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Research Article

The Construction of Type-2 Fuzzy Reasoning Relations for Type-2 Fuzzy Logic Systems

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Type-2 fuzzy reasoning relations are the type-2 fuzzy relations obtained from a group of type-2 fuzzy reasonings by using extended t-(co)norm, which are essential for implementing type-2 fuzzy logic systems. In this paper an algorithm is provided for constructing type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems. First, we give some properties of extended t-(co)norm and simplify the expression of type-2 fuzzy reasoning relations in accordance with different input subdomains under certain conditions. And then different techniques are discussed to solve the simplified expressions on the input subdomains by using the related methods on solving fuzzy relation equations. Besides, it is pointed out that the computation amount level of the proposed algorithm is the same as that of polynomials and the possibility of applying the proposed algorithm in the construction of type-2 fuzzy reasoning relations is illustrated on several examples. Finally, the calculation of an arbitrary extended continuous t-norm can be obtained as the special case of the proposed algorithm.

1. Introduction

Type-2 fuzzy sets first proposed by Zadeh in 1975 [1] are fuzzy sets equipped with ordinary fuzzy subsets of $[0, 1]$ as membership grades, henceforth called fuzzy truth values. Then Mizumoto and Tanaka [2, 3] used Zadeh's extension principle to extend minimum and maximum both based on minimum for calculating union and intersection on type-2 fuzzy sets, respectively, and showed that the results of the union and intersection keep the convexity and normality. Based on the theory of type-2 fuzzy sets, Karnik et al. [4] proposed a new fuzzy system called type-2 fuzzy system. Up to now, both the theory and application of type-2 fuzzy systems have been widely researched (see, e.g., [5–8]). What is more, type-2 fuzzy neural networks and type-2 fuzzy classification and pattern recognition have been also studied (see, e.g., [9, 10]). However, the computation process of the extended operations on the noninterval type-2 fuzzy sets is more complex than that of ordinary operations on type-1 fuzzy sets, which blocks the wide use of the noninterval type-2 fuzzy logic systems, type-2 fuzzy neural networks, and so on. In recent years, a heated wave of research about the operation on type-2 fuzzy sets has been set off. For example, Karnik and Mendel [11] further generalized these definitions of operations presented by Mizumoto and Tanaka and gave some analytical formulae for extensions of extended maximum and minimum based on minimum or product. Kawaguchi and Miyakoshi [12, 13] showed that extended continuous t-(co)norms based on arbitrary t-norm satisfy the definitions of type-2 t-(co)norms. C. L. Walker and E. A. Walker [14, 15] considered the algebras of fuzzy truth values equipped with extended maximum and minimum based on minimum. Coupland and John [16, 17] presented geometric methods for performing the operations of extended minimum and maximum based on minimum on type-2 fuzzy sets. Starczewski [18] provided analytical expressions for membership functions of five kinds of extended t-norms. Ling and Zhang [19] reconstructed the framework of set-theoretic operations on triangle type-2 fuzzy sets by presenting polygon type-2 fuzzy sets and gave manageable and simplified formulas for operations on triangle type-2 fuzzy sets. Hu and Kwong [20] discussed extended t-norm on a linearly ordered set with a unit interval and a real number set as special cases.

From the above it can be seen that these research works have well contributed to the properties of extended t-(co)norms and gave many useful results for the calculations of some kinds of extended t-(co)norms. All of these promote the structure of noninterval type-2 fuzzy logic systems since extended t-(co)norms are the important tools in the construction of type-2 fuzzy reasoning relations. Nevertheless, there are still many other extended t-norms whose membership functions lack analytical expressions or feasible algorithms. It hampers the attempt of the construction of type-2 fuzzy reasoning relations by using these extended t-norms. Besides, the work [18] leaves a key problem to us that, except for extended minimum and maximum both based on minimum, no theory guarantees that the results of general extended t-(co)norms on two type-2 fuzzy sets still satisfy the calculation conditions (e.g., convexity and normality). Moreover, there are always more than two fuzzy truth values in the calculation process of the construction of type-2 fuzzy reasoning relations; it may be time-consuming and laborious to proceed the calculation just on two fuzzy truth values each time. It is a natural idea that we can solve the computation in an integral and faster way. This paper is devoted to deal with these problems we have mentioned above. The following rows present our results: we show that the results of extended continuous t-(co)norms based on arbitrary t-norm keep the convexity and normality and simplify the expression of type-2 fuzzy reasoning relations of type-2 fuzzy logic systems with single input and single output (SISO) in accordance with different input subdomains under the condition that all the fuzzy truth values of type-2 fuzzy sets participated in the calculation are required to be convex and normal (Theorem 2). After that, we solve the simplified expressions on three input subdomains (from Theorem 3 to Theorem 9), which demonstrate an algorithm to construct type-2 fuzzy reasoning relations. The complexity of the algorithm is analyzed and it is pointed out that the computation amount level of the proposed algorithm is the same as that of polynomials. And then the possibility of applying the proposed algorithm in the construction of type-2 fuzzy reasoning relations is illustrated on several examples. Besides, the calculation of a class of extended t-norms being broader than those in [18] can be obtained as the special case of the proposed algorithm.

This paper is organized in five sections. The following section contains some preliminary knowledge and the concrete expression of type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems. In Section 3 the method for the construction of type-2 fuzzy reasoning relations is investigated under certain conditions on the basis of the properties of extended t-(co)norm and the related methods on solving method of fuzzy relation equations. Section 4 gives several examples by using the presented method. Conclusions are given in Section 5.

2. Preliminaries

A type-2 fuzzy set $\widetilde{A}$ on the domain X is characterized by a membership function $\mu_{\widetilde{A}} : X \to \mathscr{F}([0,1])$, $x \mapsto \mu_{\widetilde{A}}(x)$, where $\mathscr{F}([0,1]) = \{f \mid f : [0,1] \to [0,1]\}$, and $f \in \mathscr{F}([0,1])$ is called a fuzzy truth value. Convenience to the following writing, we denote $\mu_{\widetilde{A}}(x)$ by $\mu_{\widetilde{A}(x)}$. Moreover, f is normal if there exists an $x \in [0,1]$ such that $f(x) = 1$ and convex if, for any $x_1, x_2 \in [0,1]$ and each $\lambda \in [0,1]$, $f(\lambda x_1 + (1-\lambda)x_2) \geqslant f(x_1) \wedge f(x_2)$. Let $\mathscr{F}_{CN}([0,1])$ be the set of both convex and normal fuzzy truth values. Assume that $\widetilde{A}, \widetilde{B} \in \widetilde{\mathscr{F}}(X)$. Let $\star$ and $\star'$ be t-norm and $\diamond$ t-conorm. Union and intersection on type-2 fuzzy sets are given as follows. For $\forall x \in X$, $\forall w \in [0,1]$,

$$\begin{aligned}\widetilde{A} \cup \widetilde{B} : \mu_{(\widetilde{A}\cup\widetilde{B})(x)}(w) &\triangleq \left(\mu_{\widetilde{A}(x)} \sqcup^{(\diamond,\star')} \mu_{\widetilde{B}(x)}\right)(w) \\ &\triangleq \sup_{w=u\diamond v}\left(\mu_{\widetilde{A}(x)}(u) \star' \mu_{\widetilde{B}(x)}(v)\right), \\ \widetilde{A} \cap \widetilde{B} : \mu_{(\widetilde{A}\cap\widetilde{B})(x)}(w) &\triangleq \left(\mu_{\widetilde{A}(x)} \sqcap^{(\star,\star')} \mu_{\widetilde{B}(x)}\right)(w) \\ &\triangleq \sup_{w=u\star v}\left(\mu_{\widetilde{A}(x)}(u) \star' \mu_{\widetilde{B}(x)}(v)\right),\end{aligned} \tag{1}$$

where $\sqcup^{(\diamond,\star')}$ and $\sqcap^{(\star,\star')}$ are called *extended t-conorm* and *extended t-norm*, respectively. Let $X \times Y$ be a new domain constructed by two domains X, Y. A type-2 fuzzy set $\widetilde{R} \in \widetilde{\mathscr{F}}(X \times Y)$ is called a type-2 fuzzy relation between X and Y, where

$$\mu_{\widetilde{R}} : X \times Y \longrightarrow \mathscr{F}([0,1]), \quad (x,y) \longmapsto \mu_{\widetilde{R}}(x,y) \triangleq \mu_{\widetilde{R}(x,y)}. \tag{2}$$

In the following, we will give the expression of type-2 fuzzy relation from a group of type-2 fuzzy reasoning. This type-2 fuzzy relation is called a *type-2 fuzzy reasoning relation*. Let $\{\widetilde{A}_i\}_{1\leqslant i\leqslant N}$ and $\{\widetilde{B}_i\}_{1\leqslant i\leqslant N}$ be, respectively, type-2 fuzzy sets on input domain X and output domain Y. For a group of type-2 fuzzy reasonings in a SISO type-2 fuzzy logic system

$$\text{if } x \text{ is } \widetilde{A}_i \text{ then } y \text{ is } \widetilde{B}_i, \quad i = 1, \ldots, N, \tag{3}$$

which can be rewritten as $\{\widetilde{A}_i \to \widetilde{B}_i, i = 1, \ldots, N\}$ and induce the total type-2 fuzzy reasoning relation as follows:

$$\widetilde{R} = \bigcup_{i=1}^{N} \widetilde{R}_i = \bigcup_{i=1}^{N} \left(\widetilde{A}_i \longrightarrow \widetilde{B}_i\right). \tag{4}$$

By choosing the suitable $\sqcup^{(\vee,\star')}$ and $\sqcap^{(\star,\star')}$ we can obtain that

$$\begin{aligned}\mu_{\widetilde{R}(x,y)}(w) &= \mu_{(\bigcup_{i=1}^{N}\widetilde{R}_i)(x,y)}(w) \\ &= \left({\sqcup^{(\vee,\star')}}_{i=1}^{N}\left(\mu_{\widetilde{A}_i(x)} \sqcap^{(\star,\star')} \mu_{\widetilde{B}_i(y)}\right)\right)(w) \\ &= \sup_{\bigvee_{i=1}^{N}(u_i\star v_i)=w}\left({\mathscr{T}'}_{i=1}^{N}\left(\mu_{\widetilde{A}_i(x)}(u_i) \star' \mu_{\widetilde{B}_i(y)}(v_i)\right)\right),\end{aligned} \tag{5}$$

where $\mathscr{T}'$ and $\star'$ indicate the same t-norm. It is clear that the difficulty on the calculation of type-2 fuzzy reasoning relation is to solve the expression (5). For convenience, we first fix x and y and denote

$$\mathbf{u} = (u_1, \ldots, u_N), \qquad \mathbf{v} = (v_1, \ldots, v_N),$$
$$F(w) = \mu_{\widetilde{R}(x,y)}(w),$$

$$g_i(u_i) = \mu_{\widetilde{A}_i(x)}(u_i), \quad i = 1, \ldots, N,$$
$$h_i(v_i) = \mu_{\widetilde{B}_i(y)}(v_i), \quad i = 1, \ldots, N,$$
$$f(\mathbf{u}, \mathbf{v}) = \mathcal{T}'^{N}_{i=1}\left(g(u_i) \star' h(v_i)\right),$$
$$P_w = \left\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \bigvee_{i=1}^{N}(u_i \star v_i) = w\right\}. \tag{6}$$

Then the expression (5) can be rewritten as

$$F(w) = \sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in P_w\}. \tag{7}$$

In what follows, we mainly pay attention to working out the expression (7). When w changes, P_w and $F(w)$ change with it. Then in order to solve $F(w)$, we should reduce the range of P_w as much as possible and then obtain $F(w)$ (i.e., the maximum of $f(\mathbf{u}, \mathbf{v})$ in P_w) according to the characteristic of elements in P_w. Next, we will focus on analyzing the condition $\bigvee_{i=1}^{N}(u_i \star v_i) = w$, which is a fuzzy relation equation if $\mathbf{u}$ is regarded as a coefficient vector and $\mathbf{v}$ is regarded as an unknown vector. It is known that fuzzy relation equation was first presented by Sanchez in 1976 [21]. Following it, a lot of work has focused on the solvability conditions and the solution sets. For example, these works [22–24] have systematically introduced some theories of fuzzy relational equations. Bourke and Fisher [25] gave solution algorithms for fuzzy relational equations with max-product composition. Stamou and Tzafestas [26] discussed the resolution of composite fuzzy relation equations based on Archimedean triangular norms. Wang and Xiong [27] investigated the solution sets of a fuzzy relation equation with sup-conjunctor composition in a complete lattice. Next some conceptions and conclusions on fuzzy relation equations will be given.

