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Research Article

Nonlinear Super Integrable Couplings of Super Classical-Boussinesq Hierarchy

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Nonlinear integrable couplings of super classical-Boussinesq hierarchy based upon an enlarged matrix Lie super algebra were constructed. Then, its super Hamiltonian structures were established by using super trace identity. As its reduction, nonlinear integrable couplings of the classical integrable hierarchy were obtained.

1. Introduction

With the development of soliton theory, super integrable systems associated with Lie super algebra have aroused growing attentions by many mathematicians and physicists. It was known that super integrable systems contained the odd variables, which would provide more prolific fields for mathematical researchers and physical ones. Several super integrable systems including super AKNS hierarchy, super KdV hierarchy, super C-KdV hierarchy, and super classical-Boussinesq hierarchy have been studied [1–4]. There are some interesting results on the super integrable systems, such as Darboux transformation [5], super Hamiltonian structures [6, 7], binary nonlinearization [8], and reciprocal transformation [9].

The research of integrable couplings of the well-known integrable hierarchy has received considerable attentions [10–15]. A few approaches to construct linear integrable couplings of the classical soliton equation are presented by permutation, enlarging spectral problem, using matrix Lie algebra constructing new loop Lie algebra [16], and creating semidirect sums of Lie algebra. Recently, Ma and Zhu [17, 18] presented a scheme for constructing nonlinear continuous and discrete integrable couplings using the block type matrix algebra. However, there is one interesting question for us which is how to generate nonlinear super integrable couplings for the super integrable hierarchy.

In this paper, we would like to construct nonlinear super integrable couplings of the super soliton equations through enlarging matrix Lie super algebra. We take the Lie algebra sl(2, 1) as an example to illustrate the approach for extending Lie super algebra. Based on the enlarged Lie super algebra sl(4, 1), we work out nonlinear super integrable Hamiltonian couplings of the super classical-Boussinesq hierarchy. Finally, we will reduce the nonlinear super classical-Boussinesq integrable Hamiltonian couplings to some special cases.

2. Enlargement of Lie Superalgebra

Consider the Lie superalgebra sl(2, 1). Its basis is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad (1)$$

$$E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where E_1, E_2, E_3 are even elements and E_4, E_5 are odd ones. Their nonzero (anti)commutation relations are

$$\begin{gathered}[E_1, E_2] = 2E_2, \qquad [E_1, E_3] = -2E_3, \\ [E_2, E_3] = E_1, \qquad [E_1, E_4] = [E_2, E_5] = E_4, \\ [E_1, E_5] = [E_4, E_3] = -E_5, \qquad [E_4, E_5] = E_1, \\ [E_4, E_4] = -2E_2, \qquad [E_5, E_5] = 2E_3. \end{gathered} \tag{2}$$

Let us enlarge the Lie superalgebra sl(2, 1) to the Lie superalgebra sl(4, 1) with a basis

$$\begin{gathered}e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}, \qquad e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}, \end{gathered} \tag{3}$$

where e_1, e_2, e_3, e_4, e_5, e_6 are even and e_7, e_8 are odd. The generators of the Lie superalgebra sl(4, 1), e_i, $0 \le i \le 8$, satisfy the following (anti)commutation relations:

$$\begin{gathered}[e_1, e_2] = 2e_2, \qquad [e_1, e_3] = -2e_3, \\ [e_1, e_5] = 2e_5, \qquad [e_1, e_6] = -2e_6, \qquad [e_1, e_7] = e_7, \\ [e_1, e_8] = -e_8, \qquad [e_2, e_3] = e_1, \\ [e_2, e_4] = -2e_5, \qquad [e_2, e_6] = e_4, \qquad [e_2, e_8] = e_7, \\ [e_3, e_4] = 2e_6, \qquad [e_3, e_5] = -e_4, \\ [e_3, e_7] = e_8, \qquad [e_4, e_5] = 2e_5, \\ [e_4, e_6] = -2e_6, \qquad [e_5, e_6] = e_4, \\ [e_7, e_8] = e_1 - e_4, \qquad [e_7, e_7] = 2e_5 - 2e_2, \\ [e_8, e_8] = 2e_3 - 2e_6, \\ [e_1, e_4] = [e_2, e_5] = [e_2, e_7] = [e_3, e_6] = [e_3, e_8] = 0, \\ [e_4, e_7] = [e_4, e_8] = [e_5, e_7] = [e_5, e_8] \\ = [e_6, e_7] = [e_6, e_8] = 0. \end{gathered} \tag{4}$$

