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Research Article

A New Upper Bound for $\|A^{-1}\|$ of a Strictly α-Diagonally Dominant M-Matrix

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A new upper bound for $\|A^{-1}\|$ of a real strictly diagonally dominant M-matrix A is present, and a new lower bound of the smallest eigenvalue $\lambda_{\min}(A)$ of A is given, which improved the results in the literature. Furthermore, an upper bound for $\|A^{-1}\|$ of a real strictly α-diagonally dominant M-matrix is shown.

1. Introduction

The estimation for the bound for the norm $\|A^{-1}\|$ of a real invertible $n \times n$ matrix A is important in numerical analysis, so many researchers were devoted to studying this kind of problems. For example, Varah [1] discussed the bound for the infinity norm $\|A^{-1}\|_\infty$ of a strictly diagonally dominant matrix $A = (a_{ij})_{n\times n} \in R^{n\times n}$ and obtained the following estimation:

$$\left\|A^{-1}\right\|_\infty \le \max_i \left\{\frac{1}{|a_{ii}| - \sum_{j=1, j\ne i} |a_{ij}|}\right\}, \quad i \in N. \tag{1}$$

After that Varga [2] extended the result of [1] to H-matrices. Evidently, the upper bound for $\|A^{-1}\|_\infty$ in (1) only involves the entries in the matrix A. If the diagonal dominance of A is weak, that is, $\min\{|a_{ii}| - \sum_{j=1, j\ne i} |a_{ij}|\}$ is small, then the bound given by (1) may be large. For this reason, some authors were devoted to improving the result of (1). Recently, Cheng and Huang [3] presented a more compacted upper bound for a strictly diagonally dominant M-matrix

$$\begin{aligned}\left\|A^{-1}\right\|_\infty \le{} & \frac{1}{a_{11}(1 - u_1 d_1)} \\ & + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1 - u_i d_i)} \prod_{j=1}^{i-1} \left(1 + \frac{u_j}{1 - u_j l_j}\right)\right],\end{aligned} \tag{2}$$

and then Wang [4] further improved this bound and gave the following result:

$$\begin{aligned}\left\|A^{-1}\right\|_\infty \le{} & \frac{1}{a_{11}(1 - u_1 d_1)} \\ & + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1 - u_i d_i)} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j}\right],\end{aligned} \tag{3}$$

where notations in (2) and (3) have the same meanings as those used in this paper, which will be shown later.

In this paper, we present a new upper bound $\|A^{-1}\|_\infty$ of a strictly diagonally dominant matrix $A = (a_{ij})_{n\times n} \in R^{n\times n}$, which is better than that obtained by Wang, and a new lower bound of the smallest eigenvalue $q(A)$ of A is also obtained. In addition, an upper bound for $\|A^{-1}\|_\infty$ of a strictly α-diagonal dominant matrix is presented. To our knowledge, little has been done for upper bound of strictly α-diagonal dominant matrices. Further, examples are given to illustrate the performance of our results.

Next, we introduce some notations and definitions. As usual, let I be an identity matrix of order n. If there exists an $n \times n$ nonnegative matrix B and a real number a such that $A = aI - B$ with $a > \rho(B)$, then A is called a nonsingular M-matrix, where $\rho(B)$ is the spectral radius of the nonnegative matrix B. It is well known that the inverse matrix A^{-1} of a M-matrix A is nonnegative and, therefore, $1/\rho(A^{-1})$ is a positive eigenvalue of A related to the Perron eigenvalue of the nonnegative matrix A^{-1}. If $q(A)$ denotes the minimum of the real parts of the eigenvalues of A, that is, $q(A) = a - \rho(B)$, then $q(A) = 1/\rho(A^{-1})$. For further properties of the M-matrix A, we refer the readers to [5–7].