Let $\mathbf{a} = (a_1, \ldots, a_N)$, $\mathbf{b} = (b_1, \ldots, b_N) \in [0,1]^N$. Define the partial order

$$\mathbf{a} \leqslant \mathbf{b} \Longleftrightarrow a_i \leqslant b_i, \quad i = 1, \ldots, N. \tag{8}$$

There exists no partial order relation between $\mathbf{a}$ and $\mathbf{b}$ if and only if $[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}] = \emptyset$.

Define

$$\mathbf{a} \vee \mathbf{b} \triangleq (a_1 \vee b_1, \ldots, a_N \vee b_N),$$
$$\mathbf{a} \wedge \mathbf{b} \triangleq (a_1 \wedge b_1, \ldots, a_N \wedge b_N). \tag{9}$$

The single fuzzy relation equation constituted by composite relation $\vee - \star$ is as follows:

$$(a_1 \star x_1) \vee (a_2 \star x_2) \vee \cdots \vee (a_N \star x_N) = b, \tag{10}$$

where $\mathbf{a} = (a_1, \ldots, a_N) \in [0,1]^N$ is the coefficient vector, $b \in [0,1]$ is known, and $\mathbf{x} = (x_1, \ldots, x_N) \in [0,1]^N$ is unknown. Let $\mathcal{X}_\star$ be the solution set of (10). The greatest and minimal elements in $\mathcal{X}_\star$ are, respectively, called the greatest and minimal solutions of (10). Denote

$$\mathcal{I}_\star(a, b) = \sup\{x \in [0,1] \mid a \star x \leqslant b\},$$
$$\mathcal{L}_\star(a, b) = \inf\{x \in [0,1] \mid a \star x \geqslant b\}. \tag{11}$$

Define $\inf \emptyset = 1$. Moreover, some necessary interpretations about the two operations are presented in the following.

(1) $\mathcal{I}_\star(a, b) \geqslant b$ since $a \star b \leqslant 1 \star b = b$.

(2) If $a \star c \leqslant b$ then $\mathcal{I}_\star(a, b) \geqslant c$; if $a \star c \geqslant b$, then $\mathcal{L}_\star(a, b) \leqslant c$.

(3) Both $\mathcal{I}_\star(a, b)$ and $\mathcal{L}_\star(a, b)$ are monotone decreasing about the first variable, that is,

$$a_1 \leqslant a_2 \Longrightarrow \mathcal{I}_\star(a_1, b) \geqslant \mathcal{I}_\star(a_2, b),$$
$$\mathcal{L}_\star(a_1, b) \geqslant \mathcal{L}_\star(a_2, b), \tag{12}$$

since $\{x \in [0,1] \mid a_1 \star x \leqslant b\} \supseteq \{x \in [0,1] \mid a_2 \star x \leqslant b\}$ and $\{x \in [0,1] \mid a_1 \star x \geqslant b\} \subseteq \{x \in [0,1] \mid a_2 \star x \geqslant b\}$.

Let

$$G_b = \{i \in \{1, \ldots, N\} \mid a_i \geqslant b\} = \{k_j, j = 1, \ldots, |G_b|\}. \tag{13}$$

In this work it is assumed that $\star$ is continues and the following results presented in [27] are fitted for (10) on $[0,1]$.

Lemma 1. *Let $\star$ be a continuous t-norm. Then the following items are equivalent.*

(1) *$\mathcal{X}_\star \neq \emptyset$ if and only if $G_b \neq \emptyset$; that is, there exists $i \in \{1, \ldots, N\}$, such that $a_i \geqslant b$ if and only if (10) has the greatest solution $\mathbf{x}^* = (x_1^*, \ldots, x_N^*) \triangleq (\mathcal{I}_\star(a_1, b), \ldots, \mathcal{I}_\star(a_N, b))$.*

(2) *If $\mathcal{X}_\star \neq \emptyset$, then (10) has the minimum solutions where the jth minimum solution $\mathbf{x}_j^0 = (x_{j1}^0, \ldots, x_{jN}^0)$ $(1 \leqslant j \leqslant |G_b|)$ is*

$$x_{ji}^0 = \begin{cases} \mathcal{L}_\star\left(a_{k_j}, b\right), & i = k_j, \\ 0, & \text{otherwise}, \end{cases} \quad i = 1, \ldots, N. \tag{14}$$

Furthermore, the solution set of (10) can be written as

$$\mathcal{X}_\star = \bigcup_{j=1}^{|G_b|}\left[\mathbf{x}_j^0, \mathbf{x}^*\right]. \tag{15}$$

3. The Construction of Type-2 Fuzzy Reasoning Relations

In this section, we will demonstrate the solving process for the expression (7) gradually. First, we will simplify the expression (7) in accordance with three subdomains of w. Importantly, for two of these subdomains we will, respectively, reduce P_w into its subdomains P_{w1} and P_{w2} but keeping the values of $F(w)$ without change (Theorem 2). Then all the elements in P_{w1} and P_{w2} will be found out (Theorem 3). Following it, P_{w1} and P_{w2} will be further reduced into smaller subsets $\mathfrak{X}_1$ and $\mathfrak{X}_2$ still keeping the values of $F(w)$ without change, respectively (Theorem 5). Finally, some theorems about how to get the exact value of $F(w)$ will be presented on the basis of the characteristics of the $f(\mathbf{u}, \mathbf{v})$ on $\mathfrak{X}_1$ and $\mathfrak{X}_2$ (Theorems 7 and 9).

It needs to be stated that the proposed method to solve $F(w)$ differs from the native algorithm which is just finding the maximal number of $f(\mathbf{u},\mathbf{v})$ from all the elements in P_w (or P_{w1} and P_{w2}). The native algorithm is impractical due to its huge computation. But what form of the elements in P_w is the key to solving the problem (7). Let $g \in \mathscr{F}_{CN}([0,1])$. Denote $[g]_1 = \{u \in [0,1] \mid g(u) = 1\}$.

Theorem 2. *Let $w \in [0,1], g_1,\ldots,g_N, h_1,\ldots,h_N \in \mathscr{F}_{CN}([0,1])$, where $[g_1]_1 = [m_{g_1}, n_{g_1}],\ldots,[g_N]_1 = [m_{g_N}, n_{g_N}]$, $[h_1]_1 = [m_{h_1}, n_{h_1}],\ldots,[h_N]_1 = [m_{h_N}, n_{h_N}]$. Denote*

$$\alpha = \bigvee_{i=1}^{N}\left(m_{g_i} \star m_{h_i}\right), \qquad \beta = \bigvee_{i=1}^{N}\left(n_{g_i} \star n_{h_i}\right),$$
$$\mathbf{m_g} = \left(m_{g_1},\ldots,m_{g_N}\right), \qquad \mathbf{m_h} = \left(m_{h_1},\ldots,m_{h_N}\right),$$
$$\mathbf{n_g} = \left(n_{g_1},\ldots,n_{g_N}\right), \qquad \mathbf{n_h} = \left(n_{h_1},\ldots,n_{h_N}\right),$$
$$P_{w1} = \left\{(\mathbf{u},\mathbf{v}) \in [0,1]^{2N} \mid \bigvee_{i=1}^{N}\left(u_i \star v_i\right) = w, \mathbf{u} \leqslant \mathbf{m_g}, \mathbf{v} \leqslant \mathbf{m_h}\right\},$$
$$P_{w2} = \left\{(\mathbf{u},\mathbf{v}) \in [0,1]^{2N} \mid \bigvee_{i=1}^{N}\left(u_i \star v_i\right) = w, \mathbf{u} \geqslant \mathbf{n_g}, \mathbf{v} \geqslant \mathbf{n_h}\right\}. \tag{16}$$

Then the following items hold.

(1) *If $w \in [0,\alpha]$, then $F(w) = \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in P_{w1}\}$.*

(2) *If $w \in [\alpha,\beta]$, then $F(w) = 1$.*

(3) *If $w \in [\beta,1]$, then $F(w) = \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in P_{w2}\}$.*

Before the proof of Theorem 2, several conclusions and their proofs will be given in the following and the conclusion (a) is from [18].

(a) Let $w \in [0,1], f, g \in \mathscr{F}_{CN}([0,1])$, where $[f]_1 = [m_f, n_f]$ and $[g]_1 = [m_g, n_g]$. Assume that $\star$ is continuous. Denote

$$L = \left\{(u,v) \in [0,1]^2 \mid u \star v = w\right\},$$
$$L_1 = \left\{(u,v) \in [0,1]^2 \mid u \star v = w, u \leqslant m_f, v \leqslant m_g\right\}, \tag{17}$$
$$L_2 = \left\{(u,v) \in [0,1]^2 \mid u \star v = w, n_f \leqslant u, n_g \leqslant v\right\}.$$

Then the following items hold.

(1) If $w \in [0, m_f \star m_g]$, then $(f\sqcap^{(\star,\star')}g)(w) = \sup\{f(u)\star' g(v) \mid (u,v) \in L_1\}$.

(2) If $w \in [m_f \star m_g, n_f \star n_g]$, then $(f\sqcap^{(\star,\star')}g)(w) = 1$.

(3) If $w \in [n_f \star n_g, 1]$, then $(f\sqcap^{(\star,\star')}g)(w) = \sup\{f(u)\star' g(v) \mid (u,v) \in L_2\}$.

(b) Suppose that the conditions is the same as that of (a). Denote

$$C_1 = \left\{(u,v) \in [0,1]^2 \mid u \vee v = w, u \leqslant m_f, v \leqslant m_g\right\},$$
$$C_2 = \left\{(u,v) \in [0,1]^2 \mid u \vee v = w, n_f \leqslant u, n_g \leqslant v\right\}. \tag{18}$$

Then the following items hold.

(1) If $w \in [0, m_f \vee m_g]$, then $(f\sqcup^{(\vee,\star')}g)(w) = \sup\{f(u)\star' g(v) \mid (u,v) \in C_1\}$.

(2) If $w \in [m_f \vee m_g, n_f \vee n_g]$, then $(f\sqcup^{(\vee,\star')}g)(w) = 1$.

(3) If $w \in [n_f \vee n_g, 1]$, then $(f\sqcup^{(\vee,\star')}g)(w) = \sup\{f(u)\star' g(v) \mid (u,v) \in C_2\}$.