Define a loop superalgebra corresponding to the Lie superalgebra sl(4, 1), denoted by

$$\begin{aligned}\widetilde{\mathrm{sl}}(4,1) &= \mathrm{sl}(4,1) \otimes [\lambda, \lambda^{-1}] \\ &= \{e_i\lambda^m, e_i \in \mathrm{sl}(4,1), i = 1, 2, \ldots, 8; m = 0, \pm 1, \pm 2, \ldots\}. \end{aligned} \tag{5}$$

The corresponding (anti)commutative relations are given as

$$[e_i\lambda^m, e_j\lambda^n] = [e_i, e_j]\lambda^{m+n}, \quad \forall e_i, e_j \in \mathrm{sl}(4,1). \tag{6}$$

3. Nonlinear Super Integrable Couplings of the Super Classical-Boussinesq Hierarchy

Let us start from an enlarged spectral problem associated with sl(4, 1)

$$\varphi_x = U\varphi, \tag{7}$$

where

$$\begin{aligned}U &= -e_1(1) - \frac{1}{q}e_1(0) + re_2(0) - e_3(0) - u_1e_4(0) \\ &\quad + u_2e_5(0) + \alpha e_7(0) + \beta e_8(0) \\ &= \begin{pmatrix} -\lambda - \frac{1}{4}q & r & -u_1 & u_2 & \alpha \\ -1 & \lambda + \frac{1}{4}q & 0 & u_1 & \beta \\ 0 & 0 & -\lambda - \frac{1}{4}q - u_1 & r + u_2 & 0 \\ 0 & 0 & -1 & \lambda + \frac{1}{4}q + u_1 & 0 \\ \beta & -\alpha & -\beta & \alpha & 0 \end{pmatrix}, \end{aligned} \tag{8}$$

where r, s, u_1, and u_2 are even potentials but α and β are odd ones. In order to obtain super integrable couplings of super integrable hierarchy, we first solve the adjoint representation (8)

$$V_x = [U, V] \tag{9}$$

with

$$V = Ae_1(0) + Be_2(0) + Ce_3(0) + Ee_4(0) + Fe_5(0) + Ge_6(0) + \rho e_7(0) + \delta e_8(0) = \begin{pmatrix} A & B & E & F & \rho \\ C & -A & G & -E & \delta \\ 0 & 0 & A+E & B+F & 0 \\ 0 & 0 & C+G & -A-E & 0 \\ \delta & -\rho & -\delta & \rho & 0 \end{pmatrix}, \quad (10)$$

where A, B, C, E, F, and G are commuting fields, ρ and δ are anticommuting fields, and α, β are anticommuting fields.

Substituting

$$\begin{aligned} A &= \sum_{m\geq 0} A_m\lambda^{-m}, \qquad B = \sum_{m\geq 0} B_m\lambda^{-m}, \\ C &= \sum_{m\geq 0} C_m\lambda^{-m}, \qquad F = \sum_{m\geq 0} F_m\lambda^{-m}, \\ G &= \sum_{m\geq 0} G_m\lambda^{-m}, \qquad \rho = \sum_{m\geq 0} \rho_m\lambda^{-m}, \\ \sigma &= \sum_{m\geq 0} \sigma_m\lambda^{-m} \end{aligned} \quad (11)$$