An $n \times n$ matrix $A = (a_{ij})$ is called a strictly diagonally dominant matrix if $|a_{ii}| > \sum_{j=1, j\neq i} |a_{ij}|$ for $i \in N$. Let

$$R_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad r_i(A) = \sum_{j=1}^{n} |a_{ij}|,$$

$$d_i = \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad J(A) = \{i \in N : d_i < 1\},$$

$$u_i = \frac{1}{|a_{ii}|} \sum_{j=i+1}^{n} |a_{ij}|,$$

$$l_k = \max_{k\le i\le n} \frac{\sum_{j\neq i, k\le j\le n} |a_{ij}|}{|a_{ii}|}, \quad l_n = u_n = 0, \tag{4}$$

$$w_{ij} = \frac{|a_{ij}|}{|a_{ii}| - \sum_{k\neq i,j} |a_{ik}|}, \quad i \neq j,\ j < k \le n,$$

$$w_i = \max_{j\neq i} \{w_{ij}\}, \quad C_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ji}|,$$

$$m_{ij} = \frac{|a_{ij}| + \sum_{k\neq i,j} |a_{ik}| w_k}{|a_{ii}|}, \quad i \neq j,\ j < k \le n,$$

where N is the set of positive integers. For an $n \times n$ matrix A, the principal matrix of A formed by rows and columns with indices between n_1 and n_2 is denoted by $A^{(n_1,n_2)}$.

Definition 1 (see [8]). $A \in R^{n\times n}$ is weakly chained diagonally dominant if, for all $i \in N$, $d_i \le 1$ and $J(A) \neq \emptyset$ and for all $i \in N$, $i \notin J(A)$, there exist indices $i_1, i_2, \ldots, i_k$ in N with $a_{i_r i_{r+1}} \neq 0$, $0 \le r \le k-1$, where $i_0 = i$ and $i_k \in J(A)$.

Definition 2 (see [9]). Let $A \in R^{n\times n}$, A is strictly diagonally dominant if $J(A) = N$.

Obviously, if $A \in R^{n\times n}$ is a strictly diagonally dominant matrix, then A be a weakly chained diagonally dominant matrix.

Definition 3 (see [9]). $A \in R^{n\times n}$ is an L-matrix if, for all $i, j \in N$ with $i \neq j$, $a_{ij} \le 0$ and $a_{ii} > 0$.

Definition 4 (see [10]). Let $A \in R^{n\times n}$; if there exist $\alpha \in [0, 1]$, such that

$$|a_{ii}| \ge \alpha R_i(A) + (1-\alpha) C_i(A), \tag{5}$$

for all $i \in N$, then A is said to be an α-diagonal dominant matrix, denoted by D_n^{α}.

Remark 5. By Definition 4, we know that A is just a diagonal dominant matrix while $\alpha = 1$.

Definition 6. If all the inequalities in (5) strictly hold, then A is said to be strictly α-diagonal dominant matrix (SD_n^{α}).

2. Estimation for an Upper Bound for $\|A^{-1}\|_\infty$ of Strictly Diagonally Dominant M-Matrix

We state some lemmas before giving a new upper bound for $\|A^{-1}\|_\infty$.

Lemma 7 (see [3]). *Let $A = (a_{ij})$ be an $n \times n$ weakly chained diagonally dominant M-matrix, $B = A^{(2,n)}$, $A^{-1} = (\alpha_{ij})_{i,j=1}^{n}$, and $B^{-1} = (\beta_{ij})_{i,j=2}^{n}$. Then, for $i, j = 1, 2, \ldots, n$,*

$$\alpha_{11} = \frac{1}{\Delta},$$
$$\alpha_{i1} = \frac{1}{\Delta} \sum_{k=2}^{n} \beta_{ik} (-a_{k1}),$$
$$\alpha_{1j} = \frac{1}{\Delta} \sum_{k=2}^{n} \beta_{kj} (-a_{1k}), \tag{6}$$
$$\alpha_{ij} = \beta_{ij} + \alpha_{1j} \sum_{k=2}^{n} \beta_{ik} (-a_{k1}),$$

where

$$\Delta = a_{11} - \sum_{k=2}^{n} a_{1k} \sum_{i=2}^{n} \beta_{ki} a_{i1} > 0. \tag{7}$$

Furthermore, if $J(A) = N$, then $\Delta \ge a_{11}(1 - d_1 l_1) \ge a_{11}(1 - d_1)$.