Proof. This proof is similar as that of (a) in [18] since $\vee$ is also monotone increasing in the first and second variables. □

(c) Let $w_1, w_2, \tau_1, \tau_2 \in [0,1]$, where $w_1 < w_2$. Assume that $\star$ is continuous. Denote

$$\mathscr{M}_1 = \left\{(a,b) \in [0,1]^2 \mid a \star b = w_1, a \leqslant \tau_1, b \leqslant \tau_2\right\},$$
$$\mathscr{M}_2 = \left\{(c,d) \in [0,1]^2 \mid c \star d = w_2, c \leqslant \tau_1, d \leqslant \tau_2\right\},$$
$$\mathscr{M}_3 = \left\{(a,b) \in [0,1]^2 \mid a \star b = w_1, \tau_1 \leqslant a, \tau_2 \leqslant b\right\}, \tag{19}$$
$$\mathscr{M}_4 = \left\{(c,d) \in [0,1]^2 \mid c \star d = w_2, \tau_1 \leqslant c, \tau_2 \leqslant d\right\}.$$

Then for every $(a,b) \in \mathscr{M}_1$ [resp. $\mathscr{M}_3$], there exists $(c,d) \in \mathscr{M}_2$ [resp. $\mathscr{M}_4$] such that $a \leqslant c$ and $b \leqslant d$.

Proof. Let $(a,b) \in \mathscr{M}_1$ and $(u,v) \in \mathscr{M}_2$. Since $w_1 < w_2$, by the monotonicity of $\star$, we have $a \leqslant u$ or $b \leqslant v$. Assume that $a \leqslant u$. If $b \leqslant v$, then the conclusion is obvious. For the case of $b > v$, there is

$$a \star b \leqslant u \star v \leqslant u \star b, \tag{20}$$

that is,

$$a \star b \leqslant w_2 \leqslant u \star b. \tag{21}$$

By the continuity of $\star$, it can be inferred that there exists $z \in [a,u]$ such that $w_2 = z \star b$. Let $(c,d) = (z,b)$. Then there are $c \star d = w_2, a \leqslant c \leqslant \tau_1$, and $b \leqslant d \leqslant \tau_2$. Clearly $(c,d) \in \mathscr{M}_2$. Similarly, we can prove that if $b \leqslant v$ and $a > u$, there exists $x \in [b,v]$ such that $w_2 = a \star x$. Let $(c,d) = (a,x)$. Then there are $c \star d = w_2, a \leqslant c \leqslant \tau_1$, and $b \leqslant d \leqslant \tau_2$. To sum up, we can conclude that for every $(a,b) \in \mathscr{M}_1$, there exists $(c,d) \in \mathscr{M}_2$ such that $a \leqslant c, b \leqslant d$. In a similar way, we can prove that for every $(a,b) \in \mathscr{M}_3$, there exists $(c,d) \in \mathscr{M}_4$ such that $a \leqslant c$ and $b \leqslant d$. □

(d) Let $f, g \in \mathscr{F}_{CN}([0,1])$, where $[f]_1 = [m_f, n_f]$ and $[g]_1 = [m_g, n_g]$. Assume that $\star$ and $\diamond$ are continuous. Then $f\sqcap^{(\star,\star')}g, f\sqcup^{(\diamond,\star')}g \in \mathscr{F}_{CN}([0,1])$. Furthermore, $[f\sqcap^{(\star,\star')}g]_1 = [m_f \star m_g, n_f \star n_g]$ and $[f\sqcup^{(\diamond,\star')}g]_1 = [m_f \diamond m_g, n_f \diamond n_g]$.

Proof. Let $[f]_1 = [m_f, n_f]$, $[g]_1 = [m_g, n_g]$, $w_1, w_2, w_3 \in [0,1]$, where $w_1 \leqslant w_2 \leqslant w_3$. By the continuity of $\star$ and conclusion (a), we obtain that if $w_2 \in [m_f \star m_g, n_f \star n_g]$, then $(f \sqcap^{(\star,\star')} g)(w_2) = 1$. For the converse, let $w \in [0, m_f \star m_g)$. If there exists $w_0 \in [0, m_f \star m_g)$ such that $(f \sqcap^{(\star,\star')} g)(w_0) = 1$, that is, there exists $(u_0, v_0) \in [0,1]^2$ such that $w_0 = u_0 \star v_0 < m_f \star m_g$ and $f(u_0) \star' g(v_0) = 1$, then there is $f(u_0) = g(v_0) = 1$. By the monotonicity of $\star$, we have $u_0 < m_f$ or $v_0 < m_g$. Without loss of generality, we can assume that $u_0 < m_f$. Thus $f(u_0) < f(m_f) = 1$, which leads to a contradiction. Therefore, $(f \sqcap^{(\star,\star')} g)(w) < 1$. In a similar way, we can prove that if $w \in (n_f \star n_g, 1]$, then $(f \sqcap^{(\star,\star')} g)(w) < 1$. To sum up, we have $[f \sqcap^{(\star,\star')} g]_1 = [m_f \star m_g, n_f \star n_g]$.

Now we will give the proof of convexity. It is obvious that

$$\left(f \sqcap^{(\star,\star')} g\right)(w_2) \geqslant \left(f \sqcap^{(\star,\star')} g\right)(w_1) \wedge \left(f \sqcap^{(\star,\star')} g\right)(w_3). \tag{22}$$

If $w_2 \in [0, m_f \star m_g)$, then $w_1 \leqslant w_2 \leqslant m_f \star m_g$. From conclusion (a), it can be inferred that the values of $(f \sqcap^{(\star,\star')} g)(w_1)$ and $(f \sqcap^{(\star,\star')} g)(w_2)$ can be obtained on $[0, m_f] \times [0, m_g]$. Denote

$$\begin{aligned} \mathcal{N}_1 &= \left\{(a,b) \in [0,1]^2 \mid a \star b = w_1, a \leqslant m_f, b \leqslant m_g\right\}, \\ \mathcal{N}_2 &= \left\{(c,d) \in [0,1]^2 \mid c \star d = w_2, c \leqslant m_f, d \leqslant m_g\right\}. \end{aligned} \tag{23}$$

Let $(u_1, v_1) \in \mathcal{N}_1$ satisfy

$$f(u_1) \star' g(v_1) = \left(f \sqcap^{(\star,\star')} g\right)(w_1). \tag{24}$$

By conclusion (c), there exists $(u_2, v_2) \in \mathcal{N}_2$ such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$. Because $f(u_1) \leqslant f(u_2)$ and $g(v_1) \leqslant g(v_2)$ by the convexity of f and g, we have

$$f(u_1) \star' g(v_1) \leqslant f(u_2) \star' g(v_2) \leqslant \left(f \sqcap^{(\star,\star')} g\right)(w_2). \tag{25}$$

From (24) and (25), we get

$$\left(f \sqcap^{(\star,\star')} g\right)(w_1) \leqslant \left(f \sqcap^{(\star,\star')} g\right)(w_2) \tag{26}$$

which implies that (22) holds. If $w_2 \in (n_f \star n_g, 1]$, then $n_f \star n_g \leqslant w_2 \leqslant w_3$. In a similar way, we can prove that $(f \sqcap^{(\star,\star')} g)(w_3) \leqslant (f \sqcap^{(\star,\star')} g)(w_2)$. Thus (22) holds. To sum up, there is $f \sqcap^{(\star,\star')} g \in \mathcal{F}_C([0,1])$. It is easy to prove that the conclusion (c) is valid if $\star$ is replaced with $\diamond$ since $\diamond$ is also monotone increasing in the first and second variables. Therefore, in a similar way, we can give the proof of $f \sqcup^{(\diamond,\star')} g \in \mathcal{F}_{CN}([0,1])$. □

Next we will give the proof of Theorem 2.

Proof. It is known that $F(w) = (\sqcup^{(\vee,\star')N}_{i=1} (f_i \sqcap^{(\star,\star')} g_i))(w)$. From conclusion (d) it can be obtained that $[f_i \sqcap^{(\star,\star')} g_i] \in \mathcal{F}_{CN}([0,1])$ and $[f_i \sqcap^{(\star,\star')} g_i]_1 = [m_{f_i} \star m_{g_i}, n_{f_i} \star n_{g_i}]$, $i = 1, \ldots, N$. Moreover, denote

$$\begin{aligned} G_{i1} &= \left\{(u_i, v_i) \in [0,1]^2 \mid u_i \star v_i = z_i, u_i \leqslant m_{g_i}, v_i \leqslant m_{h_i}\right\}, \\ G_{i2} &= \left\{(u_i, v_i) \in [0,1]^2 \mid u_i \star v_i = z_i, u_i \geqslant n_{g_i}, v_i \geqslant n_{h_i}\right\}. \end{aligned} \tag{27}$$

From conclusion (a), we obtain that if $z_i \in [0, m_{f_i} \star m_{g_i}]$, then $(f_i \sqcap^{(\star,\star')} g_i)(z_i) = \sup\{f_i(u_i) \star' g_i(v_i) \mid (u_i, v_i) \in G_{i1}\}$; if $z_i \in [n_{f_i} \star n_{g_i}, 1]$, then $(f_i \sqcap^{(\star,\star')} g_i)(z_i) = \sup\{f_i(u_i) \star' g_i(v_i) \mid (u_i, v_i) \in G_{i2}\}$. From conclusion (d) we have $\sqcup^{(\vee,\star')N}_{i=1} (f_i \sqcap^{(\star,\star')} g_i) \in \mathcal{F}_{CN}([0,1])$ and $[\sqcup^{(\vee,\star')N}_{i=1} (f_i \sqcap^{(\star,\star')} g_i)]_1 = [\bigvee_{i=1}^N (m_{f_i} \star m_{g_i}), \bigvee_{i=1}^N (n_{f_i} \star n_{g_i})] = [\alpha, \beta]$. Denote $\mathbf{z} = (z_1, \ldots, z_N)$ and

$$\begin{aligned} E_1 &= \left\{\mathbf{z} \in [0,1]^N \mid \bigvee_{i=1}^N z_i = w, z_i \leqslant m_{f_i} \star m_{g_i}, i = 1, \ldots, N\right\}, \\ E_2 &= \left\{\mathbf{z} \in [0,1]^N \mid \bigvee_{i=1}^N z_i = w, z_i \geqslant n_{f_i} \star n_{g_i}, i = 1, \ldots, N\right\}. \end{aligned} \tag{28}$$

From the above discussion and conclusion (b), we have that if $w \in [0, \alpha]$, then

$$\begin{aligned} F(w) &= \sup\left\{\mathcal{T}'^N_{i=1} \left(f_i \sqcap^{(\star,\star')} g_i\right)(z_i) \mid \mathbf{z} \in E_1\right\} \\ &= \sup\left\{\mathcal{T}'^N_{i=1} \left(f_i \sqcap^{(\star,\star')} g_i\right)(z_i) \mid \bigvee_{i=1}^N z_i = w, z_i \leqslant m_{f_i} \star m_{g_i}, i = 1, \ldots, N\right\} \\ &= \sup\left\{\mathcal{T}'^N_{i=1} (g_i(u_i) \star h_i(v_i)) \mid \bigvee_{i=1}^N (u_i \star v_i) = w, u_i \leqslant m_{g_i}, v_i \leqslant m_{h_i}, i = 1, \ldots, N\right\} \\ &= \sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in P_{w1}\}. \end{aligned} \tag{29}$$

