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Research Article

An Iterative Learning Control Design Method for Nonlinear Discrete-Time Systems with Unknown Iteration-Varying Parameters and Control Direction

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An iterative learning control (ILC) scheme is designed for a class of nonlinear discrete-time dynamical systems with unknown iteration-varying parameters and control direction. The iteration-varying parameters are described by a high-order internal model (HOIM) such that the unknown parameters in the current iteration are a linear combination of the counterparts in the previous certain iterations. Under the framework of ILC, the learning convergence condition is derived through rigorous analysis. It is shown that the adaptive ILC law can achieve perfect tracking of system state in presence of iteration-varying parameters and unknown control direction. The effectiveness of the proposed control scheme is verified by simulations.

1. Introduction

Iterative learning control (ILC) is an effective control method in improving the transient response and tracking performance of controlled system when the control task is performed repeatedly in a finite time interval [1]. The main idea of ILC is to modify the control input profile by using the deviation of the system output and the desired trajectory so that the track performance can be improved continuously along the iteration axis. Recently, more and more attentions have been put towards ILC design under more general problem settings as well as application of the well-established ILC schemes to industrial and engineering processes [2–8].

Traditional framework of ILC design needs the strict repeatability of processes, which however is hard to be met in practice. As a result, ILC design with iteration-varying factors is a problem of considerable importance in both theory and practical applications [9]. For example, the iteration-varying initial state [10, 11], reference [12, 13], and disturbances [14, 15] have been frequently encountered. In practice, along the iterative axis, these factors can be described by high-order internal models (HOIMs) [16]; that is, the iteration-varying factors in the current iteration are linear combinations of the counterparts in the previous certain iterations [17]. It is worth noticing that although HOIM information has been considered to expedite the learning convergence of ILC in [9, 16, 17], there have been no works addressing ILC design of nonlinear discrete-time systems with iteration-varying HOIM-type uncertainties.

The main contribution of the paper lies in the fact that HOIM-based ILC scheme is proposed for a class of nonlinear discrete-time systems with unknown control direction [18–21]. The learning convergence condition is derived through rigorous analysis. It is shown that the proposed adaptive ILC law can achieve perfect tracking of system state in presence of iteration-varying parameters and unknown control direction. The paper is organized as follows. In Section 2, the problem formulation is given. In Section 3, an adaptive ILC scheme is proposed to achieve perfect tracking. of system output. In Section 4, the learning convergence of the proposed control scheme is addressed rigorously. In Section 5, the effectiveness of the proposed control scheme is verified by simulations. Section 6 concludes the work.

2. Problem Formulation

Consider the following discrete-time system:

$$\begin{aligned} x_{i,k}(t+1) &= x_{i+1,k}(t), \quad i = 1,\ldots,n-1, \\ x_{n,k}(t+1) &= \theta_k(t) f(\mathbf{x}_k(t),t) + b(t) u_k(t), \end{aligned} \tag{1}$$

where $x_{i,k}(t) \in \mathscr{R}$ denotes the ith state variable at the tth time instant of the kth iteration, $t \in [0,1,\ldots,T]$, $\mathbf{x}_k \triangleq [x_{1,k},\ldots,x_{n,k}]^T \in \mathscr{R}^n$ is the state vector with random initial condition $\mathbf{x}_k(0)$ in each iteration k, $u_k(t) \in \mathscr{R}$ is the system input, $\theta_k(t)$ is an unknown iteration-varying bounded parameter, $f(\mathbf{x}_k(t),t)$ is a known nonlinear regressor function, and $b(t) \geq b_{\min} > 0$, $\forall t$ is the unknown time-varying input gain function. The case that $b(t) < 0$, $\forall t$, can be considered similarly by redefining the input profile.

Define the desired trajectory as $\mathbf{x}_d(t) = [x_{1,d}(t),\ldots,x_{n,d}(t)]^T$, $t \in [0,1,\ldots,T]$, and assume that $\mathbf{x}_d(t)$ is bounded and generated by the following reference model:

$$\begin{aligned} x_{i,d}(t+1) &= x_{i+1,d}(t), \quad i = 1,\ldots,n-1, \\ x_{n,d}(t+1) &= g(\mathbf{x}_d(t),t), \end{aligned} \tag{2}$$

where the function g is continuous with respect to its arguments.

Then, the tracking error at the kth iteration is $e_{i,k}(t) \triangleq x_{i,k}(t) - x_{i,d}(t)$. From (1) and (2), it follows that

$$\begin{aligned} e_{i,k}(t+1) &= e_{i+1,k}(t), \quad i = 1,\ldots,n-1, \\ e_{n,k}(t+1) &= \theta_k(t) f(\mathbf{x}_k(t),t) + b(t) u_k(t) \\ &\quad - g(\mathbf{x}_d(t),t). \end{aligned} \tag{3}$$

The control target is to find a sequence of system input $u_k(t)$ so that the system state $\mathbf{x}_k(t)$ of (1) can converge to the desired trajectory $\mathbf{x}_d(t)$ asymptotically along the iteration axis.