Lemma 8 (see [11]). *A weakly chained diagonally dominant L-matrix is a nonsingular M-matrix.*

Lemma 9 (see [11]). *Let $A = (a_{ij})$ be an $n \times n$ weakly chained diagonally dominant M-matrix; then $B = A^{(2,n)}$ is an $(n-1) \times (n-1)$ weakly chained diagonally dominant M-matrix; that is, $B^{-1} = (\beta_{ij})$ exists and $\beta_{ij} \ge 0$ $(i, j = 2, 3, \ldots, n)$.*

Lemma 10 (see [11]). *Let $A = (a_{ij})$ be an $n \times n$ weakly chained diagonally dominant M-matrix, $A^{-1} = (\alpha_{ij})$. Then, for $i \neq j$,*

$$\alpha_{ij} \le d_i \alpha_{jj} \le \alpha_{jj}. \tag{8}$$

Lemma 11 (see [11]). *Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant M-matrix; then*

$$\Delta \ge a_{11}(1 - d_1 l_1) > a_{11}(1 - d_1) > 0. \tag{9}$$

Lemma 12 (see [2]). *Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant M-matrix; then, for $A^{-1} = (\alpha_{ij})_{i,j=1}^{n}$, we have*

$$\frac{1}{a_{ii}} \le \alpha_{ii} \le \frac{1}{a_{ii} - \sum_{j\neq i} |a_{ij}| m_{ji}}. \tag{10}$$

Lemma 13 (see [1]). *Let $A = (a_{ij})$ be an $n \times n$ weakly chained diagonally dominant M-matrix, $A^{-1} = (\alpha_{ij})$, and $q = q(A)$, $N = 1, 2, \ldots, n$. Then*

$$q \le \min_{i\in N}\{a_{ii}\}, \quad q \le \max_{i\in N}\left\{\sum_{j\in N} a_{ij}\right\}, \quad q \ge \min_{i\in N}\left\{\sum_{j\in N} a_{ij}\right\},$$
$$\frac{1}{M} \le q \le \frac{1}{m}, \tag{11}$$

where

$$M = \max_{i\in N}\left\{\sum_{j\in N}\alpha_{ij}\right\} = \left\|A^{-1}\right\|_\infty, \qquad m = \min_{i\in N}\left\{\sum_{j\in N}\alpha_{ij}\right\}. \tag{12}$$

Now we give an upper bound for $\|A^{-1}\|_\infty$ and $q(A)$ of a strictly diagonally dominant M-matrix A by the following theorem.

Theorem 14. *Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant M-matrix, $A^{-1} = (\alpha_{ij})$. Then*

$$\left\|A^{-1}\right\|_\infty \le \frac{1}{a_{11} - \sum_{k=2}^n |a_{1k}| m_{k1}} + \sum_{i=2}^n\left[\frac{1}{a_{ii} - \sum_{k\ne i, i\le k\le n}^n |a_{ik}| m_{ki}}\prod_{j=1}^{i-1}\frac{1}{1-u_j l_j}\right]. \tag{13}$$

Proof. We prove this theorem by induction.

(1) Let $r_i = \sum_{j=1}^n \alpha_{ij}$, $B = A^{(2,n)}$, $M_A = \|A^{-1}\|_\infty$, and $M_B = \|B^{-1}\|_\infty$. Then

$$M_A = \max\{r_i : i \in N\},$$
$$M_B = \max\left\{\sum_{j=2}^n \beta_{ij} : 2 \le i \le n\right\}. \tag{14}$$

By Lemmas 7, 11, and 12, we know that

$$\begin{aligned} r_1 &= \alpha_{11} + \sum_{j=2}^n \alpha_{1j} \\ &= \frac{1}{\Delta} + \sum_{j=2}^n \frac{1}{\Delta}\sum_{k=2}^n \beta_{kj}(-a_{k1}) \\ &= \frac{1}{\Delta}\left(1 + \sum_{k=2}^n (-a_{k1})\sum_{j=2}^n \beta_{kj}\right) \\ &\le \frac{1}{\Delta}(1 + a_{11}\cdot d_1 \cdot M_B) \le \frac{1}{\Delta} + \frac{d_1 M_B}{1 - d_1 l_1} \\ &\le \frac{1}{\Delta} + \frac{M_B}{1 - d_1 l_1} \\ &\le \frac{1}{a_{11} - \sum_{k=2}^n |a_{1k}| m_{k1}} + \frac{M_B}{1 - d_1 l_1}. \end{aligned} \tag{15}$$

