas needed in proving Lemma 2.2 are scattered in [25], we will give a complete proof of Lemma 2.2 in the Appendix.

Proof of Theorem 2.1. For convenience of notations, we will denote $N = J_1^{-1}[a,b]$.

(i) It follows from the continuity of J_1 that N is closed.

(ii) Suppose that π explodes in N. We are going to derive a contradiction. Let $u_0 \in X^\theta$ be such that $\pi(t, u_0) \in N$ for all $t \in [0, T)$ with $0 < T < \infty$, and for some $t_n \in (0, T)$, $t_n \to T$, $\|\pi(t_n, u_0)\|_{X^\theta} \to \infty$. By Lemma 2.2, $\|\pi(t, u_0)\|_{L^\infty} \le M$ for all $t \in [T/2, t_n], n \ge 1$. Hence, by Theorem 4.3 of [26], $\|\pi(t_n, u_0)\|_{X^\sigma} \le M_1$ for all $t \in [2T/3, t_n], n \ge 1$ with some $M_1 > 0$ and $\sigma \in (\theta, 1)$. Since X^σ imbeds continuously (in fact, compactly) into X^θ, we arrive at a contradiction. Thus, π does not explode in N.

(iii) Let $u_n \in X^\theta$, $t_n \in R^+$ satisfy $t_n \to \infty$ and $\{\pi(t, u_n) : t \in [0, t_n]\} \subset N$. We may assume that $t_n > 1$ for all n. Then using Lemma 2.2 and Theorem 4.3 of [26] as in (ii), we deduce that $\|\pi(t, u_n)\|_{X^\sigma} \le M_1$ for all n and $t \in [1, t_n]$, where M_1 is a positive number and $\sigma \in (\theta, 1)$. Since X^σ imbeds compactly into X^θ, $\{\pi(t_n, u_n)\}$ is precompact in X^θ and hence has a convergent subsequence.

The proof of Theorem 2.1 is complete. ∎

Theorem 2.2. *Suppose that (H_1), (H_2) are satisfied, and $\gamma < 1 + (4q + 8)/(3N)$. Then there exists $a_0 < 0$ such that J has no critical point in $J^{-1}(-\infty, a_0]$. If further J has no critical point in $J^{-1}[b_0, \infty)$ for some $b_0 >$*

a_0, $\gamma < 1 + (4q+8)/(3N)$ and K is the maximal invariant set of π in $J_1^{-1}[a_0, b_0]$, then the generalized Morse index $h(\pi, K)$ is well-defined and

$$H_q(h(\pi, K)) = 0, \quad q = 0, 1, 2, ...$$

Proof. Let $S^\infty = \{u \in W_B^{1,2}(D) : \|u\|_{W_B^{1,2}} = 1\}$. Then it follows from (H_2) that for any $u \in S^\infty$, $J(tu) \to -\infty$ as $t \to \infty$. Moreover, there exists $a_0 < 0$ such that $t > 0, u \in S^\infty$ and $J(tu) \le a_0$ imply $(d/dt)J(tu) < 0$. A detailed proof of these facts can be found in [27] where the Dirichlet boundary condition is considered, but the proof for the Neumann problem is the same. As in [27], one finds easily from these two facts that, for $a \le a_0$,

$$H_q(W_B^{1,2}, J^{-1}(-\infty, a]) = H_q(W_B^{1,2}, S^\infty) = 0, \quad q = 0, 1, 2.... \tag{2.1}$$

It is well known that under (H_1) and (H_2), J satisfies the P.S. condition. Hence if J has no critical point in $J^{-1}[b_0, \infty)$, then for any $b \ge b_0$, $J^{-1}(-\infty, b]$ is a strong deformation retractor of $W_B^{1,2}$. Thus, by (2.1), for $a \le a_0, b \ge b_0$,

$$\begin{aligned} H_q(J^{-1}(-\infty, b], J^{-1}(-\infty, a]) &= H_q(W_B^{1,2}, J^{-1}(-\infty, a]) \\ &= 0, \quad q = 0, 1, 2 \dots . \end{aligned} \tag{2.2}$$

Now let $a < a_1 < a_0$, $b_1 > b > b_0$. Then $J^{-1}(-\infty, a]$ and $J^{-1}(-\infty, b]$ are strong deformation retractors of $J^{-1}(-\infty, a_1)$ and $J^{-1}(-\infty, b_1)$ respectively. Therefore,

$$H_q(J^{-1}(-\infty, b], J^{-1}(-\infty, a]) = H_q(J^{-1}(-\infty, b_1), J^{-1}(-\infty, a_1)).$$

By Palais' theorem (see, Theorem 3.2 in [8]),

$$H_q(J_1^{-1}(-\infty, b_1), J_1^{-1}(-\infty, a_1)) = H_q(J^{-1}(-\infty, b_1), J^{-1}(-\infty, a_1)).$$

Thus, using (2.2), we have

$$H_q(J_1^{-1}(-\infty, b_1), J_1^{-1}(-\infty, a_1)) = 0, \; q = 0, 1, 2, ... \tag{2.3}$$

By Theorem 2.1, for any $a < a_0$ and $b > b_0$, $N \equiv J_1^{-1}[a, b]$ is strongly π-admissible. We show that N is an isolating neighborhood of K. It suffices to show that K is the maximal invariant set of π in N and that $K \subset J_1^{-1}(a_0, b_0)$. Let K_1 be the maximal invariant set of π in N and $u \in K_1$ (If K_1 is empty, then the conclusion of the theorem holds trivially). Then $u = u(t, x)$ is a full solution of (1.2). Since N is strongly π-admissible, it is easy to see that the omega limit set $\omega(u)$ and the alpha limit set $\alpha(u)$ are both nonempty. Since J is a Lyapunov functional for (1.2), $\omega(u)$ and $\alpha(u)$ consist of equilibria of (1.2), i.e., solutions of (1.1). But solutions of (1.1) are critical points of J. Therefore, by the assumptions, for any $u_1 \in \alpha(u)$ and $u_2 \in \omega(u)$, $J(u_1), J(u_2) \in (a_0, b_0)$. It follows that $J(u(t, \cdot)) \in [J(u_1), J(u_2)] \in (a_0, b_0)$ for any $t \in (-\infty, \infty)$. This shows that $u \in J_1^{-1}(a_0, b_0)$. Thus $K = K_1$ and N is an isolating neighborhood of K. Therefore $h(\pi, K)$ is well-defined.

Since J_1 is a quasi-potential of π, by Corollary 4.7 of [26],

$$H_q(h(\pi, K)) = H_q(J_1^{-1}(-\infty, a], J_1^{-1}(-\infty, b]), \quad q = 0, 1, 2, ... \tag{2.4}$$

A simple variant of the argument on page 59 of [8] shows that, if $a < a_1 \le a_0$ and $b_1 > b \ge b_0$, then $J_1^{-1}(-\infty, a]$ and $J_1^{-1}(-\infty, b]$ are strong deformation retractors of $J_1^{-1}(-\infty, a_1)$ and $J_1^{-1}(-\infty, b_1)$ respectively. Thus

$$H_q(J_1^{-1}(-\infty, a], J_1^{-1}(-\infty, b]) = H_q(J_1^{-1}(-\infty, a_1), J_1^{-1}(-\infty, b_1)).$$

This and (2.3), (2.4) imply

$$H_q(h(\pi, K)) = 0, \quad q = 0, 1, 2, ...$$

The proof is complete. ▪

Remark 2.1. All our above results in this section are true for much more general problems than (1.2). We could replace Δ by a general second order self-adjoint uniformly elliptic operator. f can also be dependent on x. Of course, the functional J should be adjusted accordingly.

Now we are able to prove a weaker version of Proposition 1.1. We remark that though we give a unified generalized Conley index approach, a number of subcases in Proposition 2.1 can be proved by the standard topological and variational methods.

Proposition 2.1. *Suppose that* $(H_1) - (H_3)$ *are satisfied and* $\gamma < 1 + (4q + 8)/(3N)$*. Then* (1.4) *has at least* i *nonconstant solutions outside* (α, β) *if* $(f'(\alpha), f'(\beta)) \in G_i$*,* $i = 1, 2, 3$.

Proof. It follows from (H_3) that (1.4) has no constant solution outside $[\alpha, \beta]$. Moreover, there exist unique $\alpha_1 > \alpha$ and $\beta_1 < \beta$ such that $\alpha_1 \le \beta_1$ and

(2.5)
$$f(\alpha_1) = f(\beta_1) = 0, \; f(u) > 0 \text{ for } \; x \in (\alpha, \alpha_1), \; f(u) < 0 \text{ for } \; u \in (\beta_1, \beta).$$

We observe two facts by using (2.5). First, it follows from (2.5) and the elliptic maximum principle that any nonconstant solution of (1.4) which is outside $[\alpha_1, \beta_1]$ must be outside $[\alpha, \beta]$.

Second, for any $\alpha' \in (\alpha, \alpha_1)$ and $\beta' \in (\beta_1, \beta)$, $u \equiv \alpha'$ and $v \equiv \beta'$ are strict lower and upper solutions of (1.4) respectively. Fix such a pair and define

$$N_0 = [\alpha', \beta']_{X^\theta} \equiv \{u \in X^\theta : \alpha' \le u \le \beta'\}.$$

As in [10], one can easily show that N_0 is a strongly π-admissible isolating neighborhood of K_0, where K_0 is the maximal invariant set of π in N_0. Moreover

$$H_q(h(\pi, K_0)) = \delta_{0,1} G. \tag{2.6}$$

(2.6) can be proved by either a simple homotopy argument (see Remark 3.2 later) or by calculating $h(\pi, K_0)$ directly (observing that N_0 is in fact an isolating block of K_0).

We have three cases to consider:

(i) $(f'(\alpha), f'(\beta)) \in G_1$, (ii) $(f'(\alpha), f'(\beta)) \in G_2$, (iii) $(f'(\alpha), f'(\beta)) \in G_3$.

We give the detailed proof for case (ii) only. The proofs for the other cases are similar or simpler, and therefore are left to the interested readers.

Now suppose that $(f'(\alpha), f'(\beta)) \in G_2$. We may assume that $f'(\alpha) \in (\lambda_1, \lambda_2)$ and $f'(\beta) \in (\lambda_3, \infty)$ as the remaining case can be proved analogously.