We shall make some assumptions first.

Assumption 1 (see [16]). The iteration-varying parameter $\theta_k(t)$ satisfies

$$\theta_k(t) = a_1\theta_{k-1}(t) + \cdots + a_m\theta_{k-m}(t), \tag{4}$$

where $a_1,\ldots,a_m$, $m \geq 1$, are known constant parameters and the initial parameters $\theta_0(t),\ldots,\theta_{1-m}(t)$ are unknown functions that are linearly independent. In other words, $\theta_k(t)$ satisfies HOIM with order m.

Assumption 2. The nonlinear function $f(\mathbf{x}_k(t),t)$ satisfies the linear growth condition; that is, $|f(\mathbf{x}_k(t),t)| \leq c_1 + c_2\|\mathbf{x}_k(t)\|$, where c_1 and c_2 are positive constants.

The following lemma will be used in deriving the learning convergence of the proposed control scheme.

Lemma 3 (the Key Technical Lemma [22]). *If*

$$\lim_{t\to\infty} \frac{s^2(t)}{b_1(t) + b_2(t)\sigma^T(t)\sigma(t)} = 0, \tag{5}$$

where $b_1(t)$, $b_2(t)$, and $s(t)$ are real scalar sequence and $\sigma(t)$ is a real vector sequence, and the following two conditions hold:

(1) *uniform boundedness conditions $0 < b_1(t) < K < \infty$ and $0 < b_2(t) < K < \infty$ for all $t > 0$;*

(2) *linear boundedness condition*

$$\|\sigma(t)\| \leq C_1 + C_2 \max_{0<\tau<t} |s(\tau)|, \tag{6}$$

where $0 < C_1 < \infty$ and $0 < C_2 < \infty$, then we have

$$\lim_{t\to\infty} s(t) = 0, \tag{7}$$

and $\|\sigma(t)\|$ is bounded. $\|\cdot\|$ denotes the Euclid norm.

Define $\varphi_j(t) = \theta_{j-m}(t)$, where $j = 1,2,\ldots,m$ and $\boldsymbol{\theta}_k^*(t) \triangleq [\theta_{k-m+1},\ldots,\theta_k(t)]^T$ with $\boldsymbol{\theta}_0^*(t) = [\varphi_1(t),\ldots,\varphi_m(t)]^T$. Then we can rewrite (4) as

$$\boldsymbol{\theta}_{k+1}^*(t) = B\boldsymbol{\theta}_k^*(t), \tag{8}$$

where

$$B \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_2 & a_1 \end{pmatrix}. \tag{9}$$

Repeating (8), we obtain

$$\boldsymbol{\theta}_k^*(t) = B\boldsymbol{\theta}_{k-1}^*(t) = \cdots = B^k\boldsymbol{\theta}_0^*(t). \tag{10}$$

Let $\boldsymbol{\beta}_k \triangleq [\beta_{1,k},\cdots,\beta_{m,k}]^T$ be the last row of the matrix B^k; it renders to

$$\theta_k(t) = \beta_{1,k}\varphi_1(t) + \cdots + \beta_{m,k}\varphi_m(t) = \boldsymbol{\beta}_k^T\boldsymbol{\theta}_0^*(t). \tag{11}$$

Notice that $\boldsymbol{\beta}_1 = [a_m,\ldots,a_1]$. Owing to the boundedness of $\theta_k(t)$, $\beta_{j,k}$ is also bounded; that is, there exists $\beta > 0$ such that $|\beta_{j,k}| \leq \beta$ for $j = 1,2,\ldots,m$ and $k = 1,2,\ldots$. As such, the last equation of system (1) can be written in a more compact form:

$$\begin{aligned} x_{n,k}(t+1) &= \sum_{j=1}^{m} \varphi_j(t)\beta_{j,k} f(\mathbf{x}_k(t),t) + b(t)u_k(t) \\ &= \boldsymbol{\varphi}^T(t)\boldsymbol{\eta}_k(t) + b(t)u_k(t), \end{aligned} \tag{12}$$

where $\boldsymbol{\varphi}(t) \triangleq [\varphi_1(t),\ldots,\varphi_m(t)]^T$ and $\boldsymbol{\eta}_k(t) \triangleq [\beta_{1,k}f(\mathbf{x}_k(t)),\ldots,\beta_{m,k}f(\mathbf{x}_k(t))]^T$.