Let $2 \le i \le n$. By (8) and the second equality in (6), we have

$$\sum_{k=2}^n \beta_{ik}(-a_{k1}) = \Delta\alpha_{i1} \le \Delta d_i \alpha_{11} = d_i < 1. \tag{16}$$

From (8) with $2 \le j \le n$, we have

$$\alpha_{ij} \le \beta_{ij} + \alpha_{1j} d_i < \beta_{ij} + \alpha_{1j}. \tag{17}$$

Thus, for $2 \le i \le n$, we obtain

$$\begin{aligned} r_i &= \alpha_{i1} + \sum_{j=2}^n \alpha_{ij} \\ &\le d_i\alpha_{11} + \sum_{j=2}^n\left(\beta_{ij} + \alpha_{1j} d_i\right) \\ &= d_i\alpha_{11} + \sum_{j=2}^n \beta_{ij} + \sum_{j=2}^n \alpha_{1j} d_i \\ &\le r_1 d_i + \sum_{j=2}^n \beta_{ij} \\ &\le r_1 l_1 + M_B \\ &\le \left\{\frac{1}{\Delta} + \frac{d_1 M_B}{1 - d_1 l_1}\right\} l_1 + M_B \\ &\le \frac{l_1}{\Delta} + \frac{d_1 l_1 M_B}{1 - d_1 l_1} + M_B \\ &\le \frac{1}{\Delta} + \frac{M_B}{1 - d_1 l_1} \\ &\le \frac{1}{a_{11} - \sum_{k=2}^n |a_{1k}| m_{k1}} + \frac{M_B}{1 - d_1 l_1}. \end{aligned} \tag{18}$$

So by (15) and (18), we get

$$\left\|A^{-1}\right\|_\infty \le \frac{1}{a_{11} - \sum_{k=2}^n |a_{1k}| m_{k1}} + \frac{M_B}{1 - u_1 l_1}. \tag{19}$$

(2) Applying induction with respect to k of $A^{(k,n)}$ in (19) finishes the proof. □

From Theorem 14 and Lemma 13, the following theorem can be obtained easily.

Theorem 15. *Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant M-matrix. Then the smallest eigenvalue of A is*

$$q(A) \ge \left\{\frac{1}{a_{11} - \sum_{k=2}^n |a_{1k}| m_{k1}} + \sum_{i=2}^n\left[\frac{1}{a_{ii} - \sum_{k\ne i, i\le k\le n}^n |a_{ik}| m_{ki}}\prod_{j=1}^{i-1}\frac{1}{1-u_j l_j}\right]\right\}^{-1}. \tag{20}$$

Theorem 16. *Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant M-matrix. Then the bound in* (13) *is sharper than that in* (3), *that is,*

$$\frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k \neq i, i \le k \le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right] \le \frac{1}{a_{11}(1 - u_1 d_1)} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1 - u_i l_i)} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right]. \quad (21)$$

Proof. Since A is a strictly diagonally dominant matrix, $0 \le d_k < 1$, $m_{ki} \le d_i < 1$, and $1 \le j \le n-1$, then we have

$$\frac{1}{a_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \le \frac{1}{a_{ii}(1 - u_i d_i)}. \quad (22)$$

The results follow Lemma 12. Inequality (21) shows that the bound in (13) is better than that in (3).