A simple variant of the proof of Proposition 2 in [12] (where the Dirichlet boundary condition is considered) shows that $u = \beta_1$ is a strict local minimum of J restricted to the set $S = \{u \in W^{1,2}(D) : u \geq \beta_1\}$, and (1.4) has a solution $u_1 > \beta_1$ which is of mountain pass type. We may assume that (1.4) has only finitely many solutions in S. Then it follows as in [12] that

$$C_q(J, u_1) = \delta_{1,q} G.$$

Here, and in what follows, $C_q(J, u)$ denotes the critical groups of the critical point u of J. By Proposition 2.1 in [10] (applied to the Neumann boundary condition case),

$$H_q(h(\pi, \{u_1\})) = C_q(J, u_1).$$

Thus we have

$$H_q(h(\pi, \{u_1\})) = \delta_{1,q} G. \tag{2.7}$$

Since $f'(\beta) \in (\lambda_3, \infty)$, we can use the shifting theorem (see, e.g., [22]) to calculate the critical groups of the critical point $u_2 = \beta$, and obtain

$$C_q(J, u_2) = 0, \quad q = 0, 1, 2.$$

Now use this and Proposition 2.1 in [10] as above, we obtain

$$H_q(h(\pi, \{u_2\})) = 0, \quad q = 0, 1, 2. \tag{2.8}$$

This implies in particular that $u_2 \neq u_1$. As $u_3 = \alpha \notin S$, we also have $u_3 \neq u_1$. Moreover, it follows from $f'(\alpha) \in (\lambda_1, \lambda_2)$ that

$$H_q(h(\pi, \{u_3\})) = \delta_{1,q} G. \tag{2.9}$$

Now clearly u_1 is a nonconstant solution of (1.4) outside (α, β). We need to show that there is at least one more such solution. Suppose that this is not the case. Then one easily sees that the solution set of (1.4) is compact in X^θ. Hence we can find $a < b$ such that $a < J(u) < b$ for any solution u of (1.4). Moreover, if K is the maximal invariant set of π in $J_1^{-1}[a, b]$, then by Theorem 2.2, $h(\pi, K)$ is well-defined and

$$H_q(h(\pi, K)) = 0, \quad q = 0, 1, 2, ... \tag{2.10}$$

By changing the subscripts in u_1, u_2 and u_3 we can suppose that $J(u_1) \leq J(u_2) \leq J(u_3)$. We show next that $\{K_0, \{u_1\}, \{u_2\}, \{u_3\}\}$ is a Morse decomposition of K. For convenience, we denote $K_i = \{u_i\}$, $i = 1, 2, 3$. It suffices to show that, for any $u \in K$, either $u \in K_i$ for some $0 \leq i \leq 3$ or $\alpha(u) \in K_j$ and $\omega(u) \in K_i$ for some $0 \leq i < j \leq 3$. Now let $u \in K$ and $u \notin \cup_{i=0}^3 K_i$. Then u cannot be an equilibrium of π. Moreover, since $t \to J(u(t, \cdot))$ is strictly decreasing, $\alpha(u)$ and $\omega(u)$ are different sets and consist of solutions of (1.4). Since both $\alpha(u)$ and $\omega(u)$ are connected, each of them is contained in one and only one K_i. Choose $w \in \alpha(u)$. If $w \in K_j$, $j > 0$, then we necessarily have $\alpha(u) = K_j$, and $J(\omega(u)) < J(\alpha(u))$. Hence we must have $\omega(u) \subset K_i$, $i < j$. It remains to check the case that $w \in K_0$. In this case, $w \in [\alpha_1, \beta_1]$. It follows from the parabolic maximum principle and the fact that $u = \alpha', u = \beta'$ are lower and upper solutions of (1.4) that $\alpha' < u(t, x) < \beta'$ for all $t \in (-\infty, \infty)$ and $x \in D$, i.e., $u \in K_0$. But this contradicts our initial assumption. Therefore the last case never occurs. This proves that $\{K_0, K_1, K_2, K_3\}$ is a Morse decomposition of K.

Next we use the Morse equation for this Morse decomposition to deduce a contradiction and thereby complete the proof. We substitute (2.6)-(2.10) into the following Morse equation

$$\Sigma_{q=0}^{\infty} \mathrm{rank} H_q(h(\pi, K))t^q$$

$$= \Sigma_{i=0}^{3}\Sigma_{q=0}^{\infty} \ \mathrm{rank} H_q(h(\pi, K_i))t^q - (1+t)Q(t),$$

where $Q(t) = \Sigma_{q=0}^{\infty} d_q t^q, \ \ d_q \geq 0$, and obtain by comparing the coefficients:

$$0 = 1 - d_0, 0 = 1 + 1 - d_1 - d_0, 0 = -d_2 - d_1.$$

This gives $d_0 = d_1 = 1, d_2 = -1$, contradicting $d_2 \geq 0$. The proof is complete. ∎

Remark 2.2. As in [26], one can use irreducibility properties of the generalized Conley index to discuss the existence of connecting orbits of (1.2). For example, under the conditions of Proposition 2.1, corresponding to each isolated solution of (1.4) with non-trivial Conley index, there is a bounded non-equilibrium entire solution of (1.2) (with Neumann boundary conditions) having this solution of (1.4) as its alpha or omega limit set.

3. The Gradient Flow

In this section, we set up the framework for the gradient flow η so that the generalized Conley index can be used.

We note first that $\sigma(\eta)J'(\eta)$ is locally Lipschitz in η and $\|\sigma(\eta)J'(\eta)\|_{W_B^{1,2}} \leq 1$. Therefore η is defined on the whole $R \times W_B^{1,2}$.

Fix any $k > 0$ and define $A : W_B^{1,2} \to W_B^{1,2}$ by

$$A(u) = (-\Delta + k)^{-1}(f(u) + ku),$$

where $(-\Delta + k)^{-1}$ denotes the inverse of $-\Delta + k$ under the homogeneous boundary conditions $Bu = 0$. It is well known that, under condition (H_1), A is compact, and if we use the equivalent norm

$$\|u\| = \int_D [|\nabla u|^2/2 + ku^2]dx$$

in $W_B^{1,2}(D)$, then

$$J'(u) = u - A(u).$$

For any $u_0 \in W_B^{1,2}$, let

$$w(t, u_0) = \int_0^t \sigma(\eta(\xi, u_0))d\xi.$$

Then

$$\eta(t, u_0) = e^{-w(t,u_0)}u_0 + e^{-w(t,u_0)} \int_0^t e^{w(s,u_0)} A(\eta(s, u_0))\sigma(\eta(s, u_0))ds. \tag{3.1}$$

Let $X_0 = W_B^{2,p}$, $p \geq 2$ is such that $W_B^{2,p}$ imbeds compactly into $C^1(\overline{D})$. Then it is well known (see, e.g., [8] or [21]) that there exist a finite sequence of Banach spaces $X_1, ..., X_k$ such that

$$W_B^{2,p} = X_0 \hookrightarrow X_1 \hookrightarrow ... \hookrightarrow X_k = W_B^{1,2},$$

the imbedding of X_{i-1} into X_i is compact, and $A : X_i \to X_{i-1}$ is continuous and maps bounded sets into bounded sets, $i = 1, ..., k$.

Using this and (3.1), one sees that if $\eta(t, u_0)$ is a solution of (1.3), then $\eta(t, u_0) \in X_i$ for all t if $u_0 \in X_i$, $t \to \eta(t, u_0)$ is continuously differentiable in X_i and $(t, u_0) \to \eta(t, u_0)$ is continuous in $R \times X_i$.

Lemma 3.1. *Let (H_1) be satisfied, $\{u_n\} \subset X_i$ for some fixed $1 \leq i \leq k$ and $0 < T \leq \infty$. Suppose that*

$$M_1 = \sup\{\|\eta(t, u_n)\|_{X_k} : 0 \leq t < T, n \geq 1\} < \infty$$

and

$$M_2 = \sup\{\|u_n\|_{X_i} : n \geq 1\} < \infty.$$

Then there exists C depending only on i, M_1 and M_2 such that

$$\|\eta(t, u_n) - e^{-w(t,u_n)} u_n\|_{X_{i-1}} \leq C, \quad \forall t \in [0, T), n \geq 1.$$

Proof. By (3.1),

$$\|\eta(t, u_n) - e^{-w(t,u_n)} u_n\|_{X_{k-1}} = e^{-w(t,u_n)} \| \int_0^t e^w A(\eta)\sigma(\eta) ds\|_{X_{k-1}}$$
$$\leq e^{-w(t,u_n)} \int_0^t e^w \|A(\eta)\|_{X_{k-1}} \sigma(\eta) ds.$$

Since A is a bounded operator from X_k to X_{k-1}, we can find $C_1 = C_1(M_1)$ such that

$$\|A(u)\|_{X_{k-1}} \leq C_1, \quad \forall u \in X_k, \|u\|_{X_k} \leq M_1.$$

Using this we obtain

$$e^{-w(t,u_n)} \int_0^t e^w \|A(\eta)\|_{X_{k-1}} \sigma(\eta) ds$$
$$\leq e^{-w(t,u_n)} \int_0^t e^{w(s,u_n)} C_1 \sigma(\eta(s, u_n)) ds$$
$$= e^{-w(t,u_n)} \int_0^{w(t,u_n)} C_1 e^{\xi} d\xi = C_1(1 - e^{-w(t,u_n)})$$
$$\leq C_1, \quad \forall t \in [0, T), \ n \geq 1.$$

Thus

$$\|\eta(t, u_n) - e^{-w(t,u_n)} u_n\|_{X_{k-1}} \leq C_1, \quad \forall t \in [0, T),$$

and

$$\|\eta(t, u_n)\|_{X_{k-1}} \leq C_1 + \|u_n\|_{X_{k-1}} \leq K_1, \ \forall t \in [0, T), n \geq 1.$$

Repeating the above argument $k - i$ times, we obtain

$$\|\eta(t, u_n)\|_{X_i} \leq K_{k-i}, \quad \forall t \in [0, T), n \geq 1$$

where K_{k-i} depends only on i and M_1, M_2. Now use the argument once more, and the proof is complete. ∎

Theorem 3.1. *Let (H_1) be satisfied. Then any bounded closed set in $W_B^{1,2}$ is strongly η-admissible.*

Proof. Let U be any bounded closed set in $W_B^{1,2}$. Since η is a global flow, it does not explode in U. Therefore, it suffices to show that $\{\eta(t_n, u_n)\}$ has a convergent subsequence whenever $\eta([0, t_n], u_n) \subset U$ and $t_n \to \infty$. Since U is bounded, it is easy to see that $\delta = \min\{\sigma(u) : u \in U\}$ is positive. It follows that

$$w(t_n, u_n) = \int_0^{t_n} \sigma(\eta(s, u_n))ds \geq \delta t_n \to \infty.$$

By (3.1) and Lemma 3.1, there exists $C > 0$ such that

$$\|\eta(t_n, u_n) - e^{-w(t_n,u_n)} u_n\|_{X_{k-1}} \leq C \quad \text{for all} \quad n.$$

But X_{k-1} imbeds compactly into $X_k = W_B^{1,2}$. Thus $\{\eta(t_n, u_n) - e^{-w(t_n,u_n)} u_n\}$ has a convergent subsequence in $W_B^{1,2}$. Since $e^{-w(t_n,u_n)} u_n \to 0$ in $W_B^{1,2}$, we conclude that $\{\eta(t_n, u_n)\}$ has a convergent subsequence. The proof is complete. ∎

The following result establishes the relationship between the P.S. condition and η-admissibility.

Theorem 3.2. *Suppose that (H_1) is satisfied. Then J satisfies the P.S. condition if and only if, for any $-\infty < a < b < \infty$, $J^{-1}[a, b]$ is strongly η-admissible.*

Proof. We show first that strongly η-admissibility implies the P.S. condition. Suppose that $J^{-1}[a, b]$ is strongly η-admissible for any finite interval $[a, b]$. Let $\{u_n\} \subset W_B^{1,2}$ be a P.S. sequence: $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$. It follows easily that we can find $t_n > 0$, $t_n \to \infty$ and a finite interval $[a, b]$ such that $\eta([-t_n, 0], u_n) \subset J^{-1}[a, b]$. Let $v_n = \eta(-t_n, u_n)$. Then $\eta([0, t_n], v_n) \subset J^{-1}[a, b]$. Since $t_n \to \infty$, and $J^{-1}[a, b]$ is strongly η-admissible, $u_n = \eta(t_n, v_n)$ has a convergent subsequence. This proves that J satisfies the P.S. condition.

Next we prove the converse. Suppose that J satisfies the P.S. condition. We need to show that $\{\eta(t_n, u_n)\}$ has a convergent subsequence whenever $\eta([0, t_n], u_n) \subset J^{-1}[a, b]$ and $t_n \to \infty$. The proof of Theorem 3.1 shows that it suffices to prove that $\{\eta(t, u_n) : 0 \leq t \leq t_n, n \geq 1\}$ is bounded in $W_B^{1,2}$. Suppose that this set is not bounded. Then, by passing to a subsequence, we can find $s_n \in [0, t_n]$ such that

$$R_n \equiv \|\eta(s_n, u_n)\|_{W_B^{1,2}} \to \infty.$$

Since J satisfies the P.S. condition, its critical points in $J^{-1}[a, b]$ form a bounded set K, say $K \subset B_{R_0-1} = \{u \in W_B^{1,2} : \|u\|_{W_B^{1,2}} \leq R_0 - 1\}$. Moreover, the P.S. condition implies that, for some $\delta \in (0, 1)$, $\|J'(u)\|_{W_B^{1,2}} \geq \delta$ for all $u \in J^{-1}[a, b] \setminus B_{R_0}$. By definition,

$$\min\{1, \|J'(u)\|_{W_B^{1,2}}\} \leq \sigma(u)\|J'(u)\|_{W_B^{1,2}} \leq 1.$$

Thus, if we denote $v_n = \eta(s_n, u_n)$, then

$$\|\eta(t, v_n) - v_n\|_{W_B^{1,2}} \le |t|, \quad \forall t.$$

It follows that

$$\|\eta(t, v_n)\|_{W_B^{1,2}} \ge \|v_n\|_{W_B^{1,2}} - |t| > R_0 \ \text{ whenever } |t| < R_n - R_0.$$

Since $t_n \to \infty$, we have, subject to a subsequence, either (i) $s_n \to \infty$, or (ii) $t_n - s_n \to \infty$.