Remark 4. From system (12), the estimation of the iteration-varying parameter $\theta_k(t)$ is transformed to that of the iteration-invariant parameter $\boldsymbol{\varphi}(t)$. It implies that the parametric updating law and the control law can be, more conveniently, designed in the iteration domain. This is the main reason why the iteration-varying parameters satisfying HOIM can be addressed along the proposed way.

3. Controller Design

In this section, by making full use of the HOIM information of the parametric uncertainties $\theta_k(t)$, an ILC controller is designed for the considered nonlinear discrete-time system (1). Notice that the dynamics of $x_{n,k}$ in (1) has been reformulated as (12), where the parametric uncertainties $\boldsymbol{\varphi}(t)$ and $b(t)$ are iteration-invariant.

The control law is given as

$$\begin{aligned} u_k(t) &= \frac{-\sum_{j=1}^{m}\widehat{\varphi}_{j,k}(t)\beta_{j,k}f(\mathbf{x}_k(t)) + x_{n,d}(t+1)}{\operatorname{proj}(\widehat{b}_k(t))} \\ &= \frac{-\widehat{\boldsymbol{\varphi}}_k^T(t)\boldsymbol{\eta}_k(t) + x_{n,d}(t+1)}{\operatorname{proj}(\widehat{b}_k(t))}, \end{aligned} \tag{13}$$

where $\widehat{\varphi}_{j,k}(t)$, $j = 1,2,\ldots,m$, and $\widehat{b}_k(t)$ are the estimates of $\varphi_j(t)$ and $b(t)$ at the kth iteration, respectively, $\widehat{\boldsymbol{\varphi}}_k(t) \triangleq [\widehat{\varphi}_{1,k}(t),\ldots,\widehat{\varphi}_{m,k}(t)]^T$, and $\operatorname{proj}(\cdot)$ is a projection operator defined as [23]

$$\operatorname{proj}(\widehat{b}_k(t)) = \begin{cases} \widehat{b}_k(t), & \text{if } |\widehat{b}_k(t)| \geq b_{\min}, \\ \widehat{b}_{k-1}(t), & \text{otherwise}, \end{cases} \tag{14}$$

where $b_{\min}$ is the lower bound of the unknown control gain $b(t)$. By using the projection operator function, the possible singularity in (13) can be avoid. In fact, $|\operatorname{proj}(\widehat{b}_k(t))| \geq b_{\min}$ if the initial condition is chosen as $|\widehat{b}_0(t)| \geq b_{\min}$.

Observing (13), we have

$$\begin{aligned} x_{n,d}(t+1) = {} & \sum_{j=1}^{m}\widehat{\varphi}_{j,k}(t)\beta_{j,k}f(\mathbf{x}_k(t)) \\ & + \operatorname{proj}(\widehat{b}_k(t))u_k(t). \end{aligned} \tag{15}$$

Then, by the definition of state tracking error $e_{i,k}(t)$,

$$\begin{aligned} e_{n,k}(t+1) = {} & \sum_{j=1}^{m}\varphi_j(t)\beta_{j,k}f(\mathbf{x}_k(t),t) + b(t)u_k(t) \\ & - \sum_{j=1}^{m}\widehat{\varphi}_{j,k}(t)\beta_{j,k}f(\mathbf{x}_k(t)) \\ & - \operatorname{proj}(\widehat{b}_k(t))u_k(t) \\ = {} & [\widehat{b}_k(t) - \operatorname{proj}(\widehat{b}_k(t))]u_k(t) \\ & - \widehat{\boldsymbol{\phi}}_k^T(t)\boldsymbol{\xi}_k(t) + \boldsymbol{\phi}^T(t)\boldsymbol{\xi}_k(t) \\ = {} & -\widetilde{\boldsymbol{\phi}}_k^T(t)\boldsymbol{\xi}_k(t) \\ & + [\widehat{b}_k(t) - \operatorname{proj}(\widehat{b}_k(t))]u_k(t), \end{aligned} \tag{16}$$

where $\widetilde{\boldsymbol{\phi}}_k(t) \triangleq [\widetilde{\varphi}_{1,k}(t),\ldots,\widetilde{\varphi}_{m,k}(t),\widetilde{b}_k(t)]^T$, $\widetilde{\varphi}_{j,k}(t) \triangleq \widehat{\varphi}_{j,k}(t) - \varphi_j(t)$, $\widetilde{b}_k(t) \triangleq \widehat{b}_k(t) - b(t)$, $\boldsymbol{\phi}(t) \triangleq [\boldsymbol{\varphi}^T(t), b(t)]^T$, $\widehat{\boldsymbol{\phi}}_k(t) \triangleq [\widehat{\boldsymbol{\varphi}}_k^T(t), \widehat{b}_k(t)]^T$, and $\boldsymbol{\xi}_k(t) \triangleq [\boldsymbol{\eta}_k^T(t), u_k(t)]^T$.