For all i, $\max_{i \le k \le n}\{1/(a_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki})\} < \max_{i \le k \le n}\{1/a_{ii}(1 - u_i d_i)\}$, we have

$$\frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k \neq i, i \le k \le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right] < \frac{1}{a_{11}(1 - u_1 d_1)} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1 - u_i l_i)} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right]. \quad (23)$$

With the help of the above discussions, we give the upper bound for $\|A^{-1}\|_\infty$ of a real strictly α-diagonally dominant M-matrix. □

3. Estimation for an Upper Bound for $\|A^{-1}\|_\infty$ of a Strictly α-Diagonally Dominant M-Matrix

We show some notations and lemmas which are necessary to our conclusions.

Lemma 17 (see [12]). *Let $A, B \in R^{n\times n}$, A, $A-B$ be nonsingular, then*

$$(A - B)^{-1} = A^{-1} + A^{-1}B\left(I - A^{-1}B\right)^{-1}A^{-1}. \quad (24)$$

Lemma 18. *Let $A = (a_{ij}) \in R^{n\times n}$ is a strictly diagonal dominant M-matrix. If $B = (b^{ij}) \in R^{n\times n}$, with*

$$\left\|A^{-1}B\right\|_\infty \le \max_{1 \le i \le n} \kappa_0 \cdot \|B\|_\infty, \quad (25)$$

and if

$$\kappa_0 < \frac{1}{\|B\|_\infty}, \quad (26)$$

then $\|A^{-1}B\|_\infty < 1$, where

$$\kappa_0 = \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k \neq i, i \le k \le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right]. \quad (27)$$

Proof. By Theorem 14, we get

$$\left\|A^{-1}B\right\|_\infty \le \left\|A^{-1}\right\|_\infty \cdot \|B\|_\infty \le \max_{1 \le i \le n} \kappa_0 \cdot \|B\|_\infty. \quad (28)$$

It is easy to see that $\|A^{-1}B\|_\infty < 1$, if

$$\kappa_0 < \frac{1}{\|B\|_\infty}, \quad (29)$$

where

$$\kappa_0 = \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k \neq i, i \le k \le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right]. \quad (30)$$

□

Lemma 19 (see [12]). *If $\|A^{-1}\|_\infty < 1$, then $I - A$ is nonsingular and*

$$\left\|(I - A)^{-1}\right\|_\infty \le \frac{1}{1 - \|A\|_\infty}. \quad (31)$$

Theorem 20. *Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly α-diagonal dominant matrix, $\alpha \in (0, 1]$, and A be an M-matrix. If, for those $i \in N_1 \subset N$, $R_i(A) > C_i(A)$, and $\kappa_1 < 1/\max_{1 \le i \le n} \alpha(R_i(A) - C_i(A))$, then*

$$\left\|A^{-1}\right\|_\infty \le \frac{\kappa_1}{1 - \kappa_1 \max_{1 \le i \le n} \alpha\left(R_i(A) - C_i(A)\right)}, \quad (32)$$

where

$$\kappa_1 = \frac{1}{\beta_1 - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{\beta_i - \sum_{k \neq i, i \le k \le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right],$$

$$\beta_i = \max\left\{a_{ii}, a_{ii} + \alpha\left(R_i(A) - C_i(A)\right)\right\}, \quad i = 1, 2, \ldots, n. \quad (33)$$

Proof. Note that $R_i(A) > C_i(A)$. Then

$$|a_{ii}| > (1 - \alpha) R_i(A) + \alpha C_i(A) = R_i(A) - \alpha\left(R_i(A) - C_i(A)\right). \quad (34)$$

So we can split A, such that $A = B - C$, where $B = (b_{ij})$ and

$$b_{ij} = \begin{cases} a_{ii} + \alpha \left(R_i(A) - C_i(A)\right) & i = j,\ R_i(A) > C_i(A) \\ a_{ij} & \text{others}, \end{cases}$$

$$c_{ij} = \begin{cases} \alpha \left(R_i(A) - C_i(A)\right) & i = j,\ R_i(A) > C_i(A) \\ 0 & \text{others}. \end{cases} \tag{35}$$