into the above equation gives the following recursive formulas:

$$\begin{aligned} A_{m,x} &= B_m + rC_m - \beta\rho_m + \alpha\delta_m, \\ B_{m,x} &= -2rA_m - 2B_{m+1} - \frac{1}{2}qB_m - 2\alpha\rho_m, \\ C_{m,x} &= -2A_m + 2C_{m+1} + \frac{1}{2}qC_m + 2\beta\delta_m, \\ E_{m,x} &= u_2C_m + F_m + rG_m + u_2G_m - \alpha\delta_m - \beta\rho_m, \\ F_{m,x} &= -2u_2A_m - 2u_1B_m - 2rE_m - 2u_2E_m - 2F_{m+1} \\ &\quad - \frac{1}{2}qF_m - 2u_1F_m + 2\alpha\rho_m, \\ G_{m,x} &= 2u_1C_m - 2E_m + 2G_{m+1} + \frac{1}{2}qG_m + 2u_1G_m - 2\beta\delta_m, \\ \rho_{m,x} &= -\alpha A_m - \beta B_m - \rho_{m+1} - \frac{1}{4}q\rho_m + r\delta_m, \\ \delta_{m,x} &= \beta A_m - \alpha C_m - \rho_m + \delta_{m+1} + \frac{1}{4}q\delta_m. \end{aligned} \quad (12)$$

From these equations, we can successively deduce

$$\begin{aligned} & A_0 = -1, \qquad B_0 = C_0 = F_0 = G_0 = \rho_0 = \delta_0 = 0, \\ & E_0 = \varepsilon = \text{const}, \qquad A_1 = 0, \\ & B_1 = r, \qquad C_1 = -1, \\ & F_1 = u_2 - r\varepsilon - u_2\varepsilon, \qquad E_1 = 0, \\ & G_1 = \varepsilon, \qquad \rho_1 = \alpha, \qquad \delta_1 = \beta, \\ & A_2 = -\frac{1}{2}r + \alpha\beta, \qquad B_2 = -\frac{1}{2}r_x - \frac{1}{4}qr, \\ & C_2 = \frac{1}{4}q, \\ & F_2 = -\frac{1}{2}u_{2x} + \frac{1}{2}\varepsilon r_x + \frac{1}{2}\varepsilon u_{2x} - \frac{1}{4}qu_2 + \frac{1}{4}q\varepsilon u_2 + \frac{1}{4}qr\varepsilon \\ & \quad - u_1r - u_1u_2 + \varepsilon u_1u_2 + \varepsilon ru_1, \\ & E_2 = -\frac{1}{2}u_2 + \frac{1}{2}\varepsilon r + \frac{1}{2}\varepsilon u_2 - \alpha\beta, \\ & G_2 = u_1 - \frac{1}{4}q\varepsilon - u_1\varepsilon, \qquad \rho_2 = -\alpha_x - \frac{1}{4}q\alpha, \\ & \delta_2 = \beta_x - \frac{1}{4}q\beta, \\ & A_3 = \frac{1}{2}r_x + \frac{1}{4}qr - \frac{1}{2}q\alpha\beta - \beta\alpha_x + \alpha\beta_x, \\ & B_3 = \frac{1}{4}r_{xx} + \frac{1}{8}q_xr + \frac{1}{4}qr_x + \frac{1}{2}r^2 - r\alpha\beta + \frac{1}{16}q^2r + \alpha\alpha_x, \\ & C_3 = \frac{1}{8}q_x - \frac{1}{2}r + \alpha\beta - \frac{1}{16}q^2 - \beta\beta_x, \\ & E_3 = (1-\varepsilon)\left[u_1u_2 + u_1r + \frac{1}{4}qu_2 + \frac{1}{4}u_{2x}\right] \\ & \quad - \alpha\beta_x + \beta\alpha_x + \frac{1}{2}q\alpha\beta - \frac{1}{4}qr - \frac{1}{4}\varepsilon r_x, \\ & F_3 = (1-\varepsilon)\left[\frac{1}{4}u_{2xx} + \frac{1}{8}q_xu_2 + \frac{1}{4}qu_{2x} + \frac{1}{2}u_{1x}r + u_1r_x \right. \\ & \quad + \frac{1}{2}u_{1x}u_2 + u_1u_{2x} + \frac{1}{16}q^2u_2 + \frac{1}{2}qu_1r \\ & \quad \left. + \frac{1}{2}qu_1u_2 + \frac{1}{2}u_2^2 + ru_2 + ru_1^2 + u_1^2u_2\right] \\ & \quad - \frac{1}{4}\varepsilon r_{xx} - \frac{1}{8}\varepsilon q_xr - \frac{1}{4}\varepsilon qr_x + r\alpha\beta - \frac{1}{2}\varepsilon r^2 \\ & \quad - \frac{1}{16}\varepsilon q^2r - \alpha\alpha_x, \\ & G_3 = \frac{1}{2}u_{1x} - \frac{1}{8}q_x\varepsilon - \frac{1}{2}u_{1x}\varepsilon - \frac{1}{2}qu_1 - \frac{1}{2}u_2 + \frac{1}{2}r\varepsilon + \frac{1}{2}u_2\varepsilon \\ & \quad + \frac{1}{16}q^2\varepsilon + \frac{1}{2}qu_1\varepsilon - \alpha\beta - u_1^2 + u_1^2\varepsilon + \beta\beta_x, \\ & \rho_3 = \alpha_{xx} + \frac{1}{4}q_x\alpha + \frac{1}{2}q\alpha_x + \frac{1}{2}\alpha r + \frac{1}{2}\beta r_x \end{aligned}$$

