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Research Article

New Result of Analytic Functions Related to Hurwitz Zeta Function

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By using a linear operator, we obtain some new results for a normalized analytic function f defined by means of the Hadamard product of Hurwitz zeta function. A class related to this function will be introduced and the properties will be discussed.

1. Introduction

A meromorphic function is a single-valued function, that is, analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities). A simpler definition states that a meromorphic function $f(z)$ is a function of the form

$$f(z) = \frac{g(z)}{h(z)}, \tag{1}$$

where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (see [1, page 64]). A meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain. A meromorphic function with an infinite number of poles is exemplified by $\csc(1/z)$ on the punctured disk $U^* = \{z : 0 < |z| < 1\}$.

An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. For example, the Gamma function is meromorphic in the whole complex plane; see [1, 2].

In the present paper, we will derive some properties of univalent functions defined by means of the Hadamard product of Hurwitz Zeta function; a class related to this function will be introduced and the properties of the liner operator $L_a^t(\alpha, \beta) f(z)$ will be discussed.

2. Preliminaries

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{2}$$

which are analytic in the punctured unit disk U^*. For $0 \leq \beta$, we denote by $S^*(\beta)$ and $k(\beta)$ the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in U^*.

For functions $f_j(z)$ $(j = 1; 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \tag{3}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{4}$$

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} z^n, \tag{5}$$

for $\beta \neq 0, -1, -2, \ldots$, and $\alpha \in \mathbb{C}/\{0\}$, where $(\lambda)n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z^2} F_1(1, \alpha, \beta; z), \quad (6)$$

where

$${}_2F_1(b, \alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(b)_n (\alpha)_n}{(\beta)_n} \frac{z^n}{n!} \quad (7)$$

is the well-known Gaussian hypergeometric function.

We recall here a general Hurwitz-Lerch-Zeta function, which is defined in [3, 4] by the following series:

$$\Phi(z, t, a) = \frac{1}{a^t} + \sum_{n=1}^{\infty} \frac{z^n}{(n+a)^t} \quad (8)$$

($a \in \mathbb{C}/\mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $t \in \mathbb{C}$ when $z \in U = U^* \subset \{0\}$; $\Re(t) > 1$ when $z \in \partial U$).

Important special cases of the function $\Phi(z, t, a)$ include, for example, the Riemann zeta function $\zeta(t) = \Phi(1, t, 1)$, the Hurwitz zeta function $\zeta(t, a) = \Phi(1, t, a)$, the Lerch zeta function $l_t(\zeta) = \Phi(\exp^{2\pi i \xi}, t, 1)$, $(\xi \in \mathbb{R}, \Re(t) > 1)$, and the polylogarithm $L_t^i(z) = z\Phi(z, t, a)$. Recent results on $\Phi(z, t, a)$ can be found in the expositions [5, 6]. By making use of the following normalized function we define

$$\begin{aligned} G_{t,a}(z) &= (1+a)^t \left[\Phi(z, t, a) - a^t + \frac{1}{z(1+a)^t}\right] \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1+a}{n+a}\right)^t z^n, \quad (z \in U^*). \end{aligned} \quad (9)$$

Corresponding to the functions $G_{t,a}(z)$ and using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L_{t,a}(\alpha, \beta)$ on Σ by the following series:

$$\begin{aligned} L_a^t(\alpha, \beta) f(z) &= \phi(\alpha, \beta; z) * G_{t,a}(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{1+a}{n+a}\right)^t a_n z^n. \quad (10) \\ &\qquad (z \in U^*). \end{aligned}$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by many others; see, for example, [7–12].

It follows from (10) that

$$\begin{aligned} &z\left(L_a^t(\alpha, \beta) f(z)\right)' \\ &\quad = \alpha\left(L_a^t(\alpha+1, \beta) f(z)\right) - (\alpha+1) L_a^t(\alpha, \beta) f(z). \end{aligned} \quad (11)$$

In order to prove our main results, we recall the following lemma according to Yang [13].