We know $b_{ii} = a_{ii} + \alpha(R_i(A) - C_i(A)) > R_i(A) = R_i(B)$ and A is an M-matrix. Thus, B is a strictly diagonal dominant M-matrix; hence, $B^{-1} > 0$. Let $\beta_i = \max\{a_{ii}, a_{ii} + \alpha(R_i(A) - C_i(A))\}$, $i = 1, 2, \dots, n$. If $\kappa_1 < 1/\max_{1\le i\le n}\alpha(R_i(A) - C_i(A))$, by Lemma 18, we get $\|B^{-1}C\|_\infty \le 1$. By Lemmas 17 and 19 and Theorem 14, we can obtain

$$\begin{aligned} \left\|B^{-1}\right\|_\infty &\le \frac{1}{b_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{b_{ii} - \sum_{k\ne i, i\le k\le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right] \\ &\le \frac{1}{\beta_1 - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{\beta_i - \sum_{k\ne i, i\le k\le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right]. \end{aligned} \tag{36}$$

Let $\kappa_1 = 1/(\beta_1 - \sum_{k=2}^{n} |a_{1k}| m_{k1}) + \sum_{i=2}^{n}[(1/(\beta_i - \sum_{k\ne i, i\le k\le n}^{n} |a_{ik}| m_{ki}))\prod_{j=1}^{i-1} 1/(1 - u_j l_j)]$. Then

$$\begin{aligned} \left\|B^{-1}C\right\|_\infty &< \kappa_1 \max_{1\le i\le n} \sum_{j=1}^{n} \left|c_{ij}\right| \\ &< \kappa_1 \max_{1\le i\le n} \alpha \left(R_i(A) - C_i(A)\right). \end{aligned} \tag{37}$$

Further, we have

$$\begin{aligned} \left\|A^{-1}\right\|_\infty &= \left\|(B - C)^{-1}\right\|_\infty \\ &= \left\|B^{-1} + B^{-1}C\left(I - B^{-1}C\right)^{-1} B^{-1}\right\|_\infty \\ &\le \left\|B^{-1}\right\|_\infty + \left\|B^{-1}C\right\|_\infty \cdot \left\|\left(I - B^{-1}C\right)^{-1}\right\|_\infty \cdot \left\|B^{-1}\right\|_\infty \\ &\le \left\|B^{-1}\right\|_\infty + \frac{\left\|B^{-1}C\right\|_\infty}{1 - \left\|B^{-1}C\right\|_\infty} \left\|B^{-1}\right\|_\infty \\ &= \frac{1}{1 - \left\|B^{-1}C\right\|_\infty} \left\|B^{-1}\right\|_\infty \\ &\le \frac{\kappa_1}{1 - \kappa_1 \max_{1\le i\le n} \alpha \left(R_i(A) - C_i(A)\right)}, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \kappa_1 &= \frac{1}{\beta_1 - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{\beta_i - \sum_{k\ne i, i\le k\le n}^{n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_j l_j} \right], \\ \beta_i &= \max\left\{a_{ii}, a_{ii} + \alpha \left(R_i(A) - C_i(A)\right)\right\}, \quad i = 1, 2, \dots, n. \end{aligned} \tag{39}$$

The proof is complete. □

4. Examples

We illustrate our results by the following two examples.

(1) Consider the bound for $\|A^{-1}\|_\infty$ of a strictly diagonal dominant matrix A, where

$$A = \begin{pmatrix} 10 & -1 & -1 & -1 & -1 \\ -1 & 10 & -1 & -1 & -1 \\ -1 & -1 & 10 & -1 & -1 \\ -1 & -1 & -1 & 10 & -1 \\ -1 & -1 & -1 & -1 & 10 \end{pmatrix}. \tag{40}$$

Direct calculation by MATLAB R2010a gives

$$\begin{aligned} \left\|A^{-1}\right\|_\infty &= 0.1669, \\ \left\|A^{-1}\right\|_\infty &\le 214.0217 \ \left(\text{by Theorem 3.3 in [8]}\right) \\ \left\|A^{-1}\right\|_\infty &\le 175.9183 \ \left(\text{by (2)}\right) \\ \left\|A^{-1}\right\|_\infty &\le 9.2041 \ \left(\text{by (3)}\right) \\ \left\|A^{-1}\right\|_\infty &\le 6.5634 \ \left(\text{by Theorem 14 (13)}\right). \end{aligned} \tag{41}$$

It is obvious that the bound of Theorem 14 of this paper is better than other known ones. Furthermore, we can estimate $q(A)$ by Theorem 15.