$$+\frac{1}{2}\beta q r - r\beta_x + \frac{1}{16}q^2\alpha,$$

$$\delta_3 = \beta_{xx} - \frac{1}{4}q_x\beta - \frac{1}{2}q\beta_x + \frac{1}{2}\beta r + \frac{1}{16}q^2\beta - \alpha_x. \tag{13}$$

Equations (12) can be written as

$$\begin{pmatrix} 4A_{m+1} + 2E_{m+1} \\ 2C_{m+1} + G_{m+1} \\ 2A_{m+1} + 2E_{m+1} \\ C_{m+1} + G_{m+1} \\ 2\delta_{m+1} \\ -2\rho_{m+1} \end{pmatrix} = L \begin{pmatrix} 4A_m + 2E_m \\ 2C_m + G_m \\ 2A_m + 2E_m \\ C_m + G_m \\ 2\delta_m \\ -2\rho_m \end{pmatrix}, \tag{14}$$

where

$$L = \begin{pmatrix} -\frac{1}{2}\partial - \frac{1}{4}\partial^{-1}q\partial & r + \partial^{-1}r\partial & -\partial^{-1}u_1\partial & u_2 + \partial^{-1}u_2\partial & \frac{1}{2}\partial + \frac{1}{2}\partial^{-1}\alpha\partial & \frac{1}{2}\beta + \partial^{-1}\beta\partial \\ \frac{1}{2} & \frac{1}{2}\partial - \frac{1}{4} & 0 & -u_1 & -\frac{1}{2}\beta & 0 \\ 0 & 0 & L_{33} & L_{34} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\partial - u_1 - \frac{1}{4}q & 0 & 0 \\ -\beta & 2\alpha & \beta & -2\alpha & \partial - \frac{1}{4}q & -1 \\ \alpha + \beta\partial & -2\beta r & -\alpha - \beta\partial & 2\beta r & -\alpha\beta - r & -\partial - \frac{1}{4}q \end{pmatrix}, \tag{15}$$

$$L_{33} = -\frac{1}{2}\partial - \frac{1}{4}\partial^{-1}q\partial - \partial^{-1}u_1\partial, \qquad L_{34} = r + u_2 + \partial^{-1}r\partial + \partial^{-1}u_2\partial.$$