Similarly, if $w \in [\beta, 1]$, then $F(w) = \sup\{\mathcal{T}'^N_{i=1} (f_i \sqcap^{(\star,\star')} g_i)(z_i) \mid \mathbf{z} \in E_2\} = \sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in P_{w2}\}$. □

From Theorem 2, it can be seen that when $w \in [\alpha, \beta]$, we can omit the calculation process of $F(w)$ since $F(w) = 1$, and for other situations $F(w)$ can be obtained from P_{w1} or P_{w2} independently. From now on, we will focus on analyzing the cases of $w \in [0, \alpha)$ and $w \in (\beta, 1]$ and assume that

$g_i(t), h_i(t) \in \mathcal{F}_{CN}([0,1])$, where $[g_i]_1 = [m_{g_i}, n_{g_i}]$, $[h_i]_1 = [m_{h_i}, n_{h_i}]$, $i = 1, \ldots, N$. Denote

$$\begin{aligned} P_{o1} &= \left\{\mathbf{u} \in [0,1]^N \mid \mathbf{u} \leqslant \mathbf{m_g}\right\}, \\ P'_{o1} &= \left\{\mathbf{v} \in [0,1]^N \mid \mathbf{v} \leqslant \mathbf{m_h}\right\}, \\ P_{o2} &= \left\{\mathbf{u} \in [0,1]^N \mid \mathbf{u} \geqslant \mathbf{n_g}\right\}, \\ P'_{o2} &= \left\{\mathbf{v} \in [0,1]^N \mid \mathbf{v} \geqslant \mathbf{n_h}\right\}. \end{aligned} \tag{30}$$

The idea about how to find the elements in P_{w1} [P_{w2}, resp.] is to solve the fuzzy relation equation

$$(u_1 \star x_1) \vee (u_2 \star x_2) \vee \cdots \vee (u_N \star x_N) = w, \tag{31}$$

by taking $\mathbf{u} \in P_{o1}$ [resp. P_{o2}] and then obtain the solution $\mathbf{x}$ in P'_{o1} [P'_{o2}, resp.]. Thus $(\mathbf{u}, \mathbf{x}) \in P_{w1}$ [P_{w2}, resp.]. In this way, all the elements in P_{w1} [P_{w2}, resp.] can be found. Denote

$$G_w = \{i \in \{1, \ldots, N\} \mid u_i \geqslant w\}. \tag{32}$$

Now we will provide all the elements of P_{w1} and P_{w2}.

Theorem 3. *Assume that $\star$ is continuous. For every $\mathbf{u} \in P_{o1}$ or P_{o2} denote the greatest solution of* (31) *in $[0,1]^N$ as $\mathbf{x_u}^*$ and minimal solution of* (31) *in $[0,1]^N$ as $\mathbf{x_u}_1^0, \ldots, \mathbf{x_u}_{|G_w|}^0$ (if any). The following items hold.*

(1) *Suppose that $w \in [0, \alpha)$. Then for every $\mathbf{u} \in P_{o1}$ the solution set of* (31) *in P'_{o1} is $\bigcup_{j=1}^{|G_w|}[\mathbf{x_u}_j^0, \mathbf{x_u}^* \wedge \mathbf{m_h}]$ denoted by $\mathcal{V}_{\mathbf{u}}^1$ and*

$$P_{w1} = \left\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1, \mathbf{u} \leqslant \mathbf{m_g}\right\}. \tag{33}$$

(2) *Suppose that $w \in (\beta, 1]$. Then for every $\mathbf{u} \in P_{o2}$ the solution set of* (31) *in P'_{o2} is $\bigcup_{j=1}^{|G_w|}[\mathbf{x_u}_j^0 \vee \mathbf{n_h}, \mathbf{x_u}^*]$ denoted by $\mathcal{V}_{\mathbf{u}}^2$ and*

$$P_{w2} = \left\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \mathcal{V}_{\mathbf{u}}^2, \mathbf{n_g} \leqslant \mathbf{u}\right\}. \tag{34}$$

Proof. (1) From Lemma 1 it is obvious that $\mathcal{V}_{\mathbf{u}}^1$ is the solution set of (31) in P'_{o1}. For every $(\mathbf{u}, \mathbf{v}) \in \{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1, \mathbf{u} \leqslant \mathbf{m_g}\}$, we have $\mathbf{u} \in P_{o1}$ and $\mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1 \subseteq P'_{o1}$. Then from Lemma 1 it can be inferred that $\bigvee_{i=1}^N (u_i \star v_i) = w$. Thus $(\mathbf{u}, \mathbf{v}) \in P_{w1}$; that is, $\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1, \mathbf{u} \leqslant \mathbf{m_g}\} \subseteq P_{w1}$. For the converse case, let $(\mathbf{u}, \mathbf{v}) \in P_{w1}$. Then $\bigvee_{i=1}^N (u_i \star v_i) = w$. Obviously $\mathbf{v} \leqslant \mathbf{m_h}$ and $\mathbf{v}$ is a solution of (31) with the coefficient vector $\mathbf{u}$. Denote the solution set of (31) in $[0,1]^N$ as $\bigcup_{j=1}^{|G_w|}[\mathbf{x}_j^0, \mathbf{x}^*]$. Clearly there exists $j \in \{1, \ldots, |G_w|\}$ such that $\mathbf{v} \in [\mathbf{x}_j^0, \mathbf{x}^*]$. Thereby $\mathbf{v} \in [\mathbf{x}_j^0, \mathbf{x}^* \wedge \mathbf{m_h}] \subseteq \mathcal{V}_{\mathbf{u}}^1$; that is, $(\mathbf{u}, \mathbf{v}) \in \{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1, \mathbf{u} \leqslant \mathbf{m_g}\}$; that is, $\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1, \mathbf{u} \leqslant \mathbf{m_g}\} \supseteq P_{w1}$. To sum up, the conclusion (1) holds. In a similar way, we can prove the case (2). □

Corollary 4. *Assume that $\star$ is continuous. Let $\mathbf{u} \in P_{o1}$ or P_{o2}. Denote the greatest solution of* (31) *in $[0,1]^N$ as $\mathbf{x_u}^*$ and minimal solution of* (31) *in $[0,1]^N$ as $\mathbf{x_u}_1^0, \ldots, \mathbf{x_u}_{|G_w|}^0$ (if any). The following hold.*

(1) *Let $\mathbf{u} \in P_{o1}$. Equation* (31) *has a solution in P'_{o1} if and only if there exists $j \in G_w$ such that $\mathbf{x_u}_j^0 \leqslant \mathbf{m_h}$.*

(2) *Let $\mathbf{u} \in P_{o2}$. Equation* (31) *has a solution in P'_{o2} if and only if $\mathbf{n_h} \leqslant \mathbf{x_u}^*$.*

Proof. (1) Equation (31) has a solution in P'_{o1}, if and only if $\mathcal{V}_{\mathbf{u}}^1 \neq \emptyset$, and if and only if there exists $j \in G_w$ such that $\mathbf{x_u}_j^0 \leqslant \mathbf{m_h}$.

(2) Equation (31) has a solution in P'_{o2}, if and only if $\mathcal{V}_{\mathbf{u}}^2 \neq \emptyset$, and if and only if $\mathbf{n_h} \leqslant \mathbf{x_u}^*$. □

Next, on the basis of Theorem 3 we will further find subsets of P_{w1} and P_{w2} but keeping the values of $F(w)$ without change.

Theorem 5. *Assume that $\star$ is continuous. The following items hold.*

(1) *Suppose that $w \in [0, \alpha)$ and for every $\mathbf{u} \in P_{o1}$ the greatest solution of* (31) *in $[0,1]^N$ is $\mathbf{x_u}^*$ (if any). Denote*

$$\begin{aligned} \mathcal{U}_1 &= \left\{\mathbf{u} \in P_{o1} \mid \exists \mathbf{v} \in P'_{o1}, \text{s.t.}\, (\mathbf{u}, \mathbf{v}) \in P_{w1}\right\}, \\ \mathfrak{X}_1 &= \left\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} = \mathbf{x_u}^* \wedge \mathbf{m_h}, \mathbf{u} \in \mathcal{U}_1\right\}. \end{aligned} \tag{35}$$

Then

$$F(w) = \sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in \mathfrak{X}_1\}. \tag{36}$$

(2) *Suppose that $w \in (\beta, 1]$ and for every $\mathbf{u} \in P_{o2}$ minimal solutions of* (31) *in $[0,1]^N$ are $\mathbf{x_u}_j^0$, $j = 1, \ldots, |G_w|$, (if any). Denote*

$$\begin{aligned} \mathcal{U}_2 &= \left\{\mathbf{u} \in P_{o2} \mid \exists \mathbf{v} \in P'_{o2}, \text{s.t.}\, (\mathbf{u}, \mathbf{v}) \in P_{2w}\right\}, \\ \mathfrak{X}_2 &= \left\{(\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \left\{\mathbf{x_u}_j^0 \vee \mathbf{n_h}, j = 1, \ldots, |G_w|\right\}, \right. \\ &\qquad \left. \mathbf{u} \in \mathcal{U}_2\right\}. \end{aligned} \tag{37}$$

Then

$$F(w) = \sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in \mathfrak{X}_2\}. \tag{38}$$

Proof. (1) Clearly, for every $\mathbf{u} \in \mathcal{U}_1$, (31) has a solution in P'_{o1}. From Theorem 3, there is $(\mathbf{u}, \mathbf{x_u}^* \wedge \mathbf{m_h}) \in P_{w1}$; that is, $\mathfrak{X}_1 \subseteq P_{w1}$. Then

$$\begin{aligned} &\sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in \mathfrak{X}_1\} \\ &\quad \leqslant \sup\{f(\mathbf{u}, \mathbf{v}) \mid (\mathbf{u}, \mathbf{v}) \in P_{w1}\} = F(w). \end{aligned} \tag{39}$$

For the converse case, let $(\mathbf{u}, \mathbf{v}) \in P_{w1}$. Then $\mathbf{u} \in \mathcal{U}_1$ and $\mathbf{v} \in \mathcal{V}_{\mathbf{u}}^1$. We have $\mathbf{v} \leqslant \mathbf{x_u}^* \wedge \mathbf{m_h}$. Denote $\mathrm{v}_{\mathbf{u}} = \mathbf{x_u}^* \wedge \mathbf{m_h}$. From the convexity of g_i and h_i and the monotonicity of t-norm, there is

$$\begin{aligned} f(\mathbf{u}, \mathbf{v}) &= \mathcal{T}_{i=1}^N \left(g_i(u_i) \star' h_i(v_i)\right) \\ &\leqslant \mathcal{T}_{i=1}^N \left(g_i(u_i) \star' h_i(v_{\mathbf{u}i})\right) = f(\mathbf{u}, \mathbf{v}_{\mathbf{u}}). \end{aligned} \tag{40}$$

That is, for every $(\mathbf{u},\mathbf{v}) \in P_{w1}$ there exists $(\mathbf{u},\mathbf{v_u}) \in \mathfrak{X}_1$, such that $f(\mathbf{u},\mathbf{v}) \leqslant f(\mathbf{u},\mathbf{v_u})$. Thus

$$F(w) = \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in P_{w1}\} \leqslant \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in \mathfrak{X}_1\}. \quad (41)$$

Combined with (39) and (41), it can be shown that the conclusion (1) holds.