The parametric updating laws for $\widehat{\boldsymbol{\varphi}}_k(t)$ and $\widehat{b}_k(t)$ are directly given as follows:

$$\begin{aligned} \widehat{\varphi}_{j,k+1}(t) &= \widehat{\varphi}_{j,k}(t) - \frac{\beta_{j,k}f(\mathbf{x}_k(t))}{p + \boldsymbol{\xi}_k^T(t)\boldsymbol{\xi}_k(t)}\widetilde{\boldsymbol{\phi}}_k^T(t)\boldsymbol{\xi}_k(t), \\ & \qquad j = 1,2,\ldots,m, \\ \widehat{b}_{k+1}(t) &= \widehat{b}_k(t) - \frac{u_k(t)}{p + \boldsymbol{\xi}_k^T(t)\boldsymbol{\xi}_k(t)}\widetilde{\boldsymbol{\phi}}_k^T(t)\boldsymbol{\xi}_k(t), \end{aligned} \tag{17}$$

where p is a positive constant.

Remark 5. The ILC (13) with parameter updating laws (17) is an adaptive scheme, which is an extension of typical adaptive controller and repetitive control [24]. Moreover, this ILC borrows the idea of the HOIM-based ILC in [9, 16, 17, 25].

4. Convergence Analysis

In this section, the learning convergence of the proposed ILC scheme, that is, control law (13) and parametric updating laws (17), will be analyzed in a rigorous way.

Theorem 6. *For nonlinear discrete-time system (1), under Assumptions 1 and 2, control law (13) and parametric updating laws (17) ensure that*

(1) *the parametric estimation $\widehat{\boldsymbol{\phi}}_k(t)$, $t \in [0,\ldots,T]$ is always bounded for all iterations,*

(2) *the tracking errors $e_{i,k}(t)$, $\forall t \in [n-i+1,\ldots,T]$, $i = 1,\ldots,n$, will converge to zero asymptotically as $k \to \infty$.*

Proof. The whole proof is divided into two parts. Part 1 derives the boundedness of $\widehat{\boldsymbol{\phi}}_k(t)$, and Part 2 addresses the asymptotical convergence of $e_{i,k}(t)$.

Part 1 (the boundedness of $\widehat{\boldsymbol{\phi}}_k(t)$). Define the composite energy function at the kth iteration as

$$V_k(t) = \widetilde{\boldsymbol{\phi}}_k^T(t)\widetilde{\boldsymbol{\phi}}_k(t) = \sum_{j=1}^{m}\widetilde{\varphi}_{j,k}^2(t) + \widetilde{b}_k^2(t), \tag{18}$$

whose difference in two consecutive iterations is

$$\Delta V_{k+1} \triangleq V_{k+1} - V_k = \sum_{j=1}^{m}(\widetilde{\varphi}_{j,k+1}^2 - \widetilde{\varphi}_{j,k}^2) + \widetilde{b}_{k+1}^2 - \widetilde{b}_k^2. \tag{19}$$

For the first part of the right hand side of (19), applying the learning laws (17) yields

$$\begin{aligned} \sum_{j=1}^{m}(\widetilde{\varphi}_{j,k+1}^2 - \widetilde{\varphi}_{j,k}^2) = {} & \sum_{j=1}^{m}(\varphi_j - \widehat{\varphi}_{j,k+1})^2 \\ & - \sum_{j=1}^{m}(\varphi_j - \widehat{\varphi}_{j,k})^2 \\ = {} & 2\sum_{j=1}^{m}(\widehat{\varphi}_{j,k} - \varphi_j)(\widehat{\varphi}_{j,k+1} - \widehat{\varphi}_{j,k}) \end{aligned}$$

$$+\sum_{j=1}^{m}\left(\widehat{\varphi}_{j,k+1}-\widehat{\varphi}_{j,k}\right)^2$$
$$=-2\sum_{j=1}^{m}\frac{\widetilde{\varphi}_{j,k}\beta_{j,k}f\left(\mathbf{x}_k\right)}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k$$
$$+\sum_{j=1}^{m}\frac{\left(\beta_{j,k}f\left(\mathbf{x}_k\right)\right)^2}{\left(p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k\right)^2}\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2. \tag{20}$$