In case (i), choose $T_n = -\min\{R_n - R_0, s_n\}$. Then $T_n \to -\infty$ and $\eta(t, v_n) \in J^{-1}[a, b] \setminus B_{R_0}$ for all $t \in [T_n, 0]$. Therefore,

$$\begin{aligned} a - b \le J(v_n) - J(\eta(T_n, v_n)) &= \int_{T_n}^0 (d/dt) J(\eta(t, v_n)) dt \\ &= \int_{T_n}^0 -\sigma(\eta(t, v_n)) \|J'(\eta(t, v_n))\|_{W_B^{1,2}}^2 dt \\ &\le \delta^2 T_n \to -\infty, \end{aligned}$$

a contradiction.

In case (ii), we define $T_n = \min\{R_n - R_0, t_n - s_n\}$ and similarly derive

$$b - a \ge J(v_n) - J(\eta(T_n, v_n)) \ge \delta^2 T_n \to \infty,$$

again a contradiction. This finishes the proof. ∎

Remark 3.1. Under the condition (H_1), in general, J satisfies the P.S. condition (or equivalently, by Theorem 3.2, $J^{-1}[a, b]$ is strongly η-admissible) does not imply that $J^{-1}[a, b] \cap X_i$ is strongly $\eta|_{R \times X_i}$-admissible for any $i \le k - 1$. To see this, we choose f such that (H_1) and the P.S. condition are satisfied and that (1.1) has a solution u_0 in $J^{-1}[a + 1, b - 1]$ for some $a < b - 2$. By elliptic regularity, $u_0 \in X_0$. Now choose $u_n \in X_i$ satisfying that

$$\|u_n - u_0\|_{X_{i+1}} \to 0, \qquad \|u_n\|_{X_i} \to \infty.$$

Since $\eta(t, u_0) \equiv u_0$, we can find an increasing sequence $\{t_n\}$ with $t_n \to \infty$ such that $\eta([0, t_n], u_n) \subset B_1(u_0) \cap J^{-1}[a, b]$, where $B_1(u_0) = \{u \in X_{i+1} : \|u\|_{X_{i+1}} \le 1\}$. By Lemma 3.1 there exists $C > 0$ independent of n such that

$$\|\eta(t, u_n) - e^{-w(t, u_n)} u_n\|_{X_i} \le C, \quad \forall t \in [0, t_n], \ n \ge 1.$$

Since $\|u_n\|_{X_i} \to \infty$ and $u_n \to u_0$ in X_k, for any fixed j, we can find $n_j \ge j$ such that $\|u_{n_j}\|_{X_i} e^{-w(t_j, u_0)} > j$ and $|w(t_j, u_0) - w(t_j, u_{n_j})| < 1$. Thus, $\|u_{n_j}\|_{X_i} e^{-w(t_j, u_{n_j})} \to \infty$, and

$$\|\eta(t_j, u_{n_j})\|_{X_i} \ge \|u_{n_j}\|_{X_i} e^{-w(t_j, u_{n_j})} - C \to \infty.$$

Since $\eta([0, t_j], u_{n_j}) \subset \eta([0, t_{n_j}], u_{n_j}) \subset J^{-1}[a, b] \cap X_i$, and $t_{n_j} \to \infty$, our above discussion means that $J^{-1}[a, b] \cap X_i$ is not strongly $\eta|_{R \times X_i}$-admissible.

The following is a useful result.

Lemma 3.2. *Suppose that f satisfies (H_1), J satisfies the P.S. condition, and for some $u_0 \in W_B^{1,2}$ and $T \leq \infty$, $\{\eta(t, u_0) : -\infty < t < T\} \subset J^{-1}[a, b]$, where $-\infty < a < b < \infty$. Then $\{\eta(t, u_0) : -\infty < t < T\}$ is a bounded set in $X_0 = W_B^{2,p}$.*

Proof. By Theorem 3.2, $J^{-1}[a, b]$ is strongly η-admissible. It follows easily that $B \equiv \{\eta(t, u_0) : -\infty < t < T\}$ is precompact in $X_k = W_B^{1,2}$. Let $v \in B$ be an arbitrary point. By (3.1) we have

$$v = e^{w(t,v)}\eta(t, v) - \int_0^t e^{w(s,v)} A(\eta(s, v))\sigma(\eta(s, v))ds. \tag{3.2}$$

Since B is precompact in X_k, we can find $t_n \to -\infty$ and some $v_0 \in X_k$ such that $\eta(t_n, v) \to v_0$ in X_k. Using $w(t_n, v) \to -\infty$, and passing to the limit in (3.2) with $t = t_n$, we obtain

$$v = \int_{-\infty}^0 e^{w(s,v)} A(\eta(s, v))\sigma(\eta(s, v))ds. \tag{3.3}$$

Since B is precompact and hence bounded in X_k, by the properties of A, there exists C_1 such that $\sup_{u\in B} \|A(u)\|_{X_{k-1}} \leq C_1$. Thus it follows from (3.3) that

$$\|v\|_{X_{k-1}} \leq \int_{-\infty}^0 e^{w(s,v)}\sigma(\eta(s, v))C_1 ds = C_1.$$

But v is an arbitrary point in B. Therefore,

$$\sup_{u\in B} \|u\|_{X_{k-1}} \leq C_1.$$

Now, by the properties of A, we can find C_2 such that $\sup_{u\in B} \|A(u)\|_{X_{k-2}} \leq C_2$, and by (3.3), we deduce

$$\sup_{u\in B} \|u\|_{X_{k-2}} \leq C_2.$$

Clearly, Lemma 3.2 follows by repeatedly using the above argument. The proof is complete. ∎

The following result follows from Theorems 3.1, 3.2 above and Corollary 4.7 and Theorem 4.8 in Chapter III of Rybakowski [26].

Lemma 3.3. *Suppose that f satisfies (H_1). Then*

(i) If J satisfies the P.S. condition and has no critical point u in the sets $\{J(u) = a\}$ and $\{J(u) = b\}$, where $a < b$ are real numbers, then

$$H_q(h(\eta, K)) = H_q(J^{-1}(-\infty, a], J^{-1}(-\infty, b]),$$

where K is the maximal invariant set of η in $J^{-1}[a, b]$.

(ii) If u is an isolated critical point of J, then

$$H_q(h(\eta, \{u\})) = C_q(J, u).$$

Theorem 3.3. *Suppose that f satisfies (H_1), J satisfies the P.S. condition and* (1.1) *has a pair of strict lower and upper solutions $u_0, v_0 \in C^2(\overline{D})$, $u_0 < v_0$, u_0 being a lower solution and v_0 an upper solution. Then there exist real numbers $a_0 < b_0$ such that*

(i) For any $a \le a_0$ and $b \ge b_0$, the maximal invariant set K_0 of η in $[u_0, v_0]_{W_B^{1,2}} \cap J^{-1}[a, b]$ is independent of a,b.

(ii) $h(\eta, K_0)$ is well-defined and $h(\eta, K_0) = \Sigma^0$, i.e., $H_q(h(\eta, K_0)) = \delta_{q,0}G$.

Proof. By standard argument, all the solutions of (1.1) contained in $[u_0, v_0]_{W_B^{1,2}}$ form a compact set S_0 in $W_B^{1,2}$. Therefore we can find $a_0 < b_0$ such that $S_0 \subset J^{-1}(a_0, b_0)$. Let $a \le a_0,\ b \ge b_0$. By Theorem 3.2, $J^{-1}[a, b]$ is strongly η-admissible.

Let $\eta((-\infty, \infty), u_0)$ be any entire orbit in $[u_0, v_0]_{W_B^{1,2}} \cap J^{-1}[a, b]$. By Lemma 3.2, $\eta((-\infty, \infty), u_0)$ is bounded in X_0 and hence precompact in $W_B^{1,2}$. Thus both $\alpha(\eta)$ and $\omega(\eta)$ are non-empty. Since J is a quasi-potential of η, $\alpha(\eta)$ and $\omega(\eta)$ consist of equilibria of η, that is, solutions of (1.1). By our choice of a_0 and b_0, $\alpha(\eta) \cup \omega(\eta) \subset J^{-1}(a_0, b_0)$ and thus $\eta((-\infty, \infty), u_0) \subset J^{-1}(a_0, b_0)$. This implies that the maximal invariant set of η in $[u_0, v_0]_{W_B^{1,2}} \cap J^{-1}[a, b]$ does not depend on a, b. Denote this set by K_0 and the proof for (i) is finished.

We may assume that the constant k in the definition of A is chosen large enough so that $u \to f(u) + ku$ is strictly increasing for $u \in [\min_D u_0(x) - 1, \max_D v_0(x) + 1]$. Then it follows from standard argument (see, e.g., [3]) that $u_n = A^n u_0$ and $v_n = A^n v_0$ satisfy:

$$u_0 < u_1 < ... < u_n < u_{n+1} < ... \to u^*, \quad v_0 > v_1 > ... > v_n > v_{n+1} > ... \to v^*,$$

and u^*, v^* are the minimal and maximal solutions of (1.1) in $[u_0, v_0]_{W_B^{1,2}}$ respectively. Denote by N_δ the closed δ-neighborhood of $[u^*, v^*]_{W_B^{1,2}}$ in $W_B^{1,2}$. We show that, for all small $\delta > 0$, any solution of (1.1) in N_δ belongs to S_0. In fact, if w is any solution of (1.1) in N_δ, and if $w_0 \in [u^*, v^*]_{W_B^{1,2}}$ is such that $\|w - w_0\|_{W_B^{1,2}} < 2\delta$, then, since A^k is continuous from X_k to X_0, by making δ arbitrarily small, we can guarantee that $\|A^k(w) - A^k(w_0)\|_{X_0}$ is arbitrarily small. But $w = A^k(w)$ and $A^k(w_0) \in [u^*, v^*]_{C^1}$ which is in the interior of $[u_0, v_0]_{C^1}$. Therefore, $w \in [u_0, v_0]_{C^1} \subset [u_0, v_0]_{W_B^{1,2}}$. This shows that $w \in S_0$.

Fix such a small $\delta > 0$ and let K_1 be the maximal invariant set of η in $N_\delta \cap J^{-1}[a_0, b_0]$. We show that $K_1 = K_0$. Indeed, if $\eta((-\infty, \infty), u_0)$ is an arbitrary orbit in K_1, then it is precompact in $W_B^{1,2}$ and $\alpha(\eta) \cup \omega(\eta) \subset S_0$ by what we have just proved. Let $w \in \alpha(\eta)$. By Lemma 3.2, $\eta((-\infty, \infty), u_0)$ is precompact in C^1. We may assume that $\eta(t_n, u_0) \to w$ in C^1, where $t_n \to -\infty$. Since $w \in [u^*, v^*]_{C^1}$, for any fixed positive integer m, $\eta(t_n, u_0) \in [u_m, v_m]_{C^1}$ for all large n. Since $[u_m, v_m]_{W_B^{1,2}}$ is A-invariant, as in Hofer[21], it follows from a well-known result in Banach space ordinary differential equation theory that $[u_m, v_m]_{W_B^{1,2}}$ is η-invariant. It follows that $\eta(t, u_0) \in [u_m, v_m]_{W_B^{1,2}}$ for all $t \in (-\infty, \infty)$. Since this is true for any m and $u_m \to u^*, v_m \to v^*$, we obtain $\eta((-\infty, \infty), u_0) \subset [u^*, v^*]_{W_B^{1,2}}$. Similarly, any entire orbit in K_0 belongs to $[u^*, v^*]_{W_B^{1,2}}$. This shows in particular that $K_1 = K_0$.