(2) Consider the bound for $\|A^{-1}\|_\infty$ of a strictly α-diagonal dominant matrix A for $\alpha = 0.5$,

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -0.5 & 0 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0.8889 & 0.4444 & 0.6667 \\ 0.5556 & 0.7778 & 0.6667 \\ 0.2222 & 0.1111 & 0.6667 \end{pmatrix}. \tag{42}$$

Note that

$$\left\|A^{-1}\right\|_\infty \approx 2. \tag{43}$$

We know that A is not a strictly diagonal dominant matrix, and the bound of $\|A^{-1}\|_\infty$ cannot be obtained by (2) or (3), but it can be estimated by (32) in Theorem 20.

Split the matrix A such that $A = B - C$, where $B = (b_{ij})$ and $b_{11} = a_{11} + \alpha(R_1(A) - C_1(A)) = 2 + 0.5 \times (2 - 1.5) = 2.25$, $b_{22} = a_{22} + \alpha(R_2(A) - C_2(A)) = 2 + 0.5 \times (2 - 1) = 2.5$, Then

$$B = \begin{pmatrix} 2.25 & -1 & -1 \\ -1 & 2.5 & -1 \\ -0.5 & 0 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (44)$$

The bound for $\|A^{-1}\|_\infty$ can be estimated by (13) in Theorem 14 and (32) in Theorem 20 as follows:

$$\left\|A^{-1}\right\|_\infty \leq 11.4259. \quad (45)$$

Conflict of Interests

There is no conflict of interests regarding the publication of this paper.

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References

[1] J. M. Varah, "A lower bound for the smallest singular value of a matrix," *Linear Algebra and Its Applications*, vol. 11, pp. 3–5, 1975.

[2] R. S. Varga, "On diagonal dominance arguments for bounding $\|A^{-1}\|_\infty$," *Linear Algebra and Its Applications*, vol. 14, no. 3, pp. 211–217, 1976.

[3] G.-H. Cheng and T.-Z. Huang, "An upper bound for $\|A^{-1}\|_\infty$ of strictly diagonally dominant M-matrices," *Linear Algebra and Its Applications*, vol. 426, no. 2-3, pp. 667–673, 2007.

[4] P. Wang, "An upper bound for $\|A^{-1}\|_\infty$ of strictly diagonally dominant M-matrices," *Linear Algebra and Its Applications*, vol. 431, no. 5–7, pp. 511–517, 2009.

[5] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, Mass, USA, 1991.

[6] C. R. Johnson, "A Hadamard product involving M-matrices," *Linear and Multilinear Algebra*, vol. 4, no. 4, pp. 261–264, 1977.

[7] M. Fiedler, C. R. Johnson, and T. L. Markham, "A trace inequality for M-matrices and the symmertrizability of a real matrix by a positive diagonal matrix," *Linear Algebra and Its Applications*, vol. 102, pp. 1–8, 1988.

[8] P. N. Shivakumar, J. J. Williams, Q. Ye, and C. A. Marinov, "On two-sided bounds related to weakly diagonally dominant M-matrices with application to digital circuit dynamics," *SIAM Journal on Matrix Analysis and Applications*, vol. 17, no. 2, pp. 298–312, 1996.

[9] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, USA, 1994.

[10] Y. L. Zhang, H. M. Mo, and J. Z. Liu, "α-diagonal dominance and criteria for generalized strictly diagonally dominant matrices," *Numerical Mathematics*, vol. 31, no. 2, pp. 119–128, 2009.

[11] P. N. Shivakumar and K. H. Chew, "A sufficient condition for nonvanishing of determinants," *Proceedings of the American Mathematical Society*, vol. 43, pp. 63–66, 1974.

[12] S. Xu, *Theory and Methods about Matrix Computation*, Tshua University Press, Beijing, China, 1986.