For the second part of the right hand side of (19), similar procedure leads to

$$\begin{aligned}\widetilde{b}_{k+1}^2-\widetilde{b}_k^2&=2\left(\widehat{b}_k-b\right)\left(\widehat{b}_{k+1}-\widehat{b}_k\right)+\left(\widehat{b}_{k+1}-\widehat{b}_k\right)^2\\&=-2\frac{\widetilde{b}_k u_k}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k+\frac{u_k^2\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{\left(p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k\right)^2}.\end{aligned} \tag{21}$$

Now, substituting (20) and (21) into (19) renders to

$$\begin{aligned}\Delta V_{k+1}&=-2\frac{\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}+\frac{\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{\left(p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k\right)^2}\\&=\left(-2+\frac{\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}\right)\frac{\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}\le-\frac{\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}\\&\le 0,\end{aligned} \tag{22}$$

implying that

$$\widetilde{\boldsymbol{\phi}}_{k+1}^T(t)\,\widetilde{\boldsymbol{\phi}}_{k+1}(t)\le\widetilde{\boldsymbol{\phi}}_k^T(t)\,\widetilde{\boldsymbol{\phi}}_k(t)\le\widetilde{\boldsymbol{\phi}}_0^T(t)\,\widetilde{\boldsymbol{\phi}}_0(t) \tag{23}$$

or equivalently

$$\left\|\widetilde{\boldsymbol{\phi}}_{k+1}(t)\right\|^2\le\left\|\widetilde{\boldsymbol{\phi}}_k(t)\right\|^2\le\left\|\widetilde{\boldsymbol{\phi}}_0(t)\right\|^2. \tag{24}$$

Considering the boundedness of $\varphi_j(t)$, $b(t)$, $\widehat{\varphi}_{j,0}(t)$, and $\widehat{b}_0(t)$, it follows that $\|\widetilde{\boldsymbol{\phi}}_k(t)\|$ is bounded, and therefore $\widehat{\boldsymbol{\phi}}_k(t)$ is bounded, $\forall t\in[0,1,\ldots,T]$, $k=1,2,\ldots$.

Part 2 (the asymptotical convergence of $e_{i,k}(t)$). Our idea is to first prove the asymptotical convergence of $e_{n,k}(t)$, and then the asymptotical convergence of $e_{i,k}(t)$, $i\le n-1$, can be obtained immediately by the canonical form of the system.

On the one hand, we have from (22)

$$V_{k+1}(t)\le V_0(t)-\sum_{l=1}^{k}\frac{\left(\widetilde{\boldsymbol{\phi}}_l^T\boldsymbol{\xi}_l\right)^2}{p+\boldsymbol{\xi}_l^T\boldsymbol{\xi}_l}. \tag{25}$$

Since $V_0(t)$ is bounded and $V_{k+1}(t)\ge 0$, it is clear to see that

$$\lim_{k\to\infty}\frac{\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}=0. \tag{26}$$

On the other hand, noticing the learning laws (17) and once again, we have

$$\widehat{\boldsymbol{\phi}}_{k+1}(t)=\widehat{\boldsymbol{\phi}}_k(t)-\frac{\boldsymbol{\xi}_k(t)}{p+\boldsymbol{\xi}_k^T(t)\,\boldsymbol{\xi}_k(t)}\widetilde{\boldsymbol{\phi}}_k^T(t)\,\boldsymbol{\xi}_k(t), \tag{27}$$

yielding

$$\begin{aligned}\left\|\widehat{\boldsymbol{\phi}}_{k+1}-\widehat{\boldsymbol{\phi}}_k\right\|^2&=\frac{\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}{\left(p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k\right)^2}\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2\\&\le\frac{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}{\left(p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k\right)^2}\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2=\frac{\left(\widetilde{\boldsymbol{\phi}}_k^T\boldsymbol{\xi}_k\right)^2}{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}.\end{aligned} \tag{28}$$

Combining (26) and (28) leads to

$$\lim_{k\to\infty}\left\|\widehat{\boldsymbol{\phi}}_{k+1}-\widehat{\boldsymbol{\phi}}_k\right\|=0. \tag{29}$$

Since $\widehat{\boldsymbol{\phi}}_k$ includes $\widehat{b}_k$ as the last entry, the following relationship is directly obtained from

$$\lim_{k\to\infty}\left|\widehat{b}_k-\widehat{b}_{k-1}\right|=0. \tag{30}$$

By the definition of $\mathrm{proj}(\widehat{b}_k)$, (30) renders to

$$\lim_{k\to\infty}\left|\widehat{b}_k-\mathrm{proj}\left(\widehat{b}_k\right)\right|=0. \tag{31}$$

Further, observing $\boldsymbol{\xi}_k(t)\triangleq[\boldsymbol{\eta}_k^T(t),u_k(t)]^T$, it is clear that

$$\left|u_k\right|=\sqrt{u_k^2}\le\sqrt{p+\boldsymbol{\eta}_k^T\boldsymbol{\eta}_k+u_k^2}=\sqrt{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}. \tag{32}$$