The above argument also shows that $N_\delta \cap J^{-1}[a_0, b_0]$ is an isolating neighborhood of K_0. Since it is strongly η-admissible by Theorem 3.2, we conclude that $h(\eta, K_0)$ is well-defined.

Next we calculate $h(\eta, K_0)$ by a homotopy argument. First we may assume that $u^* = 0$ and $f(0) = 0$. Otherwise, we can make a change of variables $w = u - u^*$ and replace f by

$$\tilde{f}(x, w) = f(u^*(x) + w) - f(u^*(x))$$

from the very beginning. One easily sees that under this change of variables, the new f and new J satisfy all the conditions of the Lemmas and Theorems in this section so far. Therefore, this change of variables does not affect the previous results in this section.

We make a remark here for our later purpose. Since in general $\tilde{f}$ is x dependent even if f is not, we need to be a little careful here if we want to use this theorem to prove Proposition 1.1, where condition (H_3) is needed. We observe that the change of variables here does not produce trouble for the proof of Proposition 1.1, as we will use $u_0 = \alpha'$, $v_0 = \beta'$ and $u^* = \alpha_1$ is a constant in the proof of Proposition 1.1, where α', β' and α_1 are defined in the proof of Proposition 2.1.

Now go back to our calculation of $h(\eta, K_0)$ under the assumptions that $u^* = 0$ and $f(0) = 0$. We consider the following homotopy:

(3.4) $-\Delta u + ku = f_\mu(u) \equiv \mu[f(u) - ku]$ in D, $\quad Bu = 0$ on ∂D, $0 \le \mu \le 1$.

Let J_μ, η_μ and A_μ denote the corresponding J, η and A, where f is replaced by f_μ. It is easy to check that $u_0 < 0$ and $v_0 > 0$ are strict lower and upper solutions of (3.4) for any fixed $\mu \in [0, 1]$ (here we use $f(u_0) + ku_0 < 0 < f(v_0) + kv_0$). Moreover, by standard regularity argument, the set

$$S_1 = \{u \in [u_0, v_0]_{W_B^{1,2}} : u \text{ solves (3.4) for some } \mu \in [0, 1]\}$$

is compact in $W_B^{1,2}$. Thus we can find real numbers $a_1 < b_1$ such that $S_1 \subset J_\mu^{-1}[a_1 + 1, b_1 - 1]$ for all $\mu \in [0, 1]$.

For any fixed $\mu \in [0, 1]$, $u_\mu^* = \lim A_\mu^n u_0$ and $v_\mu^* = \lim A_\mu^n v_0$ are minimal and maximal solutions of (3.4) in $[u_0, v_0]_{W_B^{1,2}}$. It is easy to see that $u_\mu^* \ge u_0^* = u^*$ and $v_\mu^* \le v_0^* = v^*$ for any $\mu \in [0, 1]$. Now we use a similar argument to that used in showing that $N_\delta \cap J^{-1}[a_0, b_0]$ is a strongly η-admissible isolating neighborhood of K_0, and obtain that, for $\delta > 0$ small,

$$N \equiv N_\delta \cap_{\mu \in [0,1]} J_\mu^{-1}[a_1, b_1]$$

is a strongly η_μ-admissible isolating neighborhood of K^μ for each $\mu \in [0, 1]$, where K^μ denotes the maximal invariant set of η_μ in N, and N_δ is the closed δ-neighborhood of $[u^*, v^*]_{W_B^{1,2}}$.

It is easily seen that if $\mu_n \to \mu_0$ with $\mu_n, \mu_0 \in [0,1]$, then $\eta_n \to \eta_0$. Moreover, a simple variant of the proof of Proposition 2.1 shows that N is $\{\eta_{\mu_n}\}$-admissible. Therefore, we can use the continuation property of the generalized Conley index to conclude that $h(\eta_\mu, K^\mu)$ is independent of $\mu \in [0,1]$. In particular,

$$h(\eta, K_0) = h(\eta_1, K^1) = h(\eta_0, K^0).$$

But clearly $K^0 = \{0\}$ and $u = 0$ is a global minimum of J_0. Thus, by Lemma 3.3,

$$H_q(h(\eta_0, K^0)) = C_q(J_0, 0) = \delta_{q,0}G.$$

The proof is complete. ■

Remark 3.2. The homotopy (3.4), together with the change of variables trick, also works for the parabolic flow. In this case the argument is much more simpler because we can simply use $[u_0, v_0]_{X^\theta}$ as the common isolating neighborhood. This observation can be used to simplify the arguments in a number of places in section 3 of [10].

Remark 3.3. All the above results in this section also hold true for much more general problems than (1.1). For example, they are true for the cases indicated in Remark 2.1. Therefore, they should have many other applications.

Proof of Proposition 1.1. This is just a simple variant of that for Proposition 2.1. Therefore we just give an outline of the proof by pointing out the points where care is needed.

We make use of the previous results in this section.

First, we observe that Theorem 2.2 is still true with π replaced by η. Then we follow the proof of Proposition 2.1 with π replaced by η. Note that (2.6) is now replaced by Theorem 3.3, and (2.10) by Theorem 2.2 with π replaced by η.

Second, we notice that the only other point we need to be careful with is in checking the Morse decompositions. Since by Lemma 3.2 any entire orbit is compact in C^1, we see that any orbit approaches its alpha limit set in the norm of C^1 (actually C^0 is enough for our present purpose because we are dealing with Neumann boundary conditions). Moreover, as in the proof of Theorem 3.3, if $u_0 < v_0$ and u_0 and v_0 are lower and upper solutions of (1.4) respectively, then $[u_0, v_0]$ is η-invariant. These two facts guarantee that the argument in checking the Morse decompositions in the proof of Proposition 2.1 also works now. This finishes the outline of the proof. ■

4. A Dirichlet Problem

In this section, we study problem (1.5). Proposition 1.2 is a consequence of our results in this section.

Theorem 4.1. *Suppose that g satisfies (H_4) and $\underline{\lim}_{|u|\to\infty} g(u)/u > \lambda_1$. Then there exist $\Lambda^+, \Lambda^- \in (0,\infty)$ such that* *(i) for* $\lambda > \Lambda^+$ *(resp.* Λ^-*),* (1.5) *has no positive (resp. negative) solution;*

(ii) for $0 < \lambda < \Lambda^+$ *(resp.* Λ^-*),* (1.5) *has at least one positive (resp. negative) solution;*

(iii) for $0 < \lambda < \min\{\Lambda^+, \Lambda^-\}$*,* (1.5) *has at least one sign-changing solution.*

Proof. (i) Define

$$\Lambda^+ = \sup\{\lambda > 0 : (1.5) \text{ has a positive solution }\},$$
$$\Lambda^- = \sup\{\lambda > 0 : (1.5) \text{ has a negative solution }\}.$$

Then the conclusion follows from a simple variant of the proof of Lemma 3.1 in [2].

(ii) The same upper and lower solution argument as in [2] shows that (1.5) has a minimal positive solution u_λ for $0 < \lambda < \Lambda^+$, and a maximal negative solution v_λ for $0 < \lambda < \Lambda^-$.

(iii) This follows from a variant of the proof of Theorem 1 in [11] based on Hofer [21]. We just sketch the main steps. By truncating g outside $[-\|v_\lambda\|_\infty, \|u_\lambda\|_\infty]$, we may assume that J_λ, defined by

$$J_\lambda(u) = \int_D \left[|\nabla u|^2/2 - \lambda |u|^{r+1}/(r+1) - \int_0^u g(s)ds\right] dx,$$

satisfies the P.S. condition in $W_0^{1,2}(D)$. Let

$$C = [v_\lambda, u_\lambda]_{W_0^{1,2}} \equiv \{u \in W_0^{1,2}(D) : v_\lambda \le u \le u_\lambda\}.$$

Note first that J_λ is C^1 and the monotonicity condition for $I - J'_\lambda$ in [21] can be easily met as $\lambda|u|^{r-1}u$ is increasing with u and g is locally Lipschitz. Then as in [11], v_λ and u_λ are strict local minima of J_λ on C. It then follows from the mountain pass theorem on C (see [21]) that J_λ has a critical point w of type $-I$ in C(see [21] for the definition of type $-I$ critical points). Note that C^1 smoothness for J is enough here. Since u_λ is the minimal positive solution and v_λ is the maximal negative solution, w must be a sign-changing solution if we can show that $w \neq 0$. Since the righthand side of (1.5) is greater than au for all small positive u and some $a > \lambda_2$, a variant of the argument in Step 2 of the proof of Theorem 1 in [11] shows that 0 is a critical point of type X (see [21] for the definition of type X critical points). Hence $w \neq 0$.

The proof is complete. ∎

Remark 4.1. Note that in Theorem 4.1, there is no growth restriction on g from above. One easily sees that our method used to prove (iii) above can be used to answer affirmatively a question in Remark 2.9 of [1], where they conjecture that their growth restriction on g is not necessary for the existence of w (in their setting, one easily shows that $J_\lambda(w) < 0$).

Theorem 4.2. *Suppose that* g *satisfies* (H_4) *and* (H_5)*, and* Λ^+*,* Λ^- *are as in Theorem 4.1. Then*

(i) *for* $\lambda = \Lambda^+$ *(resp.* Λ^-*),* (1.5) *has at least one positive (resp. negative) solution;*

(ii) *for* $0 < \lambda < \Lambda^+$ *(resp* Λ^-*),* (1.5) *has at least two positive (resp. negative) solutions;*

(iii) *for* $0 < \lambda \le \max\{\Lambda^+, \Lambda^-\}$*,* (1.5) *has at least two sign-changing solutions.*

Proof. (i) We carry out the proof for $\lambda = \Lambda^+$ only; the other case can be proved analogously. By Theorem 4.1, (1.5) has a minimal positive solution u_λ for any $\lambda \in (0, \Lambda^+)$. Since

$$\lim_{u \to +\infty} (\lambda u^r + g(u))/u = a > \lambda_1,$$

uniformly in $\lambda \in (0, \Lambda^+)$, it follows from a well-known result on asymptotically linear problems (see e.g. [3]) that $\|u_\lambda\|_\infty \le C$ for all $\lambda \in (0, \Lambda^+)$ and some positive constant C. It follows from elliptic regularity that $\{u_\lambda : 0 < \lambda < \Lambda^+\}$ is precompact in C^1. Hence for some sequence $\lambda_n \to \Lambda^+$, u_{λ_n} converges to a solution u of (1.5) with $\lambda = \Lambda^+$. By [2], $u_\lambda \ge w_\lambda$, where w_λ is the unique positive solution of

$$-\Delta w = \lambda w^r, \quad w|_{\partial D} = 0,$$

which is always a lower solution to (1.5). Therefore u must be a positive solution and $u \ge w_\lambda$. This implies that (1.5) has a minimal positive solution u_λ for $\lambda = \Lambda^+$ as well.

(ii) Again we consider the case $0 < \lambda < \Lambda^+$ only. The other case is similar. Choose $\lambda' \in (\lambda, \Lambda^+)$ and define $\overline{u} = u_{\lambda'}$. Then $\overline{u}$ is an upper solution to (1.5). Let u^* be the maximal solution of (1.5) between w_λ and $\overline{u}$. Then the proof of Proposition 2 in [12] shows that (1.5) has a mountain pass solution $u > u^*$ (some details are recalled in the proof of (iii) below).

(iii) We break the proof into three steps.

Step 1. For $0 < \lambda \le \max\{\Lambda^+, \Lambda^-\}$, (1.5) has at least one sign-changing solution lying in the order interval $[v_\lambda, u_\lambda]_{W_0^{1,2}}$. Here we understand that $u_\lambda = +\infty$ if $\lambda > \Lambda^+$ and $v_\lambda = -\infty$ if $\lambda > \Lambda^-$.