(2) Clearly, for every $\mathbf{u} \in \mathscr{U}_2$, (31) has a solution in P'_{o2}. From Theorem 3, there is $(\mathbf{u},\mathbf{x_u}^0 \vee \mathbf{n_h}) \in P_{w2}$, where $\mathbf{x_u}^0 \in \{\mathbf{x_u}_j^0, j = 1,\dots,|G_w|\}$; that is, $\mathfrak{X}_2 \subseteq P_{w2}$. Thus

$$\sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in \mathfrak{X}_2\} \leqslant \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in P_{w2}\} = F(w). \quad (42)$$

For the converse case, let $(\mathbf{u},\mathbf{v}) \in P_{w2}$. Then $\mathbf{u} \in \mathscr{U}_2$ and $\mathbf{v} \in \mathscr{V}_\mathbf{u}^2$. There exists $\mathbf{x_u^0} \in \{\mathbf{x_u}_i^0, i = 1,\dots,|G_w|\}$, such that $\mathbf{x_u^0} \vee \mathbf{n_h} \leqslant \mathbf{v}$. Denote $\mathbf{v_u} = \mathbf{x_u^0} \vee \mathbf{n_h}$. From the convexity of g_i and h_i and the monotonicity of t-norm, there is

$$f(\mathbf{u},\mathbf{v}) = \mathscr{T}_{i=1}^N\left(g_i(u_i) \star' h_i(v_i)\right) \leqslant \mathscr{T}_{i=1}^N\left(g_i(u_i) \star' h_i(v_{\mathbf{u}i})\right) = f(\mathbf{u},\mathbf{v_u}). \quad (43)$$

That is, for every $(\mathbf{u},\mathbf{v}) \in P_{w2}$, there exists $(\mathbf{u},\mathbf{v_u}) \in \mathfrak{X}_2$, such that $f(\mathbf{u},\mathbf{v}) \leqslant f(\mathbf{u},\mathbf{v_u})$. Thus

$$F(w) = \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in P_{w2}\} \leqslant \sup\{f(\mathbf{u},\mathbf{v}) \mid (\mathbf{u},\mathbf{v}) \in \mathfrak{X}_2\}. \quad (44)$$

Combined with (42) and (44), conclusion (2) holds. □

If $w \in [0,\alpha)$, from Theorem 5 it can be seen that all of the elements in $\mathfrak{X}_1$ can be obtained when all of the elements in $\mathscr{U}_1$ and the greatest solutions of the corresponding equation (31) in $[0,1]^N$ are obtained. The following lemma describes the characteristics of the elements in $\mathscr{U}_1$. Denote

$$J = \left\{i \in \{1,\dots,N\} \mid m_{g_i} \star m_{h_i} \geqslant w\right\},$$
$$\check{u}_j = \inf\left\{x \in \left[0, m_{g_j}\right] \mid \mathscr{L}_\star(x,w) \leqslant m_{h_j}\right\}, \quad j \in J. \quad (45)$$

Lemma 6. *Let $w \in [0,\alpha)$ and $\mathbf{u} \in P_{o1}$. Assume that $\star$ is continuous. Then (31) has a solution in P'_{o1} if and only if there exists $j \in J$ such that*

$$\left(0,\dots,0,\check{u}_j \vee w, 0,\dots,0\right) \leqslant \mathbf{u} \leqslant \mathbf{m_g}. \quad (46)$$

Proof. For the first, we will prove that $w \leqslant \check{u}_j \vee w \leqslant m_{g_j}$, $j \in J$. It can be seen that $\mathscr{L}_\star(m_{g_j},w) \leqslant m_{h_j}$ since $m_{g_j} \star m_{h_j} \geqslant w$ for every $j \in J$. Therefore, $m_{g_j} \in \{x \in [0,m_{g_j}] \mid \mathscr{L}_\star(x,w) \leqslant m_{h_j}\}$; that is, $\check{u}_j = \inf\{x \in [0,m_{g_j}] \mid \mathscr{L}_\star(x,w) \leqslant m_{h_j}\} \leqslant m_{g_j}$. Obviously $w \leqslant \check{u}_j \vee w \leqslant m_{g_j}$.

Let $\mathbf{u}$ satisfy (46). Since $w \leqslant \check{u}_j \vee w \leqslant u_j$, it can be inferred that (31) is solvable in $[0,1]^N$ and $\mathbf{x_u^0}$ is a minimal solution in $[0,1]^N$ from Lemma 1, where $\mathbf{x_u^0} = (0,\dots,0,\mathscr{L}_\star(u_j,w),0,\dots,0)$. Then we have $\mathscr{L}_\star(u_j,w) \leqslant \mathscr{L}_\star(\check{u}_j,w) \leqslant m_{h_j}$ since $\check{u}_j \leqslant \check{u}_j \vee w \leqslant u_j \leqslant m_{g_j}$. That is, $\mathbf{x_u^0} \leqslant \mathbf{m_h}$, which verifies that (31) has a solution in P'_{o1} by Corollary 4.

For the converse case, let $\mathbf{v} \in P'_{o1}$ be a solution of (31). Then $(u_1 \star v_1) \vee \cdots \vee (u_N \star v_N) = w$. There exists $j \in \{1,\dots,N\}$ such that $u_j \star v_j = w$. Thereby $m_{g_j} \geqslant u_j \geqslant w$, $m_{h_j} \geqslant v_j \geqslant w$ and $m_{g_j} \star m_{h_j} \geqslant u_j \star v_j = w$. Thus $j \in J$. It can be seen that equation $u_j \star x = w$ is solvable and its minimal solution is $\mathscr{L}_\star(u_j,w)$ by Lemma 1. Because v_j is also a solution, we have $\mathscr{L}_\star(u_j,w) \leqslant v_j \leqslant m_{h_j}$. So $u_j \in \{x \in [0,m_{g_j}] \mid \mathscr{L}_\star(x,w) \leqslant m_{h_j}\}$. Thereby, $\check{u}_j \leqslant u_j \leqslant m_{g_j}$. To sum up, $\check{u}_j \vee w \leqslant u_j \leqslant m_{g_j}$; that is, $\mathbf{u}$ satisfies (46). □

Now we will solve the formula (5) with the situation of $w \in [0,\alpha)$. For every $\mathbf{u} \in [0,1]^N$ denote the greatest solution of (31) in $[0,1]^N$ by $\mathbf{x_u^*}$ (if any), where $\mathbf{x_u^*} = (\mathscr{I}_\star(u_1,w),\dots,\mathscr{I}_\star(u_N,w))$. From Theorem 5 and Lemma 6 it can be inferred that

$$\mathfrak{X}_1 = \Big\{(\mathbf{u},\mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} = \left(\mathscr{I}_\star(u_1,w) \wedge m_{h_1},\dots,\mathscr{I}_\star(u_N,w) \wedge m_{h_N}\right), \left(0,\dots,0,\check{u}_j \vee w,0,\dots,0\right) \leqslant \mathbf{u} \leqslant \mathbf{m_g}, j \in J\Big\}. \quad (47)$$

Denote

$$\mathscr{H}_1 = \left\{g_1(u_1) \star' \cdots \star' g_N(u_N) \star' h_1(v_1) \star' \cdots \star' h_N(v_N) \mid (\mathbf{u},\mathbf{v}) \in \mathfrak{X}_1\right\}. \quad (48)$$

Obviously $\mathscr{H}_1$ can be viewed as a union of $|J|$ subsets, where the jth subset is as follows:

$$\mathscr{G}_j = \Big\{g_1(u_1) \star' \cdots \star' g_N(u_N) \star' h_1\left(I_\star(u_1,w) \wedge m_{h_1}\right) \star' \cdots \star' h_N\left(I_\star(u_N,w) \wedge m_{h_N}\right) \mid \left(0,\dots,0,\check{u}_j \vee w,0,\dots,0\right) \leqslant \mathbf{u} \leqslant \mathbf{m_g}\Big\}, \quad j \in J. \quad (49)$$

That is, $\mathscr{H}_1 = \bigcup_{j\in J}\mathscr{G}_j$. Notice that, for any $j_1, j_2 \in J$ and $j_1 \neq j_2$, it may appear that $\mathscr{G}_{j_1} \cap \mathscr{G}_{j_2} \neq \emptyset$. However, it will not affect our final results. The following theorem provides a method to obtain $F(w)$ when $w \in [0,\alpha)$.

Theorem 7. *Let $w \in [0,\alpha)$. Assume that $\star$ is continuous. Denote*

$$F_i^1 = \sup\left\{g_i(u_i) \star' h_i\left(\mathscr{I}_\star(u_i,w) \wedge m_{h_i}\right) \mid u_i \in \left[0,m_{g_i}\right]\right\}, \quad i \in \{1,\dots,N\},$$

$$\check{F}_j^1 = \sup \left\{ g_j\left(u_j\right) \star' h_j\left(\mathcal{I}_\star\left(u_j, w\right) \wedge m_{h_j}\right) \mid u_j \in \left[\check{u}_j \vee w, m_{g_j}\right] \right\} \quad j \in J. \tag{50}$$

Then the following items hold:

(1) $\sup \mathcal{G}_j = F_1^1 \star' F_{j-1}^1 \star' \check{F}_j^1 \star' F_{j+1}^1 \star' \cdots \star' F_N^1$, $j \in J$,

(2) $F(w) = \sup\{\sup \mathcal{G}_j, j \in J\}$.

Proof. (1) Without loss of the generality, we prove the case of $\mathcal{G}_1$. Clearly, $g_i(u_i) \star' h_i(\mathcal{I}_\star(u_i, w) \wedge m_{h_i})$ is bounded in $[0, m_{g_i}]$. Then there exists $u_i' \in [0, m_{g_i}]$ such that $g_i(u_i) \star' h_i(\mathcal{I}_\star(u_i, w) \wedge m_{h_i})$ reaches the maximum F_i^1, $i \in \{1, \ldots, N\}$. Similarly, there exists $u_j'' \in [\check{u}_j \vee w, m_{g_j}]$ such that $g_j(u_j) \star' h_j(\mathcal{I}_\star(u_j, w) \wedge m_{h_j})$ reaches the maximum $\check{F}_j^1$, $j \in J$. From Lemma 6, it is easy to see that

$$\left(u_1'', u_2', \ldots, u_N', \mathcal{I}_\star\left(u_1'', w\right) \wedge m_{h_1}, \mathcal{I}_\star\left(u_2', w\right) \wedge m_{h_2}, \ldots, \mathcal{I}_\star\left(u_N', w\right) \wedge m_{h_N}\right) \in \mathfrak{X}_1. \tag{51}$$

Let $\mathbf{u}$ satisfy (46). Then there are

$$\begin{aligned} & g_1\left(u_1\right) \star' h_1\left(\mathcal{I}_\star\left(u_1, w\right) \wedge m_{h_1}\right) \\ & \quad \leqslant g_1\left(u_1''\right) \star' h_1\left(\mathcal{I}_\star\left(u_1'', w\right) \wedge m_{h_1}\right) = \check{F}_1^1, \\ & g_i\left(u_i\right) \star' h_i\left(\mathcal{I}_\star\left(u_i, w\right) \wedge m_{h_i}\right) \\ & \quad \leqslant g_i\left(u_i'\right) \star' h_i\left(\mathcal{I}_\star\left(u_i', w\right) \wedge m_{h_i}\right) = F_i^1, \\ & \qquad i = 2, \ldots, N. \end{aligned} \tag{52}$$

Thus

$$\begin{aligned} & g_1\left(u_1\right) \star' g_2\left(u_2\right) \star' \cdots \star' g_N\left(u_N\right) \\ & \quad \star' h_1\left(\mathcal{L}_\star\left(u_1, w\right) \wedge m_{h_1}\right) \\ & \quad \star' h_2\left(\mathcal{L}_\star\left(u_2, w\right) \wedge m_{h_2}\right) \\ & \quad \star' \cdots \star' h_N\left(\mathcal{L}_\star\left(u_N, w\right) \wedge m_{h_N}\right) \\ & \leqslant g_1\left(u_1''\right) \star' g_2\left(u_2'\right) \star' \cdots \star' g_N\left(u_N'\right) \\ & \quad \star' h_1\left(\mathcal{I}_\star\left(u_1'', w\right) \wedge m_{h_1}\right) \\ & \quad \star' h_2\left(\mathcal{I}_\star\left(u_2', w\right) \wedge m_{h_2}\right) \\ & \quad \star' \cdots \star' h_N\left(\mathcal{I}_\star\left(u_N', w\right) \wedge m_{h_N}\right) \\ & = \check{F}_1^1 \star' F_2^1 \star' \cdots \star' F_N^1. \end{aligned} \tag{53}$$

Therefore, $\sup \mathcal{G}_1 = \check{F}_1^1 \star' F_2^1 \star' \cdots \star' F_N^1$.