As a result,

$$\frac{\left|\widehat{b}_k-\mathrm{proj}\left(\widehat{b}_k\right)\right|\left|u_k\right|}{\sqrt{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}}\le\left|\widehat{b}_k-\mathrm{proj}\left(\widehat{b}_k\right)\right|. \tag{33}$$

Since the right hand side of (33) satisfies (31),

$$\lim_{k\to\infty}\frac{\left|\widehat{b}_k-\mathrm{proj}\left(\widehat{b}_k\right)\right|\left|u_k\right|}{\sqrt{p+\boldsymbol{\xi}_k^T\boldsymbol{\xi}_k}}=0. \tag{34}$$

Now, noticing the error dyanmics of $e_{n,k}(t+1)$, that is, (16), the relationships (27) and (35) render to

$$\lim_{k\to\infty}\frac{e_{n,k}(t+1)}{\sqrt{p+\boldsymbol{\xi}_k^T(t)\,\boldsymbol{\xi}_k(t)}}=0. \tag{35}$$

In order to prove the asymptotical convergence of $e_{n,k}$ via Lemma 3, namely, the Key Technical Lemma, it suffices to prove

$$\left\|\boldsymbol{\xi}_k(t)\right\|\le C_1+C_2\max_{t'\in[0,t-1]}\left|e_{n,k}\left(t'+1\right)\right|,\qquad t=[1,\ldots,T], \tag{36}$$

where C_1 and C_2 are certain finite constants. This will be addressed in the following.

By the definition of $\boldsymbol{\xi}_k(t)$,

$$\left\|\boldsymbol{\xi}_k(t)\right\| \le \left|u_k(t)\right| + \left\|\boldsymbol{\eta}_k(t)\right\|. \tag{37}$$

First evaluate the upper bound of $|u_k(t)|$. From the expression of the proposed controller (13),

$$\begin{aligned}\left|u_k(t)\right| &= \frac{\left|-\widehat{\boldsymbol{\varphi}}_k^T(t)\,\boldsymbol{\eta}_k(t) + x_{n,d}(t+1)\right|}{\left|\operatorname{proj}\left(\widehat{b}_k(t)\right)\right|} \\ &\le \left|\operatorname{proj}\left(b_k(t)\right)\right|^{-1}\left\|\widehat{\boldsymbol{\varphi}}_k(t)\right\|\left\|\boldsymbol{\eta}_k(t)\right\| \\ &\quad + \left|\operatorname{proj}\left(b_k(t)\right)\right|^{-1}\left|x_{n,d}(t+1)\right| \\ &\le n_1 + n_2\left\|\boldsymbol{\eta}_k(t)\right\|,\end{aligned} \tag{38}$$

where $n_1 \triangleq b_{\min}^{-1}\tau$ and $n_2 \triangleq b_{\min}^{-1}\max_{t\in[0,T]}\|\widehat{\boldsymbol{\varphi}}_k(t)\|$, and $\tau \triangleq \max_{t\in[0,T]}\|\mathbf{x}_d(t)\|$. The relationship $|\operatorname{proj}(b_k(t))| \ge b_{\min}$ is adopted in deriving (38).

Second, we evaluate the upper bound of $\|\boldsymbol{\eta}_k(t)\|$. By the definition of $\boldsymbol{\eta}_k(t)$, it follows that

$$\left\|\boldsymbol{\eta}_k(t)\right\| = \sqrt{\beta_{1,k}^2 + \cdots + \beta_{m,k}^2}\left|f\left(\mathbf{x}_k(t)\right)\right|. \tag{39}$$

Notice that $\|\mathbf{x}_k(t)\| \le \|\mathbf{e}_k(t)\| + \|\mathbf{x}_d(t)\| \le \|\mathbf{e}_k(t)\| + \tau$. Then, by Assumption 2, namely, the linear growth condition for the nonlinear regressor $f(\mathbf{x}_k(t))$, we have

$$\left|f\left(\mathbf{x}_k(t)\right)\right| \le c_1 + c_2\left\|\mathbf{x}_k(t)\right\| \le c_1 + c_2\tau + c_2\left\|\mathbf{e}_k(t)\right\|. \tag{40}$$

Combining (39) and (40) gives

$$\left\|\boldsymbol{\eta}_k(t)\right\| \le \sqrt{m}\beta\left(c_1 + c_2\tau\right) + \sqrt{m}\beta c_2\left\|\mathbf{e}_k(t)\right\|, \tag{41}$$

where β is an upper bound of $\beta_{i,k}$, $i = 1, \ldots, m$.