By Theorem 4.1, we need only prove Step 1 for the case $\min\{\Lambda^+, \Lambda^-\} \le \lambda \le \max\{\Lambda^+, \Lambda^-\}$.

For $\lambda = \min\{\Lambda^+, \Lambda^-\}$, the proof is the same as in (iii) Theorem 4.1, since we have proved in (i) that (1.5) has a minimal positive solution and a maximal negative solution. Suppose next that $\min\{\Lambda^+, \Lambda^-\} < \max\{\Lambda^+, \Lambda^-\}$. We may assume that $\Lambda^- < \Lambda^+$; the other case is similar.

Now for $\Lambda^- < \lambda \le \Lambda^+$, (1.5) has a minimal positive solution u_λ and no negative solution. Let $C = \{u \in W_0^{1,2}(D) : u \le u_\lambda\}$. We first use a trick in the proof of Proposition 2 in [12]. Denote $f(u) = \lambda |u|^{r-1} u + g(u)$ and define

$$\tilde{f}(x, u) = f(u) \ \text{ for } \ u \le u_\lambda(x); \quad \tilde{f}(x, u) = f(u_\lambda(x)) \ \text{ for } u \ge u_\lambda(x).$$

Then $\tilde{f}$ is continuous, and for any given finite interval in R^1, we can find a positive constant k such that $\tilde{f}(x, u) + ku$ is strictly increasing in u for u in that interval and all x in D. Moreover, it is easily seen that w_λ and $u_\lambda + \varepsilon \phi_1$ is a pair of lower and upper solutions to

$$-\Delta u = \tilde{f}(x, u), \qquad u|_{\partial D} = 0.$$

Here $\varepsilon > 0$ is a constant and ϕ_1 is given by

$$-\Delta\phi_1 = \lambda_1\phi_1, \quad \phi_1|_{\partial D} = 0, \phi_1 \geq 0, \quad \|\phi_1\|_\infty = 1.$$

As in [12], it follows from [6] that the functional

$$\tilde{J}_\lambda(u) = \int_D |\nabla u|^2/2dx - \int_D \int_0^u \tilde{f}(x,s)dsdx,$$

restricted to $E = C_0^1(\overline{D})$ has a local minimizer in the order interval $[w_\lambda, u_\lambda + \varepsilon\phi_1]_E$, and it is also a local minimizer of $\tilde{J}_\lambda$ in $H = W_0^{1,2}(D)$. By the definition of $\tilde{f}$, clearly u_λ is the only critical point of $\tilde{J}_\lambda$ in the above interval. Therefore, u_λ is an isolated critical point of $\tilde{J}_\lambda$ in H. One easily checks that $\tilde{J}_\lambda(-t\phi_1) \to -\infty$ as $t \to +\infty$. The argument in [12] shows that $\tilde{J}_\lambda$ satisfies the P.S. condition. Since u_λ is an isolated critical point of $\tilde{J}_\lambda$, as in [12], $\inf\{\tilde{J}_\lambda(u) : \|u - u_\lambda\|_H = \varepsilon\} > \tilde{J}_\lambda(u_\lambda)$. Thus we can use the well-known mountain pass theorem to obtain a critical point u_0 of $\tilde{J}_\lambda$. It follows from the definition of $\tilde{f}$ that $u_0 \in C$, hence it is a solution to (1.5). Clearly $u_0 \neq u_\lambda$. At this stage, we are not sure if $u_0 = 0$. If we can show that (1.5) has a solution in $C \setminus \{0, u_\lambda\}$, then this solution must be a sign-changing solution as there is no negative solution and u_λ is the minimal positive solution.

Next we use an idea of [21]. Arguing indirectly, we assume that 0 and u_λ are the only solutions of (1.5) in C. Hence we necessarily have $u_0 = 0$ and thus $\tilde{J}_\lambda(u_\lambda) < \tilde{J}_\lambda(u_0) = 0$. Using a suitable equivalent norm of H, we can assume that $I - \tilde{J}'_\lambda$ is increasing in the order interval $[-1, u_\lambda + 1]_H$. From the behaviour of the nonlinearity near 0, as before, 0 is a critical point of $\tilde{J}_\lambda$ of type X. This would contradicts the well-known characterization of a mountain pass solution if our functional is smooth enough near 0. To get around this smoothness problem, we use the proof of Lemma 3 in [21] to find a continuous map $\gamma : [0,1] \to H$ such that $\gamma(0) = u_\lambda$, $\tilde{J}_\lambda(\gamma(1)) < \tilde{J}_\lambda(u_\lambda)$ and $\tilde{J}_\lambda(\gamma(t)) < \tilde{J}_\lambda(0) = 0$ for all $t \in [0,1]$. This implies that the mountain pass solution satisfies $\tilde{J}_\lambda(u_0) < 0$, a contradiction. This finishes the proof of Step 1.

Step 2. Let $0 < \lambda \leq \min\{\Lambda^+, \Lambda^-\}$. Then (1.5) has a sign-changing solution outside the order interval $[v_\lambda, u_\lambda]_{W_0^{1,2}}$.

We use some results and techniques from [12]. Since the righthand side of (1.5) is not C^1 at $u = 0$, we cannot use [12] directly. We modify $|u|^{r-1}u$ near $u = 0$ as follows. For $\delta > 0$ small, let

$$h_\delta(u) = \delta^{r-1}u \text{ for } u \in [-\delta/2, \delta/2]; \quad h_\delta(u) = |u|^{r-1}u \text{ for } \; u \notin [-2\delta, 2\delta];$$

$$h_\delta(u) \text{ is } C^1 \text{ and increasing for } u \in [-2\delta, -\delta/2] \cup [\delta/2, 2\delta].$$

Then consider

$$-\Delta u = \lambda h_\delta(u) + g(u), \quad u|_{\partial D} = 0. \tag{4.1}$$

With care, it is easy to see that we can construct h_δ as above and such that $h_\delta(u) \leq |u|^{r-1}u$ for $u > 0$ and the reverse inequality holds for $u < 0$. It follows that u_λ is a strict upper solution of (4.1) and v_λ is a strict lower solution of (4.1). Since δ is small, we can find $\varepsilon_0 > 0$ small such that $\varepsilon\phi_1 \leq u_\lambda$

and $\varepsilon\phi_1$ is a lower solution of (4.1) for all $\varepsilon \in (0, \varepsilon_0)$. It follows that (4.1) has a minimal positive solution u^δ. Similarly, it has a maximal negative solution v^δ. Now the modified problem (4.1) satisfies all the conditions of Proposition 2 in [12]. By Propositions $2, 2', 2''$ and their proofs in [12], we conclude that (4.1) has a sign-changing solution w^δ outside the order interval $[v^\delta, u^\delta]$. We want to show that, for some $\delta_n \to 0$, w^{δ_n} converges to a sign-changing solution of (1.5) outside $[v_\lambda, u_\lambda]$; this would finish our proof of Step 2.

We show first that $v^\delta \to v_\lambda$ and $u^\delta \to u_\lambda$ as $\delta \to 0$. Since $h_{\delta_1}(u) \geq h_{\delta_2}(u)$ if $u \geq 0$ and $\delta_1 \leq \delta_2$, an easy upper and lower solution argument shows that $u^{\delta_1} \geq u^{\delta_2}$. By a simple regularity argument, $\{u^\delta : 0 < \delta \leq \delta_0\}$ is precompact in $C^1(\overline{D})$ for any fixed small positive δ_0. Thus for any sequence $\delta_n \to 0$ there is a subsequence still denoted by δ_n such that $u^{\delta_n} \to u$ in C^1. One easily sees that u is a solution to (1.5). Moreover, $u^{\delta_n} \leq u \leq u_\lambda$. Hence we must have $u = u_\lambda$ as u_λ is the minimal positive solution of (1.5). This implies that $u^\delta \to u_\lambda$ in C^1 as $\delta \to 0$. The proof that $v^\delta \to v_\lambda$ is similar.

We show next that $\|w^\delta\|_\infty \leq M$ for all small δ and some fixed positive constant M. We argue indirectly. Suppose to the contrary that there exists a sequence of positive numbers $\delta_n \to 0$ such that $\|w^{\delta_n}\|_\infty \to \infty$. Then let $w_n = w^{\delta_n}/\|w^{\delta_n}\|_\infty$, and divide (4.1) with $(\delta, u) = (\delta_n, w^{\delta_n})$ by $\|w^{\delta_n}\|_\infty$, and use $\lim_{|u|\to\infty} g(u)/u = a$ and $|h_{\delta_n}(w^{\delta_n})| \leq |w^{\delta_n}|^r$. It results

$$\Delta w_n = a w_n + o(1), \quad w_n|_{\partial D} = 0. \tag{4.2}$$

From the L^∞ boundedness of the righthand side of (4.2) we infer that $\{w_n\}$ is precompact in C^1. By passing to a subsequence we may assume that $w_n \to w$ in C^1. Now let $n \to \infty$ in (4.2) we obtain

$$-\Delta w = aw, \quad w|_{\partial D} = 0.$$

Since $\|w\|_\infty = 1$, this implies that $a = \lambda_k$ for some $k \geq 1$, which contradicts (H_5).

Thus we have proved that $\|w^\delta\|_\infty \leq M$ for all $\delta \in (0, \delta_0]$, where $\delta_0 > 0$ is small. Using this fact, we find that the righthand side of (4.1) with $u = w^\delta$ is L^∞ bounded uniformly for all small positive δ. Hence it follows from elliptic regularity that $\{w^\delta : 0 < \delta \leq \delta_0\}$ is precompact in C^1. Thus we can find a sequence of positive numbers $\delta_n \to 0$ such that $w^{\delta_n} \to w$ in C^1. One easily sees that w is a solution of (1.5). Since w^{δ_n} is outside $[v^{\delta_n}, u^{\delta_n}]_{C^1}$ and $v^{\delta_n} \to v_\lambda$, $u^{\delta_n} \to u_\lambda$ in C^1, it follows that w is not in the interior of $[v_\lambda, u_\lambda]_{C^1}$ considered as a subset of $C^1(\overline{D})$.

We prove that w changes sign. Otherwise w is a positive solution or negative solution. If it is a positive solution, then, by the maximum principle, we deduce that $w > 0$ in D and $\partial w/\partial n < 0$ on ∂D. This implies that $w^{\delta_n} > 0$ in D for all large n, which contradicts the fact that w^{δ_n} changes sign. Similarly w cannot be negative. Hence w must change sign.

It remains to show that w is outside $[v_\lambda, u_\lambda]_{W_0^{1,2}}$. It suffices to show that w is outside $[v_\lambda, u_\lambda]_{C^1}$. If this is not true, then choosing constant $k > 0$ such that $\lambda|u|^{r-1}u + g(u) + ku$ is increasing for u in the range $-\|v_\lambda\|_\infty \leq u \leq$

$\|u_\lambda\|_\infty$, we obtain that

$$-\Delta(u_\lambda - w) + k(u_\lambda - w) \geq 0, \not\equiv 0, \quad (u_\lambda - w)|_{\partial D} = 0.$$

It follows then from the maximum principle that $u_\lambda - w > 0$ in D and $\partial(u_\lambda - w)/\partial n < 0$ on ∂D. The same is true also for $w - v_\lambda$. Thus w is in the interior of $[v_\lambda, u_\lambda]_{C^1}$, which contradicts our earlier observation. This finishes the proof of Step 2.

Step 3. If $\Lambda^- < \Lambda^+$, then for $\lambda \in (\Lambda^-, \Lambda^+]$, (1.5) has a sign-changing solution not comparable with u_λ. An analogous result hold if $\Lambda^- > \Lambda^+$.