Lemma 1. *Let $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ be analytic functions in $U = U^* \cup \{0\}$ with $q(z) \neq 0$ for $z \in U$. If*

$$\Re\left\{1 + a\frac{zq'(z)}{q^2(z)}\right\} < M, \quad (z \in U), \quad (12)$$

where $a > 0$, and

$$1 < M \leq \frac{na}{2\log 2}, \quad (13)$$

then

$$\Re\left\{\frac{1}{q(z)}\right\} > 1 - \frac{2(M-1)}{na} \log 2, \quad (z \in U). \quad (14)$$

The bound in (14) is the best possible.

3. Main Results

We begin with the following theorem.

Theorem 2. *Let $\alpha + 1 > 0$, $L_a^t(\alpha, \beta) f(z) / L_a^t(\alpha+1, \beta) f(z) \neq 0$ for $z \in U^*$ and suppose that*

$$\begin{aligned} \Re\Bigg\{1 &+ \frac{L_a^t(\alpha+1, \beta) f(z)}{(\alpha+1) L_a^t(\alpha, \beta) f(z)} \left(1 + \frac{\alpha L_a^t(\alpha+1, \beta) f(z)}{L_a^t(\alpha, \beta) f(z)}\right) \\ &- \frac{L_a^t(\alpha+2, \beta) f(z)}{L_a^t(\alpha, \beta) f(z)}\Bigg\} < M, \end{aligned} \quad (15)$$

where

$$1 < M \leq \frac{n}{2(\alpha+1)\log 2}. \quad (16)$$

Then

$$\begin{aligned} \Re&\left\{\frac{L_a^t(\alpha+1, \beta) f(z)}{L_a^t(\alpha, \beta) f(z)}\right\} \\ &> 1 - \frac{2(\alpha+1)(M-1)}{n} \log 2, \quad (z \in U^*). \end{aligned} \quad (17)$$

The bound in (17) is the best possible.

Proof. Define the function $q(z)$ by

$$q(z) = \frac{L_a^t(\alpha, \beta) f(z)}{L_a^t(\alpha+1, \beta) f(z)}. \quad (18)$$

Then, clearly $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ analytic function in U^* with $q(z) \neq 0$ for $z \in U^*$. It follows from (18) and (11) that

$$\frac{zq'(z)}{q(z)} = \frac{z(L_a^t(\alpha, \beta) f(z))'}{L_a^t(\alpha, \beta) f(z)} - \frac{z(L_a^t(\alpha+1, \beta) f(z))'}{L_p^*(a+1, c) f(z)} \quad (19)$$

by making use of the familiar identity (11) in (19), we obtain

$$\frac{L_a^t(\alpha+2, \beta) f(z)}{L_a^t(\alpha+1, \beta) f(z)} = \frac{1}{\alpha+1} + \frac{1}{(\alpha+1) q(z)} - \frac{zq'(z)}{(\alpha+1) q(z)} \quad (20)$$

or, equivalent,

$$1+\frac{1}{(\alpha+1)}\frac{zq'(z)}{q^2(z)} = 1+\frac{L_a^t(\alpha+1,\beta)f(z)}{(\alpha+1)L_a^t(\alpha,\beta)f(z)}\left(1+\frac{\alpha L_a^t(\alpha+1,\beta)f(z)}{L_a^t(\alpha,\beta)f(z)}\right) - \frac{L_a^t(\alpha+2,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}. \quad (21)$$

Applying Lemma 1, with $a = 1/(1+\alpha)$, we get the required result. □

Letting $\alpha = \beta = 1$ in Theorem 2, we have the following.