Combining (38) and (41) yields

$$\begin{aligned}\left\|\boldsymbol{\xi}_k(t)\right\| &\le \left|u_k(t)\right| + \left\|\boldsymbol{\eta}_k(t)\right\| \le n_1 + \left(1 + n_2\right)\left\|\boldsymbol{\eta}_k(t)\right\| \\ &\le q_1 + q_2\left\|\mathbf{e}_k(t)\right\|,\end{aligned} \tag{42}$$

where $q_1 = n_1 + \sqrt{m}\beta(1+n_2)(c_1+c_2\tau)$ and $q_2 = \sqrt{m}\beta(1+n_2)c_2$.

Now, the remaining is to find the relationship between $\|\mathbf{e}_k(t)\|$ and the quantity $\max_{t'\in[0,t-1]}|e_{n,k}(t'+1)|$. Observing the state error dynamics (3),

$$e_{1,k}(t) = e_{2,k}(t-1) = \cdots = e_{n,k}(t-n+1), \tag{43}$$

implying

$$\left|e_{i,k}(t)\right| = \left|e_{n,k}(t-n+i)\right| \le \max_{t'\in[0,t]}\left|e_{n,k}\left(t'\right)\right|. \tag{44}$$

As such, we obtain

$$\left\|\mathbf{e}_k(t)\right\| \le \sum_{i=1}^{n}\left|e_{i,k}(t)\right| \le n\max_{t'\in[0,t]}\left|e_{n,k}\left(t'\right)\right|. \tag{45}$$

Hence, by substituting (45) into (42), we have

$$\begin{aligned}\left\|\boldsymbol{\xi}_k(t)\right\| &\le q_1 + q_2\left\|\mathbf{e}_k(t)\right\| \le q_1 + q_2 n\max_{t'\in[0,t]}\left|e_{n,k}\left(t'\right)\right| \\ &\le q_1 + q_2\left(n\left|e_{n,k}(0)\right| + n\max_{t'\in[0,t-1]}\left|e_{n,k}\left(t'+1\right)\right|\right) \\ &\le C_1 + C_2\max_{t'\in[0,t-1]}\left|e_{n,k}\left(t'+1\right)\right|, \quad t = 1, \ldots, T,\end{aligned} \tag{46}$$

where $C_1 \triangleq q_1 + C_2|e_{n,k}(0)|$ and $C_2 \triangleq q_2 n$.

At last, according to (43), the asymptotical convergence of $e_{n,k}(t)$, $\forall t \in [1, \ldots, T]$, guarantees the asymptotical convergence of $e_{i,k}(t)$, $\forall t \in [n-i+1, \ldots, T]$, $i = 1, \ldots, n-1$. The proof is complete. □

Remark 7. The learning convergence of the proposed ILC scheme, that is, control law (13) and parametric updating laws (17), is proved rigorously for any random bounded initial states. In other words, the perfect tracking can be achieved for any random bounded initial conditions. The main reason is that the desired states at $t = 1, 2, \ldots, T$ of system (2) are directly utilized to regulate control input (13) and the effect of the state at $t = 0$ can be ignored. In order to achieve perfect tracking, traditional ILC schemes restrict the initial states to be identical or convergent [9–11]. Hence, the efficiency in dealing with any random initial conditions is another contribution of our paper.

5. Simulation Example

Consider the following system:

$$\begin{aligned}x_{1,k}(t+1) &= x_{2,k}(t), \\ x_{2,k}(t+1) & \\ &= \theta_k(t)\left(\sin\left(\frac{\pi x_{1,k}(t)}{20}\right) + \cos\left(\frac{\pi x_{2,k}(t)}{30}\right)\right) \\ &\quad + b(t)\,u_k(t),\end{aligned} \tag{47}$$

where $b(t) = 3 + \sin(t)$ is the unknown time-varying control direction, $\theta_k(t) = -2\cos(0.4)\theta_{k-1}(t) - \theta_{k-2}(t)$, $\theta_{-1}(t) = 0.08\cos(\pi t/10)$, and $\theta_0(t) = -0.96\sin(1.2t) + 0.8\cos(2t)$. The desired trajectories of the system are given by (2) with $g(\mathbf{x}_d(t), t) = 1 + 0.5e^{1-\cos(0.025\pi t)}$. For demonstration, we set $T = 200$, $\widehat{\varphi}_{1,0}(t) = \widehat{\varphi}_{2,0}(t) = 0$, $\widehat{b}_0(t) = 1$, and $b_{\min} = 0.1$. In addition, the random initial condition of the system state, $\mathbf{x}_k(0)$, is shown in Figure 1.