The proof of Step 3 is a variant of that of Step 2. We will just sketch the proof. Again we consider the modified problem (4.1) first and then pass to the limit $\delta \to 0$. As before, for all small positive ε and any $\lambda \in (\Lambda^-, \Lambda^+]$, $\varepsilon\phi_1$ and u_λ is a pair of strict lower and upper solutions to (4.1), and it follows that (4.1) has a minimal positive solution u^δ. Using the Lemma in the proof of Proposition $2''$ of [12], we can find a strict upper solution $\tilde{u}_\delta$ of (4.1) satisfying $u^\delta < \tilde{u}_\delta \leq u_\lambda$ such that $\omega(\tilde{u}_\delta) = u^*_\delta$ is a positive solution of (4.1) which is comparable with any positive solution of (4.1). Here $\omega(v)$ denotes the omega limit set of the solution of the corresponding parabolic problem of (4.1) which passes through v. Now using Proposition $2'$ of [12] to (4.1) with $\underline{u} = \varepsilon\phi_1$ and $\overline{u} = \tilde{u}_\delta$ we conclude that (4.1) has a solution w^δ not comparable with at least one of $u^\delta = \omega(\varepsilon\phi_1)$ and $u^*_\delta = \omega(\tilde{u}_\delta)$. It follows that w^δ must be a sign-changing solution as any positive solution of (4.1) is comparable with both u^*_δ and the minimal positive solution u^δ.

Now as in the proof of Step 2, we can show by a compactness argument that for some $\delta_n \to 0$, $w^{\delta_n} \to w$ in C^1, and w is a solution of (1.5). Since w^{δ_n} is not comparable with u^{δ_n} which was shown in Step 2 to converge to u_λ in C^1, we conclude that $w \neq 0$. Since w^{δ_n} changes sign, we conclude that $w \neq u_\lambda$. Now as in the last part of the proof of Step 2, we find that if w and u_λ are comparable, then either $w - u_\lambda$ or $u_\lambda - w$ is in the interior of the natural positive cone of $C^1_0(\overline{D})$. But since $w^{\delta_n} \to w$, $u^{\delta_n} \to u_\lambda$ and $u^*_{\delta_n} \to u_\lambda$ in C^1, it follows that either both $w^{\delta_n} - u^{\delta_n}$ and $w^{\delta_n} - u^*_{\delta_n}$, or both $u^{\delta_n} - w^{\delta_n}$ and $u^*_{\delta_n} - w^{\delta_n}$ are in the interior of the positive cone in C^1_0 if w is comparable with u_λ. But this contradicts the fact that w^{δ_n} is not comparable with at least one of u^{δ_n} and $u^*_{\delta_n}$. This shows that w is a sign-changing solution which is not comparable with u_λ, as required. The case $\Lambda^- > \Lambda^+$ can be considered in a similar way.

The proof of Theorem 4.2 is complete. ■

Remark 4.2. Note that our proof of (iii) gives information on the location of the two sign-changing solutions, i.e., one is inside the order interval $[v_\lambda, u_\lambda]$, another is outside this interval. Here we use the convention for the notation of this order interval as in the statement of Step 1. Note also that the argument used in the proof of Step 3 also works for the case of Step 2 but it gives less information on the location of the second sign-changing solution in this case.

Theorem 4.3. *If g satisfies* (H_1), (H_2) *and* (H_4), *then the conclusions* (i)-(ii) *in Theorem* 4.2 *are still true. Moreover,*

(iii)$'$ *For* $0 < \lambda \leq \max\{\Lambda^+, \Lambda^-\}$, (1.5) *has at least one sign-changing solution.*

If we assume further that g satisfies (H_6) *(which implies* (H_1) *and* (H_2)*), then*

(iv) *for any* $0 < \lambda < \max\{\Lambda^+, \Lambda^-\}$, (1.5) *has at least two sign-changing solutions.*

Proof. (i) We consider the case $\lambda = \Lambda^+$ only. By Theorem 4.1, (1.5) has a minimal positive solution u_λ for any $\lambda \in (0, \Lambda^+)$. As in [2], one easily sees that $u_{\lambda'} \leq u_{\lambda''}$ if $0 < \lambda' \leq \lambda'' < \Lambda^+$. Also, if w_λ is defined as in the proof of Theorem 4.2, then $w_\lambda \leq u_\lambda$ and w_λ is a lower solution of (1.5) for any $\lambda \in (0, \Lambda^+)$. Fix any such λ and then let $\lambda' \in (\lambda, \Lambda^+)$. Then w_λ and $u_{\lambda'}$ is a pair of lower and upper solutions of (1.5) with $w_\lambda \leq u_{\lambda'}$. Thus, as in the proof of Lemma 4.1 in [2], one can use [6] to conclude that (1.5) has a solution $\tilde{u}_\lambda$ between w_λ and $u_{\lambda'}$ which minimizes J_λ on $[w_\lambda, u_{\lambda'}]_{C^1}$. In particular, $J_\lambda(\tilde{u}_\lambda) \leq J_\lambda(w_\lambda)$. This implies that

$$J_\lambda(\tilde{u}_\lambda) \leq M \equiv \max_{\Lambda^+/2 \leq \lambda \leq \Lambda^+} J_\lambda(w_\lambda), \quad \forall \lambda \in (\Lambda^+/2, \Lambda^+).$$

Now (H_2) infers that $\sup\{\|\tilde{u}_\lambda\|_{W_0^{1,2}} : \Lambda^+/2 < \lambda < \Lambda^+\} < \infty$. Hence this set is weakly precompact in $W_0^{1,2}$ and precompact in L^γ. Now choosing a sequence $\lambda_n \to \Lambda^+$ such that $\tilde{u}_{\lambda_n} \to u$ weakly in $W_0^{1,2}$ and strongly in L^γ, and passing to the limit in (1.5) with $(\lambda, u) = (\lambda_n, \tilde{u}_{\lambda_n})$, using (H_1), we conclude that u is a weak solution of (1.5) with $\lambda = \Lambda^+$. u must be a positive solution since $u \geq w_{\Lambda^+/2}$. A bootstrap argument then shows that u is a classical solution. Moreover, as before, (1.5) has a minimal positive solution at $\lambda = \Lambda^+$.

(ii) and (iii)$'$ These follow from simple variations of the proofs of (ii) and Step 1 in (iii) of Theorem 4.2. Note that (H_1) and (H_2) guarantee that we still have the P.S. condition and the mountain pass theorem applies as before.

(iv) We may assume that $\Lambda^- < \Lambda^+$. The proofs for the other cases are similar or simpler. By (iii)$'$ and its proof we need only show that (1.5) has a sign-changing solution not comparable with u_λ for any $0 < \lambda < \Lambda \equiv \max\{\Lambda^+, \Lambda^-\}$. We use a degree argument.

Let $\lambda, \lambda', \lambda'' \in (0, \Lambda)$ be fixed with $\lambda'' < \lambda < \lambda'$. Choose constant $k > 0$ so that $g(u_\lambda(x) + u) - g(u_\lambda(x)) + ku$ is positive when $u > 0$ and negative when $u < 0$, and it is strictly increasing for u in the range $[-\|u_\lambda - u_{\lambda''}\|_\infty - 1, \|u_{\lambda'} - u_\lambda\|_\infty + 1]$. Define $K = (-\Delta + kI)^{-1}$ with Dirichlet boundary conditions and define G by

$$G(u) = H(u_\lambda + u) - H(u_\lambda) + ku, \text{ where } H(u)(x) = \lambda|u(x)|^{r-1}u(x) + g(u(x)).$$

Then, as in [11], $A = KG$ is completely continuous as a mapping of both $H = W_0^{1,2}(D)$ to itself and $E = C_0^1(\overline{D})$ to itself. Moreover, A is strongly increasing in $[u_{\lambda''} - u_\lambda, u_{\lambda'} - u_\lambda]_E$ under the order induced by the natural positive cone P in E, and maps both P and $-P$ to themselves. Finally, u is a solution of (1.5) if and only if $u - u_\lambda$ is a fixed point of A. Therefore, it suffices to show that A has a sign-changing fixed point.

As in [11], by using a priori estimates from [14] or [18] for positive solutions of the corresponding elliptic equation of $Au = u$, we obtain via a simple homotopy that

$$deg_P(I - A, P \cap B_R, 0) = 0,$$

where $B_R = \{u \in E : \|u\|_E < R\}$ contains all the possible positive solutions of $Au = u$.

Since $u_{\lambda'}$ is a strict upper solution of (1.5), A maps the order interval $[0, u_{\lambda'} - u_\lambda]_E$ to its relative interior in P, which we denote by C_+. Since C_+ is a bounded convex set in P, it follows from the basic properties of the degree (see e.g., [3] and [23]) that

$$deg_P(I - A, C_+, 0) = 1.$$

Hence, by the additivity property of the degree,

$$deg_P(I - A, (P \cap B_R) \setminus \overline{C}_+, 0) = 0 - 1 = -1.$$

Let C_- denote the relative interior of $[u_{\lambda''} - u_\lambda, 0]_E$ in $-P$, and enlarge R when necessary, we obtain similarly

$$deg_{-P}(I - A, (-P) \cap B_R, 0) = 0, \quad deg_{-P}(I - A, C_-, 0) = 1$$

and

$$deg_{-P}(I - A, (-P) \cap B_R \setminus \overline{C}_-, 0) = -1.$$

Let C be the interior of $[u_{\lambda''} - u_\lambda, u_{\lambda'} - u_\lambda]_E$ in E. Then A maps $\overline{C}$ into C and we obtain as before that

$$deg_E(I - A, C, 0) = 1.$$

Notice that now the degree is on the whole space E.

Suppose that (1.5) has only one sign-changing solution obtained in (iii)$'$. Then necessarily, A has no sign-changing fixed point, and thus we can classify all the solutions of $Au = u$ into three types:

(a) Solutions contained in C; (b) Positive solutions outside C –they form a compact set K_+ contained in $P \cap B_R \setminus \overline{C}_+$; (c) Negative solutions outside C –they form a compact set K_- contained in $-P \cap B_R \setminus \overline{C}_-$. By the maximum principle, and compactness of K_+, one easily deduces that K_+ lies in the interior of P. Similarly, K_- is in the interior of $-P$. Hence we can find small neighborhoods N_+ and N_- of K_+ and K_- in E respectively such that they are in the interior of P and $-P$ respectively. Now by the excision property of the degree,

$$deg_P(I - A, N_+, 0) = deg_P(I - A, P \cap B_R \setminus \overline{C}_+, 0) = -1,$$

$$deg_{-P}(I - A, N_-, 0) = deg_{-P}(I - A, -P \cap B_R \setminus \overline{C}_-, 0) = -1.$$

Since N_+ and N_- are in the interior of P and $-P$ respectively, we also have, by the properties of the degree, that

$$deg_E(I - A, N_+, 0) = deg_P(I - A, N_+, 0) = -1,$$

$$deg_E(I - A, N_-, 0) = deg_{-P}(I - A, N_-, 0) = -1.$$

Finally, as in [11] section 3, we use the assumptions $(H_1), (H_2)$, the fact that the solution set of (1.5) is bounded (since we assume that there is only one sign-changing solution), and obtain

$$deg_E(I - A, B_M, 0) = 0,$$

where $M > 0$ is large enough such that B_M contains all the three types of solutions listed above. By the additivity of the degree, we obtain

$$0 = deg_E(I - A, B_M, 0)$$

$$= deg_E(I - A, C, 0) + deg_E(I - A, N_+, 0) + deg_E(I - A, N_-, 0) = -1.$$

This contradiction finishes our proof. ■

Remark 4.3. The result of Theorem 4.3 (iv) seems difficult to obtain by Morse theory or our generalized Conley index approach. The difficulty comes from the fact that $|u|^{r-1}u$ is not locally Lipschitz near 0. Note that the modification trick used in the proof of Theorem 4.2 is difficult to use for the superlinear case because a priori bounds for sign-changing solutions of the modified equations are at least difficult to obtain (if obtainable at all) and such bounds are essential for the limiting process to work.

Remark 4.4. From the proof, it is clear that condition (H_6) can be replaced by any other conditions which guarantee that the positive and negative solutions of (1.5) are a priori bounded.