(2) Clearly, $F(w) = \sup\{\sup \mathcal{G}_j \mid j \in J\}$ since $\mathcal{H}_1 = \bigcup_{j \in J} \mathcal{G}_j$ and $F(w) = \sup \mathcal{H}_1$. □

If $w \in (\beta, 1]$, then by Theorem 5 all elements in $\mathfrak{X}_2$ can be obtained when all of the elements in $\mathcal{U}_2$ and minimal solutions of the corresponding equation (31) in $[0,1]^N$ are obtained. The following lemma describes the characteristics of the elements in $\mathcal{U}_2$. Denote

$$\widehat{u}_i = \sup\left\{x \in \left[n_{g_i}, 1\right] \mid \mathcal{I}_\star(x, w) \geqslant n_{h_i}\right\}, \quad i = 1, \ldots, N. \tag{54}$$

Lemma 8. *Let $w \in [\beta, 1]$ and $\mathbf{u} \in P_{o2}$. Assume that $\star$ is continuous. Then* (31) *has a solution in P_{o2}' and if and only if there exists $i \in \{1, \ldots, N\}$ such that*

$$\begin{aligned} & \left(n_{g_1}, \ldots, n_{g_{i-1}}, n_{g_i} \vee w, n_{g_{i+1}}, \ldots, n_{g_N}\right) \\ & \quad \leqslant \mathbf{u} \leqslant \left(\widehat{u}_1, \ldots, \widehat{u}_{i-1}, \widehat{u}_i, \widehat{u}_{i+1}, \ldots, \widehat{u}_N\right). \end{aligned} \tag{55}$$

Proof. For the first, we will give the proof of $w \leqslant n_{g_i} \vee w \leqslant \widehat{u}_i$ for every $i \in \{1, \ldots, N\}$. Note that $n_{g_i} \star n_{h_i} \leqslant w$ since $\bigvee_{i=1}^N (n_{g_i} \star n_{h_i}) \leqslant w$. Then $\mathcal{I}_\star(n_{g_i}, w) \geqslant n_{h_i}$. Therefore, $\mathcal{I}_\star(n_{g_i} \vee w, w) \geqslant n_{h_i}$ since $\mathcal{I}_\star(w, w) = 1 \geqslant n_{h_i}$. We obtain that $n_{g_i} \vee w \in \{x \in [n_{g_i}, 1] \mid \mathcal{I}_\star(x, w) \geqslant n_{h_i}\}$, which indicates that $\widehat{u}_i$ always exists and $\widehat{u}_i \geqslant n_{g_i} \vee w$. Obviously $w \leqslant n_{g_i} \vee w \leqslant \widehat{u}_i$.

Let $\mathbf{u}$ satisfy (55). Because $w \leqslant n_{g_i} \vee w \leqslant u_i$, from Lemma 1 it can be inferred that (31) has a solution in $[0,1]^N$ and the greatest solution is

$$\mathbf{x}_{\mathbf{u}}^* = \left(\mathcal{I}_\star\left(u_1, w\right), \ldots, \mathcal{I}_\star\left(u_i, w\right), \ldots, \mathcal{I}_\star\left(u_N, w\right)\right). \tag{56}$$

Furthermore, for every $j \in \{1, \ldots, N\}$ we have $\mathcal{I}_\star(u_j, w) \geqslant \mathcal{I}_\star(\widehat{u}_j, w) \geqslant n_{h_j}$ since $\mathbf{u} \leqslant (\widehat{u}_1, \ldots, \widehat{u}_N)$. Then $\mathbf{x}_{\mathbf{u}}^* \geqslant \mathbf{n}_{\mathbf{h}}$, which indicates that (31) has a solution in P_{o2}' by Corollary 4.

For the conversion, assume that (31) has a solution in P_{o2}'. There exists $i_0 \in \{1, \ldots, N\}$ such that $u_{i_0} \geqslant w$. Then $u_{i_0} \geqslant n_{g_{i_0}} \vee w$ since $u_{i_0} \geqslant n_{g_{i_0}}$. That is to say, $\mathbf{u} \geqslant (n_{g_1}, \ldots, n_{g_{i_0-1}}, n_{g_{i_0}} \vee w, n_{g_{i_0+1}}, \ldots, n_{g_N})$. On the other hand, there is $\mathbf{x}_{\mathbf{u}}^* \geqslant \mathbf{n}_{\mathbf{h}}$ by Corollary 4. Thus for every $j \in \{1, \ldots, N\}$, we have $u_j \in \{x \in [n_{g_j}, 1] \mid \mathcal{I}_\star(x, w) \geqslant n_{h_j}\}$, which indicates that $u_j \leqslant \widehat{u}_j$. Therefore, $\mathbf{u} \leqslant (\widehat{u}_1, \ldots, \widehat{u}_N)$. □

Next we will solve the formula (5) with the situation of $w \in (\beta, 1]$. For every $\mathbf{u} \in [0,1]^N$, denote minimal solutions of (31) in $[0,1]^N$ by ${\mathbf{x}_{\mathbf{u}}}_j^0$, $j \in \{1, \ldots, |G_w|\}$ (if any). From Theorem 5 and Lemma 8, it can be seen that

$$\begin{aligned} \mathfrak{X}_2 = \Big\{ & (\mathbf{u}, \mathbf{v}) \in [0,1]^{2N} \mid \mathbf{v} \in \left\{ {\mathbf{x}_{\mathbf{u}}}_j^0 \vee \mathbf{n}_{\mathbf{h}}, j = 1, \ldots, \left|G_w\right| \right\}, \\ & \left(n_{g_1}, \ldots, n_{g_{i-1}}, n_{g_i} \vee w, n_{g_{i+1}}, \ldots, n_{g_N}\right) \\ & \leqslant \mathbf{u} \leqslant \left(\widehat{u}_1, \ldots, \widehat{u}_{i-1}, \widehat{u}_i, \widehat{u}_{i+1}, \ldots, \widehat{u}_N\right), \\ & i \in \{1, \ldots, N\} \Big\}. \end{aligned} \tag{57}$$

For every $\mathbf{u}$ satisfying (55) there must exist ${\mathbf{x}_{\mathbf{u}}}^0 \in \{{\mathbf{x}_{\mathbf{u}}}_j^0, j = 1, \ldots, |G_w|\}$ such that it has the following form:

$${\mathbf{x}_{\mathbf{u}}}^0 = \left(0, \ldots, 0, \mathcal{L}_\star\left(u_i, w\right), 0, 0, \ldots, 0\right). \tag{58}$$

Then

$$\mathbf{v} = \mathbf{x_u}^0 \vee \mathbf{n_h} = \left(n_{h_1}, \ldots, n_{h_{i-1}}, \mathscr{L}_\star (u_i, w) \vee n_{h_i}, n_{h_{i+1}}, \ldots, n_{h_N}\right). \tag{59}$$

Denote

$$\mathscr{H}_2 = \left\{g_1(u_1) \star' \cdots \star' g_N(u_N) \star' h_1(v_1) \star' \cdots \star' h_N(v_N) \mid (\mathbf{u}, \mathbf{v}) \in \mathfrak{X}_2\right\}, \tag{60}$$

$$\delta_i = h_1\left(n_{h_1}\right) \star' \cdots \star' h_{i-1}\left(n_{h_{i-1}}\right) \star' h_{i+1}\left(n_{h_{i+1}}\right) \star' \cdots \star' h_N\left(n_{h_N}\right), \quad i \in \{1, \ldots, N\}. \tag{61}$$

Thus $\mathscr{H}_2$ can be viewed as a union of N subsets, where the ith subset is as follows:

$$\begin{aligned}\mathscr{F}_i = \{&g_1(u_1) \star' \cdots \star' g_i(u_i) \star' \cdots \star' g_N(u_N) \\ &\star' h_i\left(\mathscr{L}_\star(u_i, w) \vee n_{h_i}\right) \\ &\star' \delta_i \mid \left(n_{g_1}, \ldots, n_{g_{i-1}}, n_{g_i} \vee w, n_{g_{i+1}}, \ldots, n_{g_N}\right) \\ &\leqslant \mathbf{u} \leqslant (\widehat{u}_1, \ldots, \widehat{u}_{i-1}, \widehat{u}_i, \widehat{u}_{i+1}, \ldots, \widehat{u}_N)\} \\ &i \in \{1, 2, \ldots, N\},\end{aligned} \tag{62}$$

that is, $\mathscr{H}_2 = \bigcup_{i=1}^N \mathscr{F}_i$. In fact, it is natural that $\mathscr{H}_2 \supseteq \bigcup_{i=1}^N \mathscr{F}_i$ since $\mathscr{H}_2 \supseteq \mathscr{F}_i$, $i = 1, \ldots, N$. For the converse, take a $g_1(u_1) \star' \cdots \star' g_N(u_N) \star' h_1(v_1) \star' \cdots \star' h_N(v_N) \in \mathscr{H}_2$. From the form of the elements in $\mathfrak{X}_2$, there exists $i_0 \in \{1, \ldots, N\}$ such that

$$\mathbf{v} = \left(n_{h_1}, \ldots, n_{h_{i_0-1}}, \mathscr{L}_\star\left(u_{i_0}, w\right) \vee n_{h_{i_0}}, n_{h_{i_0+1}}, \ldots, n_{h_N}\right), \tag{63}$$

which indicates that $u_{i_0} \geqslant w$ by Lemma 1. Clearly $u_{i_0} \geqslant n_{g_{i_0}}$. Therefore,

$$\left(n_{g_1}, \ldots, n_{g_{i_0-1}}, n_{g_{i_0}} \vee w, n_{g_{i_0+1}}, \ldots, n_{g_N}\right) \leqslant \mathbf{u} \leqslant \left(\widehat{u}_1, \ldots, \widehat{u}_{i_0-1}, \widehat{u}_{i_0}, \widehat{u}_{i_0+1}, \ldots, \widehat{u}_N\right). \tag{64}$$

From the above, we have $g_1(u_1) \star' \cdots \star' g_N(u_N) \star' h_1(v_1) \star' \cdots \star' h_N(v_N) \in \mathscr{F}_{i_0}$; that is, $g_1(u_1) \star' \cdots \star' g_N(u_N) \star' h_1(v_1) \star' \cdots \star' h_N(v_N) \in \bigcup_{i=1}^N \mathscr{F}_i$. Thus $\mathscr{H}_2 \subseteq \bigcup_{i=1}^N \mathscr{F}_i$. To sum up, we obtain that $\mathscr{H}_2 = \bigcup_{i=1}^N \mathscr{F}_i$.