Then, let us consider the spectral problem (8) with the following auxiliary problem:

$$\varphi_{t_n} = V^{(n)}\varphi \tag{16}$$

with

$$V^{(n)} = \sum_{j=0}^{n} = \begin{pmatrix} A_j & B_j & E_j & F_j & \rho_j \\ C_j & -A_j & G_j & -E_j & \delta_j \\ 0 & 0 & A_j + E_j & B_j + F_j & 0 \\ 0 & 0 & C_j + G_j & -A_j - E_j & 0 \\ \delta_j & -\rho_j & -\delta_j & \rho_j & 0 \end{pmatrix} \lambda^{n-j} + C_{n+1}e_1(0) + G_{n+1}e_4(0). \tag{17}$$

From the compatible condition $\varphi_{x,t_n} = \varphi_{t_n,x}$ according to (8) and (17), we get the zero equation:

$$U_{t_n} - V_x^{(n)} + \left[U, V^{(n)}\right] = 0, \tag{18}$$

which gives a nonlinear Lax super integrable hierarchy

$$u_{t_n} = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \\ \alpha \\ \beta \end{pmatrix}_{t_n} = \begin{pmatrix} -4C_{n+1,x} \\ 2B_{n+1} + 2rC_{n+1} \\ -G_{n+1,x} \\ 2F_{n+1} + 2u_2C_{n+1} + 2rG_{n+1} + 2u_2G_{n+1} \\ -\rho_{n+1} - \alpha C_{n+1} \\ \delta_{n+1} + \beta C_{n+1} \end{pmatrix}. \tag{19}$$

The super integrable hierarchy (19) is a nonlinear super integrable coupling for the super classical-Boussinesq hierarchy

$$\overline{u}_{t_n} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}_{t_n} = \begin{pmatrix} -4C_{n+1,x} \\ 2B_{n+1} + 2rC_{n+1} \\ -\rho_{n+1} - \alpha C_{n+1} \\ \delta_{n+1} + \beta C_{n+1} \end{pmatrix}. \tag{20}$$

4. Super Hamiltonian Structures

A direct calculation reads

$$\begin{gathered}\operatorname{Str}(U_\lambda, V) = -4A - 2E, \qquad \operatorname{Str}(U_q, V) = -A - \frac{1}{2}E,\\ \operatorname{Str}(U_r, V) = 2C + G, \qquad \operatorname{Str}(U_{u_1}, V) = -2A - 2E,\\ \operatorname{Str}(U_{u_2}, V) = C + G, \qquad \operatorname{Str}(U_\alpha, V) = 2\delta,\\ \operatorname{Str}(U_\beta, V) = -2\rho.\end{gathered} \tag{21}$$

Substituting the above results into the super trace identity in [6, 7]

$$\frac{\delta}{\delta u}\int \operatorname{Str}\left(\frac{\delta U}{\delta \lambda}V\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\operatorname{Str}\left(\frac{\delta U}{\delta u}V\right) \tag{22}$$

yields that

$$\frac{\delta}{\delta u}\int(-4A-2E)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}-A-\frac{1}{2}E\\ 2C+G\\ -2A-2E\\ C+G\\ 2\delta\\ -2\rho\end{pmatrix}. \tag{23}$$

Comparing the coefficients of λ^{-n-1} on both sides of (23),

$$\frac{\delta}{\delta u}\int(-4A_{n+1}-2E_{n+1})\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}-A_n-\frac{1}{2}E_n\\ 2C_n+G_n\\ -2A_n-2E_n\\ C_n+G_n\\ 2\delta_n\\ -2\rho_n\end{pmatrix}, \quad n\ge 0. \tag{24}$$