Remark 4.5. It is possible to show that the conclusion of (iv) in Theorem 4.3 is also true for the case that $\lambda = \max\{\Lambda^+, \Lambda^-\}$. We did not include this case in Theorem 4.3 because its proof is rather different from that of Theorem 4.3 above and is very technical.

Remark 4.6. Under the condition (H_4), if 0 is an isolated solution of (1.5), then it can be shown that the critical groups of 0 as a critical point of the corresponding functional are all trivial. It can also be proved that the fixed point index of 0 for the corresponding abstract operator is zero. Note that this latter conclusion does not follow directly from the well-known relation between the fixed point index and the critical groups because the right side of (1.5) is not smooth enough. Note also that it follows easily from this fixed point index result that 0 is never isolated if g is an odd function. This implies a partial answer to a recent question of Bartsch and Willem in [T. Bartsch and M. Willem, *On an elliptic equation with concave and convex nonlinearities*, Proc. Amer. Math. Soc. **123** (1995), 3555-3561].

5. Appendix

In this section, we prove Lemmas 2.1 and 2.2.

Proof of Lemma 2.1. We follow [19]. Define

$$g(s) = \left(\int_D u^2(s, x)dx\right)^{1/2}.$$

Then

$$(1/2)(g^2(s))' = \int_D uu_t dx = -\int_D \left[|\nabla u|^2 - uf(u)\right] dx$$

$$= -2J(u(s,\cdot)) - \int_D [F(u) - uf(u)]\, dx \geq -2J(u(s,\cdot)) + \int_D qF(u)dx - C_1$$

$$\geq -2J(u(s,\cdot)) + C_2 \int_D |u|^{q+1} dx - C_3 \geq -2J(u(s,\cdot)) + C_4 g^{q+1}(s) - C_3.$$

Thus for $s \in [T_1, T_2]$,

$$g(s)g'(s) \geq -2\max\{|a|,|b|\} + C_4 g^{q+1}(s) - C_3 = -C_5 + C_4 g^{q+1}(s).$$

Here, and in what follows, we use C_i to denote positive constants independent of u and T_1, T_2.

Suppose $T_1 \leq s \leq t \leq \max\{s+1, T_2\}$. It follows that

$$g^{2q}(s) \leq \max\{1, C_4^{-2}[2g'(s)^2 + 2C_5^2]\},$$

$$\int_s^t g^{2q}(\tau)d\tau \leq 1 + C_4^{-2} \int_s^t [2g'(\tau)^2 + 2C_5^2] d\tau$$

$$\leq C_6 + C_7 \int_s^t g'(\tau)^2 d\tau.$$

On the other hand,

$$g'(s) = g(s)^{-1} \int_D u(s,\cdot) u_t(s,\cdot) dx \leq \|u_t(s,\cdot)\|_{L^2}.$$

Hence

$$g'(s)^2 \leq \int_D u_t^2(s,\cdot)dx = -(d/ds)J(u(s,\cdot)),$$

and

$$\int_s^t g'(\tau)^2 d\tau \leq \int_s^t \int_D u_t^2(\tau,x))dxd\tau \tag{5.1}$$

$$\leq -\int_s^t (d/d\tau)J(u(\tau,\cdot))d\tau \leq b-a.$$

It follows that

$$\int_s^t g^{2q}(\tau)d\tau \leq C_6 + C_7(b-a) = C_8. \tag{5.2}$$

Using (5.1), (5.2) and the following Sobolev inequality:

$$\|g\|_{L^\infty} \leq C(\|g'\|_{L^2} + \|g\|_{L^2})^\lambda \|g\|_{L^{2q}}^{1-\lambda}, \ \lambda = 1/(q+1),$$

we conclude that

$$|g(s)| \leq C_9, \quad \forall s \in [T_1, T_2]. \tag{5.3}$$

Using

$$\int_D u(s,x)u_t(s,x)dx \geq -2J(u(s,\cdot)) + C_2 \int_D |u(s,x)|^{q+1} dx - C_3$$

and (5.3) we deduce

$$\left(\int_D |u(\tau,x)|^{q+1} dx\right)^2 \leq C_2^{-2} \left[2\left(\int_D uu_t dx\right)^2 + 2(2J(u) + C_3)^2\right]$$

$$\le C_{10}\int_D u(\tau,x)^2dx\int_D u_t^2(\tau,x)dx + C_{11} \le C_{10}C_9^2\int_D u_t^2(\tau,x)dx + C_{11}.$$

Hence , by (5.1), for any $T_1 \le s \le t \le \max\{s+1, T_2\}$,

$$\int_s^t \left(\int_D |u(\tau,x)|^{q+1}dx\right)^2 d\tau \le C_{10}C_9^2\int_s^t\int_D u_t^2(\tau,x)dxd\tau + C_{11}(t-s) \le C_{12}. \tag{5.4}$$

The estimate in Lemma 2.1 now follows from (5.1), (5.3), (5.4) and the following well known inequality (see, e.g., [7] and [19]):

$$\sup_{s\le\tau\le t}\|u(\tau,\cdot)\|_{L^r} \le C\left(\int_s^t\int_D \left(|u_t(\tau,x)|^2 + |u(\tau,x)|^2\right)dxd\tau\right)^{\frac{1-\xi}{2}} \cdot \left(\int_s^t\left(\int_D |u(\tau,x)|^{q+1}dx\right)^2 d\tau\right)^{\frac{\xi}{2q+2}},$$

where $1 \le r < (2q+4)/3$ and $\xi = (q+1)/(q+2)$.

This completes the proof of Lemma 2.1. ∎

Proof of Lemma 2.2. Let

$$D(A_0) = \{u \in C^2(\overline{D}) : Bu|_{\partial D} = 0\}, \quad A_0 = -\Delta u + u.$$

Then it is well known (see, e.g, Lemmas 1 and 3 in [25]) that, for any $p \in [1,\infty]$, A_0 is closable in $L^p(D)$ and its closure A_p is a generator of an analytic semigroup $S_p(t)$ in $L^p(D)$, $S_p(t) \subset S_{p'}(t)$ if $p \le p'$, and for $1 \le p \le p' \le \infty$,

$$\|S_p(t)u\|_{L^{p'}} \le Cm(t)^{-(\frac{N}{2p}-\frac{N}{2p'})}e^{-\lambda t}\|u\|_{L^p}, \tag{5.5}$$

for all $u \in L^p(D)$, $t \in (0,\infty)$ and some $\lambda > 0$. Here $m(t) = \min\{1,t\}$. Thus if $f_1(u) = f(u) + u$ and $u(t,x)$ is a solution of (1.2), then

$$u(t,\cdot) = S(t)u(0,\cdot) + \int_0^t S(t-s)f_1(u(s,\cdot))ds, \tag{5.6}$$

for all $t > 0$ such that $u(t,\cdot)$ is defined. Here we have dropped p from $S_p(t)$ because $u(t,\cdot) \in X^\theta \subset C^1(\overline{D})$, and thus we can regard $S(t)$ as $S_p(t)$ for any p.

Since $\gamma < 1 + (4q+8)/(3N)$, we can find $r \in (1, (2q+4)/3)$ such that $(\gamma-1)N/(2r) < 1$. Now by some elementary arguments (see the proof of Lemma 12 in [25] for details), there exists a finite sequence $r = p_0 < p_1 < \ldots < p_k = \infty$ and $\varepsilon > 0$ such that for $\delta_i = N/(2p_{i-1}) - N/(2p_i)$,

$$(\gamma-1)(\delta_i + N/(2p_i)) + \varepsilon < 1,\ \gamma/p_i < 1, \gamma\delta_i < 1,\ i = 1, \ldots, k.$$

Let u be any solution of (1.2) with $u(t,\cdot) \in J^{-1}[a,b]$ for $t \in [T_1,T_2]$. By Lemma 2.1, there exists $M_0 > 0$ independent of u and T_1, T_2 such that

$$\sup_{t\in[T_1,T_2]}\|u(t,\cdot)\|_{L^{p_0}} \le M_0.$$

Define

$$\|u\|_{p,\delta,T} = \sup\{m(t)^\delta \|u(t,\cdot)\|_{L^p} : t \in (0,T]\},$$

and use the notation $\|.\|_p = \|.\|_{L^p}$. Then using (5.5) and (5.6) we obtain, for $T \in [0,1]$,

$$\begin{aligned}\|u\|_{p_i,\delta_i,T} &\le \sup\left\{t^{\delta_i}\|S_{p_{i-1}}(t)u(0,\cdot)\|_{p_i}\right\} \\ &\quad + \sup\left\{t^{\delta_i}\int_0^t \|S_{p_i/\gamma}(t-s)f_1(u(s,\cdot))\|_{p_i} ds\right\} \\ &\le C_1\|u(0,\cdot)\|_{p_{i-1}} \\ &\quad + \sup\left\{t^{\delta_i}\int_0^t C(t-s)^{-N(\gamma-1)/(2p_i)}\|f_1(u(s,\cdot))\|_{p_i/\gamma} ds\right\} \\ &\le C_1\|u(0,\cdot)\|_{p_{i-1}} \\ &\quad + \sup\left\{t^{\delta_i}C_2\int_0^t (t-s)^{-N(\gamma-1)/(2p_i)}\|1+|u|\|_{p_i}^\gamma ds\right\}.\end{aligned}$$

Here, and in what follows, we use C_i to denote positive constants independent of u and T_1, T_2.

Since

$$\begin{aligned}&t^{\delta_i}C_2\int_0^t (t-s)^{-N(\gamma-1)/(2p_i)}\|1+|u|\|_{p_i}^\gamma ds \\ &\qquad \le t^{\delta_i}C_2\int_0^t (t-s)^{-N(\gamma-1)/(2p_i)}s^{-\delta_i\gamma}ds\|1+|u|\|_{p_i,\delta_i,T}^\gamma \\ &\qquad \le T^{1-(\gamma-1)(\delta_i+N/(2p_i))}C_2\|1+|u|\|_{p_i,\delta_i,T}^\gamma \\ &\qquad \le C_2T^\varepsilon\|1+|u|\|_{p_i,\delta_i,T}^\gamma \le C_3T^\varepsilon(1+\|u\|_{p_i,\delta_i,T})^\gamma,\end{aligned}$$

we obtain

$$\|u\|_{p_i,\delta_i,T} \le C_4(\|u(0,\cdot)\|_{p_{i-1}} + T^\varepsilon(1+\|u\|_{p_i,\delta_i,T})^\gamma).$$

Thus, for any $t_0 \in [T_1,T_2]$ and any $t \in (0,\max\{1,T_2-t_0\}]$, the quantities

$$U_i(t_0,t) = \sup\{s^{\delta_i}\|u(t_0+s,\cdot)\|_{p_i} : s \in [0,t]\} = \|u(t_0+\cdot,\cdot)\|_{p_i,\delta_i,t}$$

satisfy

$$U_i(t_0,t) \le C_4\|u(t_0,\cdot)\|_{p_{i-1}} + C_4t^\varepsilon(1+U_i(t_0,t))^\gamma, \quad i=1,...,k. \tag{5.7}$$

Now we need another elementary result from [25] (see Lemma 22 in [25]):

Let $a \ge 0$ be a constant, and $b = b(t)$ and $x = x(t)$ be continuous functions satisfying

$$b(t), x(t) \ge 0 \text{ for } t \ge 0, \quad b(0) = x(0) = 0.$$

Then

$$x(t) \le a + b(t)(1+x(t))^\gamma, \quad b(t) < b_0 \equiv (\gamma-1)^{(\gamma-1)}(1+a)^{-(\gamma-1)}\gamma^{-\gamma}, \ \forall t > 0$$

imply that

$$x(t) < y_0 \equiv (1+\gamma a)(\gamma-1)^{-1}, \forall t > 0.$$

Let $T = \max\{1, T_2 - t_0\}$ and define

$$a_i = C_4\|u(t_0,\cdot)\|_{p_{i-1}}, \ b_i(t) = C_4t^\varepsilon \text{ for } t \in [0,T], b_i(t) = C_4T^\varepsilon \text{ for } t \ge T,$$

$$x_i(t) = U_i(t_0, t) \text{ for } t \in [0, T], x_i(t) = U_i(t_0, T) \text{ for } t \geq T.$$

Then (5.7) can be rewritten as

$$x_i(t) \leq a_i + b_i(t)(1 + x_i(t))^\gamma.$$

Thus,

$$\text{(5.8)} \qquad \begin{array}{l} x_i(t) < y_i \equiv (1 + \gamma a_i)(\gamma - 1)^{-1}, \quad \forall t > 0, \text{ whenever} \\ b_i(t) \leq b_i \equiv (\gamma - 1)^{(\gamma-1)}(1 + a_i)^{-(\gamma-1)}\gamma^{-\gamma}, \quad \forall t > 0. \end{array}$$

Substituting back, we obtain from (5.8), for $i = 1, \dots, k$,

$$\text{(5.9)}_a \qquad (1 + \|u(t_0 + t, \cdot)\|_{p_i}) \leq K_0(1 + \|u(t_0, \cdot)\|_{p_{i-1}})t^{-\delta_i}$$

whenever

$$\text{(5.9)}_b \qquad t_0, t_0 + t \in [T_1, T_2], 0 \leq t \leq 1; \quad C_5 t^\varepsilon (1 + \|u(t_0, \cdot)\|_{p_{i-1}})^{\gamma-1} \leq 1.$$

Here and in the following, K_i also denote positive constants independent of u and T_1, T_2.