Notice that, for any $i_1, i_2 \in \{1, 2, \ldots, N\}$ and $i_1 \neq i_2$, it may appear that $\mathscr{F}_{i_1} \cap \mathscr{F}_{i_2} \neq \emptyset$. However, it will not affect our final results. The following theorem provides a method to obtain $F(w)$ when $w \in (\beta, 1]$.

Theorem 9. *Let $w \in (\beta, 1]$. Assume that $\star$ is continuous. Then the following items hold:*

(1) $\sup \mathscr{F}_i = \sup\{g_i(u_i) \star' h_i(\mathscr{L}_\star(u_i, w) \vee n_{h_i}) \mid u_i \in [n_{g_i} \vee w, \widehat{u}_i]\}, i = 1, \ldots, N,$

(2) $F(w) = \sup\{\sup \mathscr{F}_i, i = 1, \ldots, N\}$.

Proof. (1) Similar to the proof of Theorem 7, it can be obtained that there exists $u_i' \in [n_{g_i} \vee w, \widehat{u}_i]$ such that

$$\begin{aligned} g_i\left(u_i'\right) \star' h_i\left(\mathscr{L}_\star\left(u_i', w\right) \vee n_{h_i}\right) &= \sup\left\{g_i(u_i) \star' h_i\left(n_{h_1} \mathscr{L}_\star(u_i, w) \vee n_{h_1}\right) \mid u_i \in \left[n_{g_i} \vee w, \widehat{u}_i\right]\right\}, \\ &\qquad i \in \{1, \ldots, N\}. \end{aligned} \tag{65}$$

Without loss of the generality, we prove the situation of $\mathscr{F}_1$. From Lemma 8, it is easy to see that

$$\left(u_1', n_{g_2}, \ldots, n_{g_N}, \mathscr{L}_\star\left(u_1', w\right) \vee n_{h_1}, n_{h_2}, \ldots, n_{h_N}\right) \in \mathfrak{X}_2. \tag{66}$$

Denote

$$\begin{aligned}\delta_1 &= h_2\left(n_{h_2}\right) \star' \cdots \star' h_N\left(n_{h_N}\right), \\ \sigma_1 &= g_2\left(n_{g_2}\right) \star' \cdots \star' g_N\left(n_{g_N}\right).\end{aligned} \tag{67}$$

It is easy to see $\sigma_1 = \delta_1 = 1$. Here we shall prove that $g_1(u_1') \star' h_1(\mathscr{L}_\star(u_1', w) \vee n_{h_1}) \star \delta_1 \star \sigma_1$ is the greatest element in $\mathscr{F}_1$. Take $g_1(u_1) \star' g_2(u_2) \star' \cdots \star' g_N(u_N) \star' h_1(\mathscr{L}_\star(u_1, w) \vee n_{h_1}) \star \delta_1 \in \mathscr{F}_1$. By the convexity of g_i, we have

$$g_2(u_2) \star' \cdots \star' g_N(u_N) \leqslant g_2\left(n_{g_2}\right) \star' \cdots \star' g_N\left(n_{g_N}\right). \tag{68}$$

Moreover, from the assumptions, there is

$$\begin{aligned} g_1(u_1) \star' h_1(\mathscr{L}_\star(u_1, w)) \leqslant g_1\left(u_1'\right) \star' h_1\left(\mathscr{L}_\star\left(u_1', w\right) \vee n_{h_1}\right). \end{aligned} \tag{69}$$

Thus

$$\begin{aligned} &g_1(u_1) \star' g_2(u_2) \star' \cdots \star' g_N(u_N) \star' h_1\left(\mathscr{L}_\star(u_1, w) \vee n_{h_1}\right) \star' \delta_1 \\ &\leqslant g_1\left(u_1'\right) \star' g_2\left(n_{g_2}\right) \star' \cdots \star' g_N\left(n_{g_N}\right) \star' h_1\left(\mathscr{L}_\star\left(u_1', w\right) \vee n_{h_1}\right) \star' \delta_1. \end{aligned} \tag{70}$$

It can be obtained that

$$\begin{aligned} \sup \mathscr{F}_1 &= g_1\left(u_1'\right) \star' g_1\left(n_{g_2}\right) \star' \cdots \star' \cdots \star' g_N\left(n_{g_N}\right) \star' h_1\left(\mathscr{L}_\star\left(u_1', w\right) \vee n_{h_1}\right) \star' \delta_1 \\ &= g_1\left(u_1'\right) \star' h_1\left(\mathscr{L}_\star\left(u_1', w\right)\right) \star' \delta_1 \star' \sigma_1 \\ &= g_1\left(u_1'\right) \star' h_1\left(\mathscr{L}_\star\left(u_1', w\right)\right). \end{aligned} \tag{71}$$

(2) It is clear that $\sup \mathscr{H}_2 = \sup\{\sup \mathscr{F}_i, i = 1, \ldots, N\}$ since $\mathscr{H}_2 = \bigcup_v^N \mathscr{F}_i$ and $F(w) = \sup \mathscr{H}_2$. □

Up to now, we can get all the values of expression (5). That is to say, for every fixed $x \in X$ and $y \in Y$, we can calculate the function $\mu_{\widetilde{R}(x,y)}(w)$ when $\star$ is continuous and $\mu_{\widetilde{A}(x)}$ and $\mu_{\widetilde{B}(y)}$ are both convex and normal. On the basis of Theorems 2, 7, and 9, we will give the implementation procedures in the following.

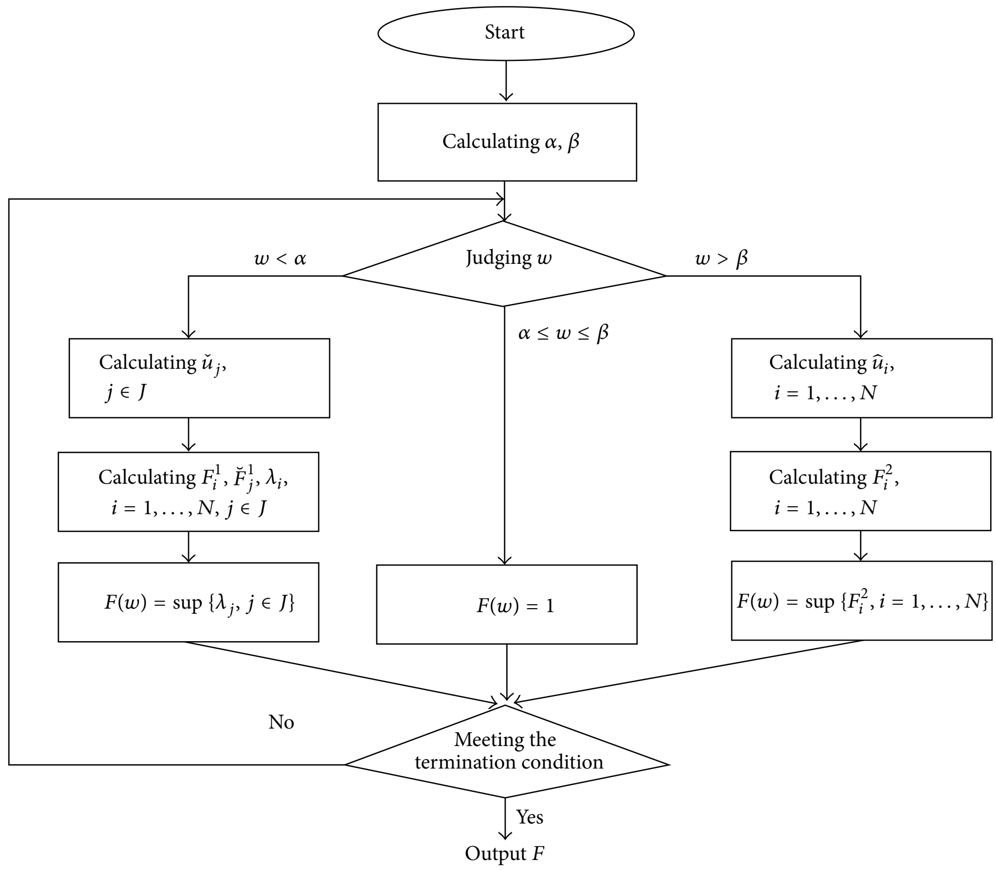

Figure 1: The structure of Algorithm 10.

Algorithm 10. Consider the following (Figure 1):
Step 1. Calculate $\alpha = \bigvee_{i=1}^{N}(m_{g_i} \star m_{h_i})$, $\beta = \bigvee_{i=1}^{N}(n_{g_i} \star n_{h_i})$. For every $w \in [0,1]$ employ Step 2~Step 4.
Step 2. When $w \in [0,\alpha)$, step size Δ and calculate the variables

$$J = \left\{ i \in \{1,\ldots,N\} \mid m_{g_i} \star m_{h_i} \geqslant w \right\},$$
$$\check{u}_j = \inf\left\{ x \in \left[0, m_{g_j}\right] \mid \mathscr{L}_\star(x,w) \leqslant m_{h_j} \right\}, \quad j \in J. \tag{72}$$

Find the greatest value of $g_i(u_i) \star' h_i(\mathscr{I}_\star(u_i,w) \wedge m_{h_i})$ in $[0, m_{g_i}]$, denoted by F_i^1, $i \in \{1,\ldots,N\}$; find the greatest value of $g_j(u_j) \star' h_j(\mathscr{I}_\star(u_j,w) \wedge m_{h_j})$ in $[\check{u}_j \vee w, m_{g_j}]$, denoted by $\check{F}_j^1$, $j \in J$. Let

$$\lambda_j = F_1^1 \star' F_{j-1}^1 \star' \check{F}_j^1 \star' F_{j+1}^1 \star' \cdots \star' F_N^1, \quad j \in J. \tag{73}$$

Then $F(w) = \sup\{\lambda_j, j \in J\}$.
Step 3. When $w \in [\alpha,\beta]$, $F(w) = 1$.
Step 4. When $w \in (\beta, 1]$, step size Δ and calculate the variables

$$\hat{u}_i = \sup\left\{ x \in \left[n_{g_i}, 1\right] \mid \mathscr{I}_\star(x,w) \geqslant n_{h_i} \right\}, \quad i = 1,\ldots,N. \tag{74}$$

For every $i \in \{1,\ldots,N\}$, find the greatest value of $g_i(u_i) \star' h_i(\mathscr{L}_\star(u_i,w) \vee n_{h_i})$ in $[n_{g_i} \vee w, \hat{u}_i]$, denoted by F_i^2, $i \in \{1,\ldots,N\}$. Then $F(w) = \sup\{F_i^2, i = 1,\ldots,N\}$.

Remark 11. The above algorithm can be applied in calculating extended continuous t-norm based on arbitrary t-norm on two type-2 fuzzy sets once setting $N = 1$ and extended maximum based on arbitrary t-norm on N type-2 fuzzy sets once setting $h_i(u) = 1$, $i = 1,\ldots,N$. Hence the type-2 fuzzy reasoning relations of type-2 fuzzy logic systems with multiple input and single output can be calculated.