From the initial values in (12), we obtain $\gamma = 0$. Thus, we have

$$\begin{gathered}\frac{\delta H_n}{\delta u} = \begin{pmatrix}-A_{n+1}-\frac{1}{2}E_{n+1}\\ 2C_{n+1}+G_{n+1}\\ -2A_{n+1}-2E_{n+1}\\ C_{n+1}+G_{n+1}\\ 2\delta_{n+1}\\ -2\rho_{n+1}\end{pmatrix},\\ H_n = \int\frac{4A_{n+2}+2E_{n+2}}{n+1}dx,\\ n\ge 0.\end{gathered} \tag{25}$$

It then follows that the nonlinear super integrable couplings (22) possess the following super Hamiltonian form:

$$u_{t_n} = K_n(u) = J\frac{\delta H_n}{\delta u}, \tag{26}$$

where

$$J = \begin{pmatrix}0 & -4\partial & 0 & 4\partial & 0 & 0\\ -4\partial & 0 & \partial & 0 & -\alpha & \beta\\ 0 & \partial & 0 & -2\partial & 0 & 0\\ 4\partial & 0 & -2\partial & 0 & -\alpha & \beta\\ 0 & \alpha & 0 & -\alpha & 0 & \frac{1}{2}\\ 0 & -\beta & 0 & \beta & \frac{1}{2} & 0\end{pmatrix} \tag{27}$$

is a super Hamiltonian operator and H_n $(n \ge 0)$ are Hamiltonian functions.

5. Reductions

Taking $\alpha = \beta = 0$, the hierarchy (26) reduces to a nonlinear integrable coupling of the super classical-Boussinesq hierarchy.

When $n = 2$ in (26), we obtain the nonlinear super integrable couplings of the second-order super classical-Boussinesq equations

$$\begin{aligned}q_{t_2} &= -\frac{1}{2}q_{xx} + 2r_x - 4\alpha_x\beta - 4\alpha\beta_x + \frac{1}{2}qq_x + 4\beta\beta_{xx} + 4\beta_x^2,\\ r_{t_2} &= \frac{1}{2}r_{xx} + \frac{1}{2}q_x r + \frac{1}{2}qr_x + 2\alpha\alpha_x - 2r\beta\beta_x,\\ u_{1,t_2} &= \frac{1}{2}(1-\varepsilon)\Big[u_{2x} + q_x u_1 + qu_{1x} + 2u_1u_{1x} - u_{1xx}\\ &\qquad + r_x - \frac{1}{4}q_{xx} + \frac{1}{4}qq_x\Big] - \frac{1}{2}r_x + \frac{1}{8}q_{xx}\\ &\qquad - \frac{1}{8}qq_x + \alpha_x\beta + \alpha\beta_x - \beta\beta_{xx},\\ u_{t_2} &= (1-\varepsilon)\Big[2u_{1x}r + 2u_{1x}u_2 + 2u_1u_{2x} + \frac{1}{2}u_{2xx}\\ &\qquad + \frac{1}{2}q_xu_2 + \frac{1}{2}qu_{2x} + \frac{1}{2}qr_x\\ &\qquad + \frac{1}{2}q_xr + \frac{1}{2}r_{xx} + 2r_xu_1\Big]\\ &\qquad + 2r\beta\beta_x - \frac{1}{2}q_xr - \frac{1}{2}qr_x - \frac{1}{2}r_{xx} - 2\alpha\alpha_x\alpha_{t_2}\\ &= -\alpha_{xx} - \frac{1}{2}q\alpha_x - \frac{3}{8}q_x\alpha - \frac{1}{2}\beta r_x + \alpha\beta\beta_x,\\ \beta_{t_2} &= \beta_{xx} - \frac{1}{8}q_x\beta - \frac{1}{2}q\beta_x - \alpha_x.\end{aligned} \tag{28}$$