The tracking performance is shown in Figures 2 and 3. More clearly, Figure 2 gives the maximum tracking error of $x_{2,k}(t)$, $t \in [1, 200]$, along the iteration axis. It can be seen that the tracking error is decreased significantly in 10 iterations and becomes invisible after 50 iterations. For illustration, the state profile of the system in the 70th iteration and its desired trajectory are given simultaneously in Figure 3. All these simulation results verify the effectiveness of the proposed ILC scheme.

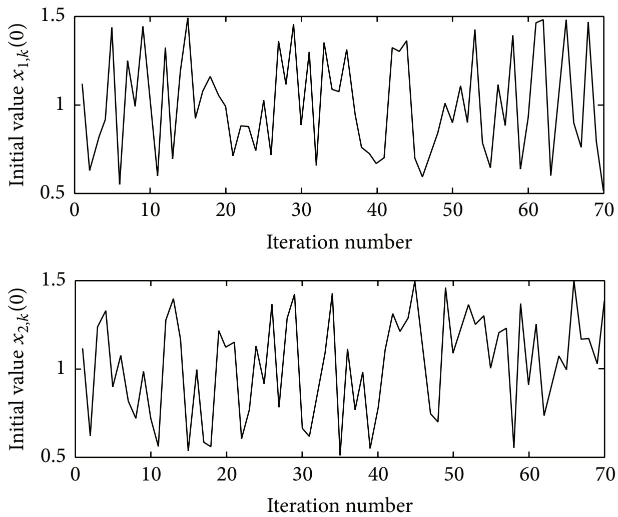

Figure 1: The random initial condition of the system state, $\mathbf{x}_k(0)$, versus the iteration number.

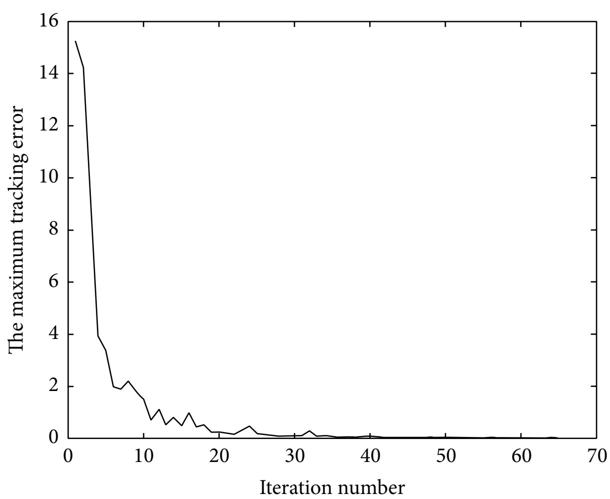

Figure 2: The profile of maximum tracking error of $x_{2,k}(t)$, $t \in [1, 200]$, in the iteration domain.

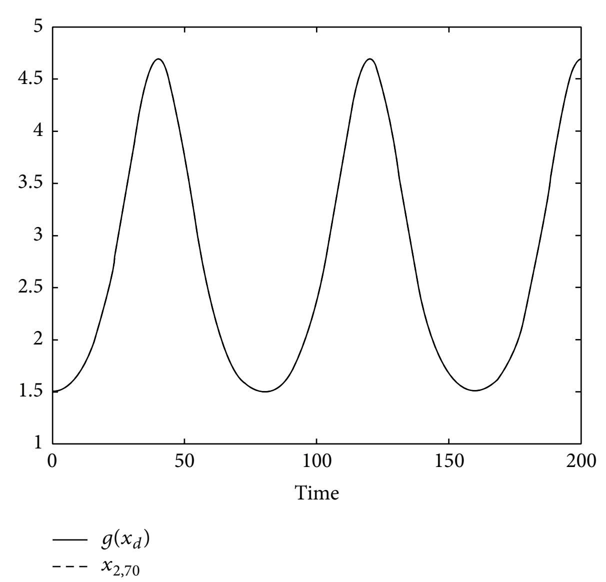

Figure 3: The state tracking profile, $x_{2,70}(t)$, $t \in [1, 200]$, and its desired trajectory.

6. Conclusions

In this paper, an iterative learning control scheme is presented for a class of nonlinear discrete-time systems with unknown iteration-varying parameters and unknown control direction, where the unknown iteration-varying parameters are assumed to satisfy a structure of high-order internal model (HOIM). By making full use of the information embedded in the HOIM, two efficient parametric updating laws are proposed to learn the system uncertainties. The learning convergence of the proposed control scheme is ensured through rigorous analysis. Our next research phase is to exploit the ILC design for systems with HOIM-type uncertainties but without linear growth conditions, as well as its applications.

Competing Interests

The authors declare that they have no competing interests.

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