Remark 12. It can be seen from the operation steps above that the presented method to calculate the formula (5) is much simpler than the native algorithm (i.e., finding the maximum of $f(\mathbf{u},\mathbf{v})$ from all of the combination $(\mathbf{u},\mathbf{v})$ in P_w (or P_{w1} and P_{w2})) which is a huge operation process undoubtedly. Take w from $[0,1]$ with step size Δ_0. Then the amount of computation is no more than

$$4N + \frac{\alpha}{\Delta_0} \cdot \left(\sum_{i=1}^{N} \frac{4m_{g_i}}{\Delta} + N^2 \right) + \frac{1-\beta}{\Delta_0} \cdot \left(\sum_{i=1}^{N} \frac{(1-\hat{u}_i) + 3\left(\hat{u}_i - n_{g_i} \vee w\right)}{\Delta} + N \right), \tag{75}$$

where $\mathcal{I}_\star(a,b)$, $\mathcal{L}_\star(a,b)$, $\star$ or $\star'$ is considered one computation and the step size Δ is small enough. According to above analysis, we can draw the following conclusions: the computation amount level of the proposed algorithm is the same as that of polynomials.

4. Examples

In this section some concrete examples for the construction of type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems on the proposed algorithm will be given. All of them are realized by using MATLAB2010 (b).

Example 1. Let input domain $X = \{x\}$ and output domain $Y = \{y\}$. Then each type-2 fuzzy reasoning relation $\widetilde{R}_i$ $(i \in \{1,\ldots,N\})$ and the total type-2 fuzzy reasoning relation $\widetilde{R}$ are only defined on $X \times Y = \{(x,y)\}$. In the group of type-2 fuzzy reasonings (3) we choose $\widetilde{A}_i$, $\widetilde{B}_i$, $i = 1,\ldots,7$ as follows:

$$\begin{aligned}
\mu_{\widetilde{A}_1(x)}(u) &= \exp\left\{-\frac{(u-0.3)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{A}_2(x)}(u) &= \exp\left\{-\frac{(u-0.34)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{A}_3(x)}(u) &= \exp\left\{-\frac{(u-0.36)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{A}_4(x)}(u) &= \exp\left\{-\frac{(u-0.4)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{A}_5(x)}(u) &= \exp\left\{-\frac{(u-0.5)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{A}_6(x)}(u) &= \exp\left\{-\frac{(u-0.55)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{A}_7(x)}(u) &= \exp\left\{-\frac{(u-0.6)^2}{2\times 0.2^2}\right\},\\
\mu_{\widetilde{B}_1(y)}(v) &= \exp\left\{-\frac{(v-0.5)^2}{2\times 0.1^2}\right\},\\
\mu_{\widetilde{B}_2(y)}(v) &= \exp\left\{-\frac{(v-0.55)^2}{2\times 0.1^2}\right\},\\
\mu_{\widetilde{B}_3(y)}(v) &= \exp\left\{-\frac{(v-0.6)^2}{2\times 0.1^2}\right\},\\
\mu_{\widetilde{B}_4(y)}(v) &= \exp\left\{-\frac{(v-0.65)^2}{2\times 0.1^2}\right\},\\
\mu_{\widetilde{B}_5(y)}(v) &= \exp\left\{-\frac{(v-0.68)^2}{2\times 0.1^2}\right\},\\
\mu_{\widetilde{B}_6(y)}(v) &= \exp\left\{-\frac{(v-0.7)^2}{2\times 0.1^2}\right\},\\
\mu_{\widetilde{B}_7(y)}(v) &= \exp\left\{-\frac{(v-75)^2}{2\times 0.1^2}\right\}.
\end{aligned} \tag{76}$$

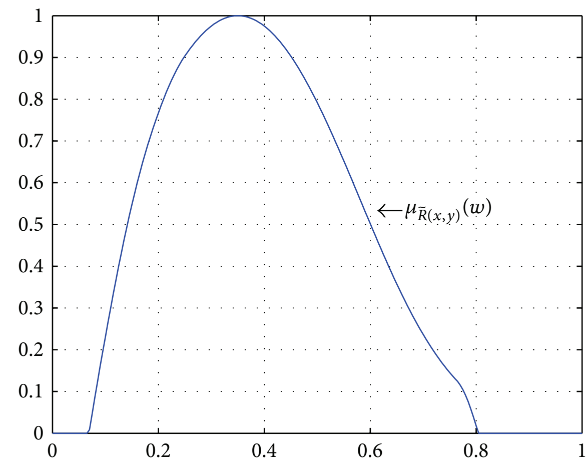

FIGURE 2: The function graph of $\mu_{\widetilde{R}(x,y)}(w)$ in (77).

Choose $\sqcup^{(\vee,\star')} = \sqcup^{(\vee,\oplus)}$ and $\sqcap^{(\star,\star')} = \sqcap^{(\oplus,\oplus)}$, where $a \oplus b = 0 \vee (a+b-1)$, $\widetilde{A}_i$, and $\widetilde{B}_i$ as stated in Example 1, $i = 1,\ldots,7$. Then expression (5) is reduced as

$$\begin{aligned}
&\mu_{\widetilde{R}(x,y)}(w)\\
&= \sup_{\bigvee_{i=1}^{7}(0\vee(u_i+v_i-1))=w}\left(\bigoplus_{i=1}^{7}\left(0\vee\left(\mu_{\widetilde{A}_i(x)}(u_i)\right.\right.\right.\\
&\qquad\left.\left.\left.+\mu_{\widetilde{B}_i(y)}(v_i)-1\right)\right)\right),
\end{aligned} \tag{77}$$

where $\bigoplus$ and $\oplus$ indicate the same t-norm. Here we shall calculate (77) by using our method. Clearly $\mathcal{I}_\oplus(a,b) = \mathcal{L}_\oplus(a,b) = 1 \wedge (b-a+1)$ and $\alpha = \beta = 0.35$. The function graph of $\mu_{\widetilde{R}(x,y)}(w)$ in (77) is shown in Figure 2.

Example 2. Choose $\sqcup^{(\vee,\star')} = \sqcup^{(\vee,\odot)}$ and $\sqcap^{(\star,\star')} = \sqcap^{(\oplus,\odot)}$, where

$$a \odot b = \begin{cases} a \wedge b, & a \vee b = 1,\\ 0, & a \vee b < 1.\end{cases} \tag{78}$$

Let $\widetilde{A}_i$ and $\widetilde{B}_i$ be the same as stated in Example 1, $i = 1,\ldots,7$. Then expression (5) is reduced as

$$\begin{aligned}
&\mu_{\widetilde{R}(x,y)}(w)\\
&= \sup_{\bigvee_{i=1}^{7}(0\vee(u_i+v_i-1))=w}\left(\bigodot_{i=1}^{7}\left(\mu_{\widetilde{A}_i(x)}(u_i)\odot\mu_{\widetilde{B}_i(y)}(v_i)\right)\right),
\end{aligned} \tag{79}$$

where $\bigodot$ and $\odot$ indicate the same t-norm. Here we will calculate (79) by using our method. Clearly $\alpha = \beta = 0.35$. The function graph of $\mu_{\widetilde{R}(x,y)}(w)$ in (79) is shown in Figure 3.

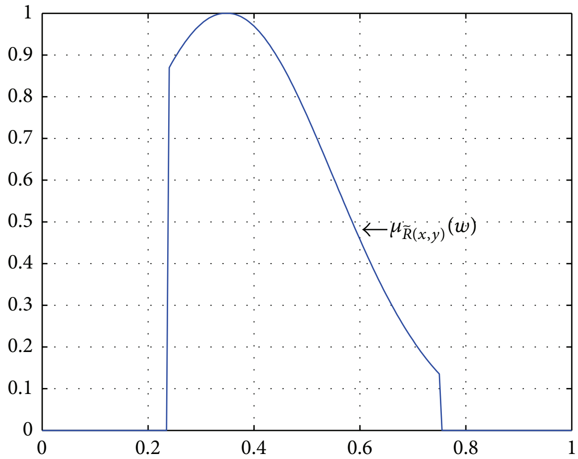

Figure 3: The function graph of $\mu_{\widetilde{R}(x,y)}(w)$ in (79).

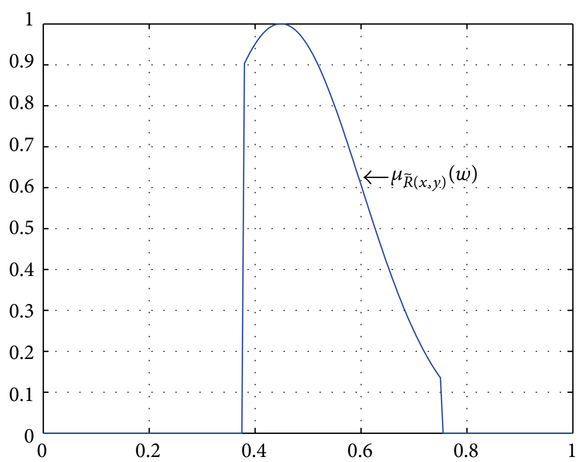

Figure 4: The function graph of $\mu_{\widetilde{R}(x,y)}(w)$ in (80).

Example 3. Choose $\sqcup^{(\vee,\star')} = \sqcup^{(\vee,\odot)}$ and $\sqcap^{(\star,\star')} = \sqcap^{(\cdot,\odot)}$, where $a \cdot b = ab$. Let $\widetilde{A}_i$ and $\widetilde{B}_i$ be the same as stated in Example 1, $i = 1, \ldots, 7$. Then expression (5) is reduced as

$$\mu_{\widetilde{R}(x,y)}(w) = \sup_{\bigvee_{i=1}^{7}(u_i v_i)=w} \left(\bigodot_{i=1}^{7} \left(\mu_{\widetilde{A}_i(x)}(u_i) \odot \mu_{\widetilde{B}_i(y)}(v_i) \right) \right). \quad (80)$$

Here we will calculate (80) by using our method. Clearly $\mathscr{I}_{\cdot}(a,b) = \mathscr{L}_{\cdot}(a,b) = 1 \wedge (b/a)$ and $\alpha = \beta = 0.45$. The function graph of $\mu_{\widetilde{R}(x,y)}(w)$ in (80) is shown in Figure 4.

5. Conclusions

In this paper, an algorithm for constructing type-2 fuzzy reasoning relations of SISO type-2 fuzzy logic systems has been given under certain conditions. The results may serve to establish many new type-2 fuzzy logic systems by using different extended t-(co)norms. An important conclusion has been given that the results of extended continuous t-(co)norms based on arbitrary t-norm keep the convexity and normality, which guarantees the operation conditions of extended t-(co)norms for the next turn. It can be seen that the proposed algorithm deals with the antecedents and consequents of the group of type-2 fuzzy reasoning in an integral way and the computation amount level of the proposed algorithm is the same as that of polynomials, which indicates that the proposed algorithm may be well applied in type-2 fuzzy logic systems. Besides, it can be seen that the calculations of an extended continuous t-norm based on arbitrary t-norms can be obtained as the special case of the proposed algorithm, which is a new idea to calculate the membership functions of a class of extended t-norm. However, all the fuzzy truth values of type-2 fuzzy sets that participated in the calculation are required to be convex and normal. Obviously, by using our proposed algorithm more applications about noninterval type-2 fuzzy logic system and type-2 fuzzy neural network could be attemptable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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