Taking $\alpha = \beta = 0$ in (28), we obtain the nonlinear super integrable couplings of the second-order super classical-Boussinesq equation

$$\begin{aligned}
q_{t_2} &= -\frac{1}{2}q_{xx} + 2r_x + \frac{1}{2}qq_x, \qquad r_{t_2} = \frac{1}{2}r_{xx} + \frac{1}{2}q_x r + \frac{1}{2}qr_x,\\
u_{1,t_2} &= \frac{1}{2}(1-\varepsilon)\left[u_{2x} + q_x u_1 + qu_{1x} + 2u_1u_{1x} - u_{1xx} + r_x - \frac{1}{4}q_{xx} + \frac{1}{4}qq_x\right]\\
&\quad - \frac{1}{2}r_x + \frac{1}{8}q_{xx} - \frac{1}{8}qq_x,\\
u_{t_2} &= (1-\varepsilon)\left[2u_{1x}r + 2u_{1x}u_2 + 2u_1u_{2x} + \frac{1}{2}u_{2xx} + \frac{1}{2}q_xu_2 + \frac{1}{2}qu_{2x} + \frac{1}{2}qr_x + \frac{1}{2}q_xr + \frac{1}{2}r_{xx} + 2r_xu_1\right]\\
&\quad - \frac{1}{2}q_xr - \frac{1}{2}qr_x - \frac{1}{2}r_{xx}.
\end{aligned} \tag{29}$$

Letting $\varepsilon = 0$ in (28), we have

$$\begin{aligned}
q_{t_2} &= -\frac{1}{2}q_{xx} + 2r_x - 4\alpha_x\beta - 4\alpha\beta_x + \frac{1}{2}qq_x + 4\beta\beta_{xx} + 4\beta_x^2,\\
r_{t_2} &= \frac{1}{2}r_{xx} + \frac{1}{2}q_xr + \frac{1}{2}qr_x + 2\alpha\alpha_x - 2r\beta\beta_x,\\
u_{1,t_2} &= \frac{1}{2}\left[u_{2x} + q_xu_1 + qu_{1x} + 2u_1u_{1x} - u_{1xx}\right] + \alpha_x\beta + \alpha\beta_x - \beta\beta_{xx},\\
u_{t_2} &= \left[2u_{1x}r + 2u_{1x}u_2 + 2u_1u_{2x} + \frac{1}{2}u_{2xx} + \frac{1}{2}q_xu_2 + \frac{1}{2}qu_{2x} + 2r_xu_1\right] + 2r\beta\beta_x - 2\alpha\alpha_x,\\
\alpha_{t_2} &= -\alpha_{xx} - \frac{1}{2}q\alpha_x - \frac{3}{8}q_x\alpha - \frac{1}{2}\beta r_x + \alpha\beta\beta_x,\\
\beta_{t_2} &= \beta_{xx} - \frac{1}{8}q_x\beta - \frac{1}{2}q\beta_x - \alpha_x.
\end{aligned} \tag{30}$$

When setting $\varepsilon = 1$, $u_1 = -(1/4)q$, $u_2 = r$ in (28), we obtain the second-order super classical-Boussinesq equations