Choose $h_0 \in (0, 1]$ such that

$$C_5 h_0^\varepsilon (1 + M_0)^{\gamma-1} \leq 1.$$

Since $\|u(t, \cdot)\|_{p_0} \leq M_0$ for all $t \in [T_1, T_2]$, we have

$$\text{(5.10)} \qquad C_5 h_0^\varepsilon (1 + \|u(t_0, \cdot)\|_{p_0})^{\gamma-1} \leq 1, \ \forall t_0 \in [T_1, T_2].$$

Let h be any number in $(0, h_0]$ satisfying $T_1 + kh \leq T_2$. Since any $t \in [T_1 + h, T_2]$ can be written as $t = t_0 + h$ with $t_0 \in [T_1, T_2]$, by (5.9) and (5.10), we have

$$\begin{aligned} 1 + \|u(t, \cdot)\|_{p_1} = 1 + \|u(t_0 + h, \cdot)\|_{p_1} &\leq K_0(1 + \|u(t_0, \cdot)\|_{p_0})h^{-\delta_1} \\ &\leq K_0(1 + M_0)h^{-\delta_1} \equiv K_1 h^{-\delta_1}, \quad \forall t \in [T_1 + h, T_2]. \end{aligned}$$

Let $\alpha_1 = \max\{\delta_1(\gamma - 1)/\varepsilon, 1\}$ and choose $h_1 \in (0, 1)$ such that

$$C_5 h_1^\varepsilon (K_1)^{\gamma-1} \leq 1.$$

Then, for $t_0 \in [T_1 + h, T_2]$,

$$\begin{aligned} &C_5(h_1 h^{\alpha_1})^\varepsilon (1 + \|u(t_0, \cdot)\|_{p_1})^{\gamma-1} \\ &\leq C_5 h_1^\varepsilon h^{\delta_1(\gamma-1)}(K_1 h^{-\delta_1})^{\gamma-1} \\ &\leq C_5 h_1^\varepsilon (K_1)^{\gamma-1} \leq 1. \end{aligned}$$

Hence, by (5.9),

$$\begin{aligned} 1 + \|u(t_0 + h_1 h^{\alpha_1}, \cdot)\|_{p_2} &\leq K_0(1 + \|u(t_0, \cdot)\|_{p_1})(h_1 h^{\alpha_1})^{-\delta_2} \\ &\leq K_0(K_1 h^{-\delta_1})h_1^{-\delta_2} h^{-\alpha_1\delta_2} \\ &\leq K_2 h^{-\delta_1 - \alpha_1\delta_2} = K_2 h^{-\beta_2}, \quad \forall t_0 \in [T_1 + h, T_2]. \end{aligned}$$

Any $t \in [T_1 + 2h, T_2]$ can be written as $t = t_0 + h_1 h^{\alpha_1}$ with $t_0 \in [T_1 + h, T_2]$. Thus it follows

$$1 + \|u(t, \cdot)\|_{p_2} \leq K_2 h^{-\beta_2}, \quad \forall t \in [T_1 + 2h, T_2].$$

Now let $\alpha_2 = \max\{\beta_2(\gamma - 1)/\varepsilon, 1\}$ and choose $h_2 \in (0, 1)$ such that

$$C_5 h_2^\varepsilon (K_2)^{\gamma-1} \leq 1.$$

Then, for $t_0 \in [T_1 + 2h, T_2]$,

$$C_5(h_2h^{\alpha_2})^{\varepsilon}(1 + \|u(t_0,\cdot)\|_{p_2})^{\gamma-1}$$

$$\le C_5h_2^{\varepsilon}h^{\beta_2(\gamma-1)}(K_2h^{-\beta_2})^{\gamma-1}$$

$$\le C_5h_2^{\varepsilon}(K_2)^{\gamma-1} \le 1.$$

Hence, by (5.9),

$$1 + \|u(t_0 + h_2h^{\alpha_2},\cdot)\|_{p_3} \le K_0(1 + \|u(t_0,\cdot)\|_{p_2})(h_2h^{\alpha_2})^{-\delta_3}$$

$$\le K_0(K_2h^{-\beta_2})h_2^{-\delta_3}h^{-\alpha_2\delta_3}$$

$$\le K_3h^{-\beta_2-\alpha_2\delta_3} = K_3h^{-\beta_3}, \quad \forall t_0 \in [T_1 + 2h, T_2].$$

Any $t \in [T_1+3h, T_2]$ can be written as $t = t_0+h_2h^{\alpha_2}$ with $t_0 \in [T_1+2h, T_2]$. Thus

$$1 + \|u(t,\cdot)\|_{p_3} \le K_3h^{-\beta_3}, \quad \forall t \in [T_1 + 3h, T_2].$$

We continue the above argument, and arrive at

$$1 + \|u(t,\cdot)\|_{p_k} \le K_kh^{-\beta_k}, \quad \forall t \in [T_1 + kh, T_2].$$

Hence

$$\|u(t,\cdot)\|_{\infty} \le K_kh^{-\beta_k}, \quad \forall t \in [T_1 + kh, T_2]. \tag{5.11}$$

For any $t \in (T_1, T_2]$, we have either (i) $t - T_1 > kh_0$ or (ii) $t - T_1 \le kh_0$. In case (i), $T_2 - T_1 > kh_0$ and hence we can take $h = h_0$. Now since $t \in [T_1 + kh_0, T_2]$, we use (5.11) with $h = h_0$ and obtain

$$\|u(t,\cdot)\|_{\infty} \le K_kh_0^{-\beta_k}.$$

In case (ii), we can take $h = (t - T_1)/k$ and obtain from (5.11) that

$$\|u(t,\cdot)\|_{\infty} = \|u(T_1 + kh,\cdot)\|_{\infty} \le K_kh^{-\beta_k} = K_kk^{\beta_k}(t - T_1)^{-\beta_k}.$$

Combining these two cases, we can write

$$\|u(t,\cdot)\|_{\infty} \le K_kk^{\beta_k}(\min\{t - T_1, kh_0\})^{-\beta_k}, \quad \forall t \in (T_1, T_2].$$

Clearly there exists $K > 0$ depending only on kh_0 such that $m(t - T_1) \le K \min\{t - T_1, kh_0\}$ for all $t > T_1$. Therefore, if we take $M = K_kk^{\beta_k}K^{-\beta_k}$ and $\delta = \beta_k$, then

$$\|u(t,\cdot)\|_{\infty} \le [m(t - T_1)]^{-\delta}M, \quad \forall t \in (T_1, T_2].$$

Since such M and δ are independent of u and T_1, T_2, our proof of Lemma 2.2 is complete. ∎

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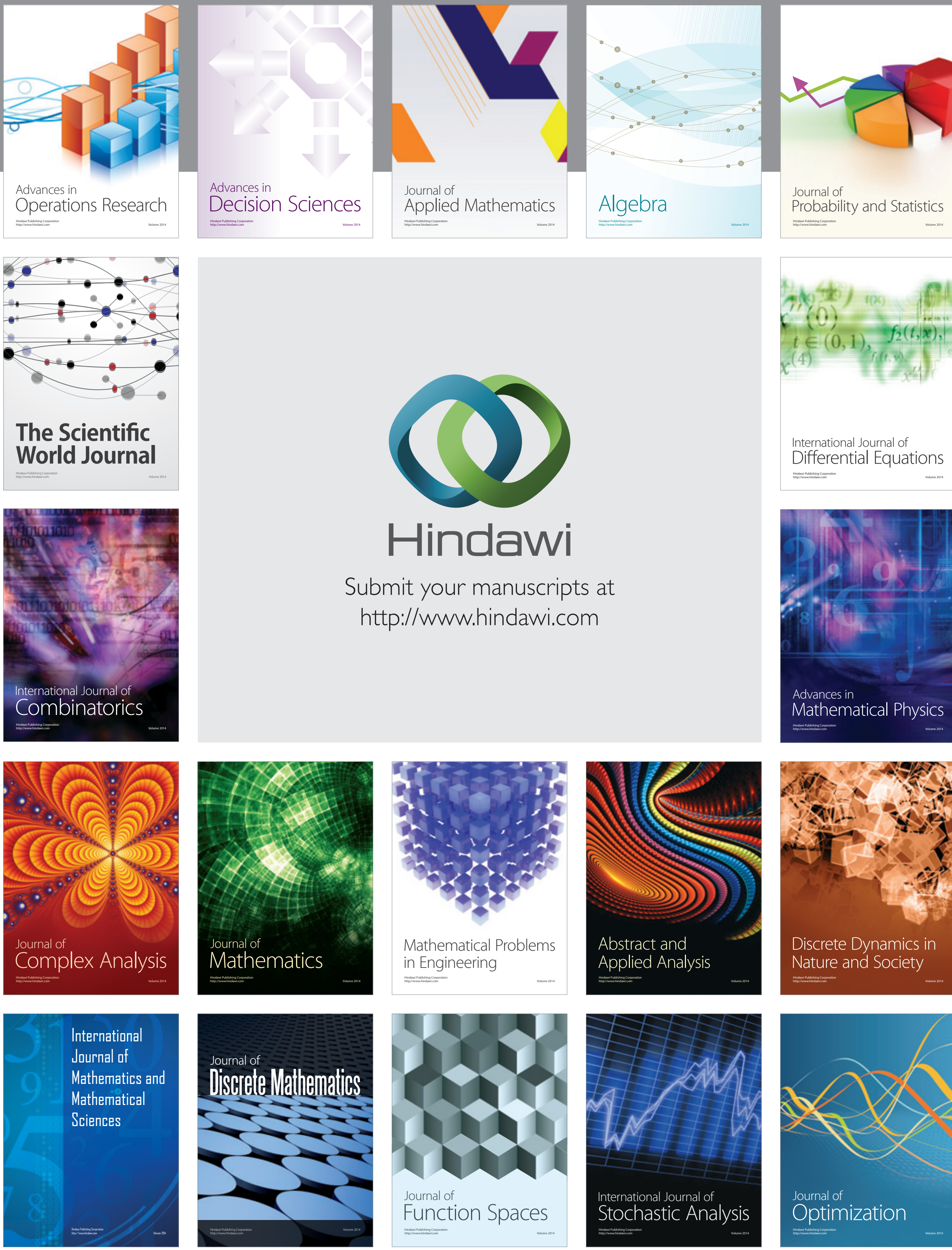
School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia
E-mail address: dancer_n@maths.su.oz.au

Department of Mathematics, Statistics and Computer Science, University of New England, Armidale, NSW 2351, Australia
E-mail address: ydu@neumann.une.edu.au



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