$$\begin{aligned}
q_{t_2} &= -\frac{1}{2}q_{xx} + 2r_x - 4\alpha_x\beta - 4\alpha\beta_x + \frac{1}{2}qq_x + 4\beta\beta_{xx} + 4\beta_x^2,\\
r_{t_2} &= \frac{1}{2}r_{xx} + \frac{1}{2}q_xr + \frac{1}{2}qr_x + 2\alpha\alpha_x - 2r\beta\beta_x,\\
\alpha_{t_2} &= -\alpha_{xx} - \frac{1}{2}q\alpha_x - \frac{3}{8}q_x\alpha - \frac{1}{2}\beta r_x + \alpha\beta\beta_x,\\
\beta_{t_2} &= \beta_{xx} - \frac{1}{8}q_x\beta - \frac{1}{2}q\beta_x - \alpha_x.
\end{aligned} \tag{31}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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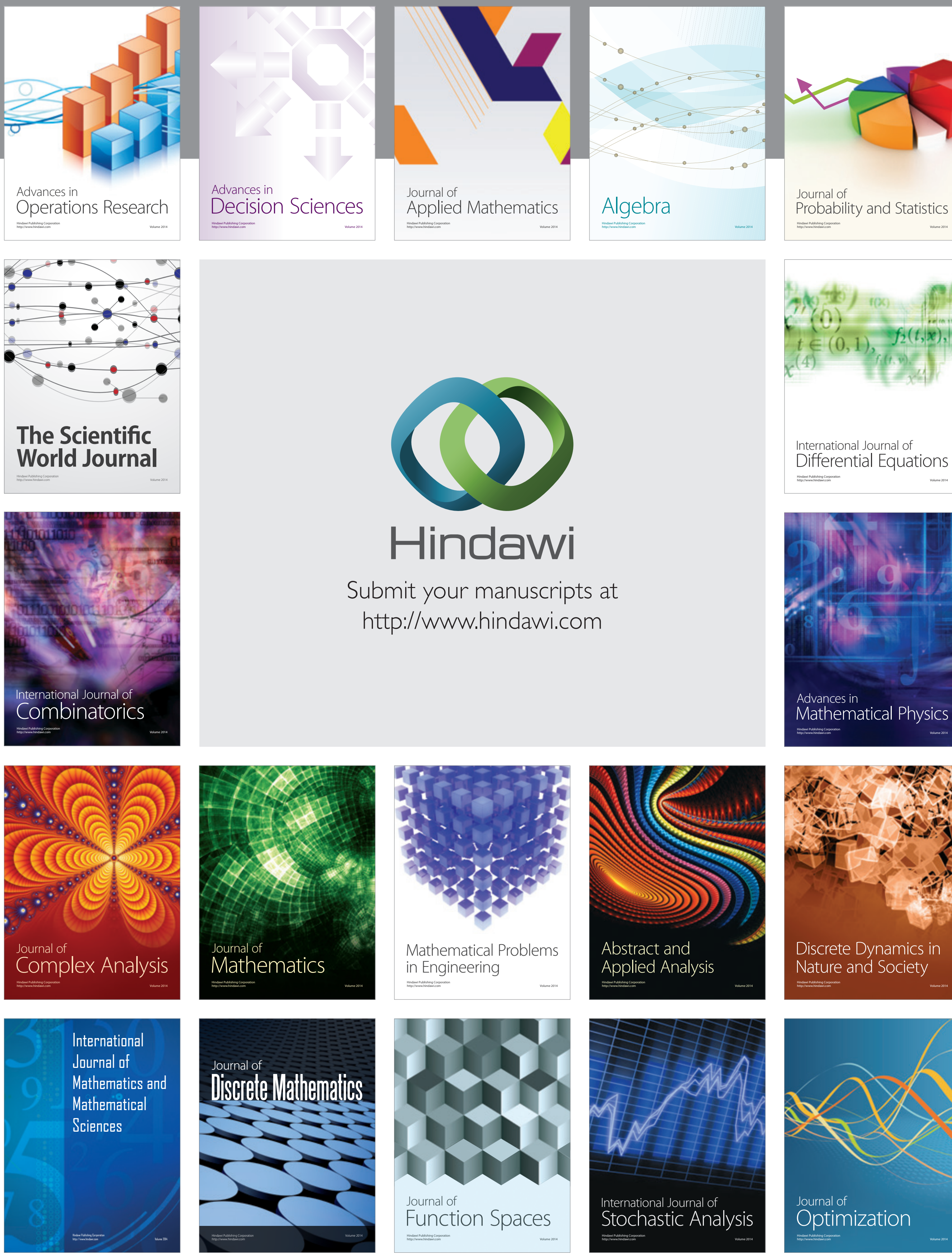

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