Corollary 3. *Let $G_{t,a}(z)/z(G_{t,a}(z))' \neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{1+\frac{z(G_{t,a}(z))'}{2G_{t,a}(z)}\left(1+\frac{z(G_{t,a}(z))'}{G_{t,a}(z)}\right) - \frac{z(G_{t,a}(z))'+(1/2)(G_{t,a}(z))''}{G_{t,a}(z)}\right\} < M, \quad (22)$$

where

$$1 < M \le \frac{n}{4\log 2}. \quad (23)$$

Then

$$\Re\left\{\frac{z(G_{t,a}(z))'}{G_{t,a}(z)}\right\} > 1-\frac{4(M-1)}{n}\log 2, \quad (z\in U^*). \quad (24)$$

The bound in (24) *is the best possible.*

Letting $M = 1 + n/4\log 2$ in Corollary 3, we have the following.

Corollary 4. *Let $G_{t,a}(z)/z(G_{t,a}(z))' \neq 0$ and $t = 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{1+\frac{zf'(z)}{2f(z)}\left(1+\frac{zf'(z)}{f(z)}\right)-\frac{zf'(z)+(1/2)z^2f''(z)}{f(z)}\right\} < 1+\frac{n}{4\log 2}. \quad (25)$$

Then $f(z)$ is starlike in U^*.

Theorem 5. *Let $\delta(\alpha+1) > 0$, $zL_a^t(\alpha+1,\beta)f(z) \neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{\left(zL_a^t(\alpha+1,\beta)f(z)\right)^{\delta}\left(\frac{L_a^t(\alpha+2,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right)\right\} < M, \quad (26)$$

where

$$1 < M \le \frac{n}{2\delta(\alpha+1)\log 2}. \quad (27)$$

Then

$$\Re\left(zL_a^t(\alpha+1,\beta)f(z)\right)^{\delta} > 1-\frac{2\delta(\alpha+1)(M-1)}{n}\log 2, \quad (z\in U^*). \quad (28)$$

The bound in (28) *is the best possible.*

Proof. Define the function $q(z)$ by

$$q(z) = \left(zL_a^t(\alpha+1,\beta)f(z)\right)^{\delta}. \quad (29)$$

Then, clearly $q(z) = 1+q_nz^n+q_{n+1}z^{n+1}+\cdots$ analytic function in U^* with $q(z)\neq 0$ for $z \in U^*$. It follows from (29) that

$$\frac{zq'(z)}{\delta q(z)} = \frac{z(L_a^t(\alpha+1,\beta)f(z))'}{L_a^t(\alpha+1,\beta)f(z)} - 1. \quad (30)$$

by making use of the familiar identity (11) in (30), we get

$$\frac{L_a^t(\alpha+2,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)} - 1 = \frac{1}{\delta(\alpha+1)}\frac{zq'(z)}{q(z)}, \quad (31)$$

or, equivalent

$$1+\frac{1}{\delta(\alpha+1)}\frac{zq'(z)}{q^2(z)} = \left(zL_a^t(\alpha+1,\beta)f(z)\right)^{\delta}\left(\frac{L_a^t(\alpha+2,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right). \quad (32)$$

Applying Lemma 1, with $a = 1/(1+\alpha)$, we get the required result. □

Letting $\alpha = \beta = 1$ in Theorem 5, we have

Corollary 6. *Let $\delta > 0, G_{t,a}(z)\neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{(zG_{t,a}(z))^{\delta}\left(\frac{z(G_{t,a}(z))'+(1/2)(G_{t,a}(z))''}{G_{t,a}(z)}\right)\right\} < M, \quad (33)$$

where

$$1 < M \le \frac{n}{4\delta\log 2}. \quad (34)$$

Then

$$\Re(zG_{t,a}(z))^{\delta} > 1-\frac{4\delta(M-1)}{n}\log 2, \quad (z\in U^*). \quad (35)$$

The bound in (35) *is the best possible.*

Letting $\delta = 1$, $M = 1+n/8\log 2$, and $t = 0$ in Corollary 6, we have the following.

Corollary 7. *Let $f'(z) \neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{zf(z)'\left(1+\frac{zf''(z)}{2f'(z)}\right)\right\} < 1+\frac{n}{8\log 2}. \quad (36)$$

Then

$$\Re\left\{zf(z)'\right\} > 0, \quad (z \in U^*). \quad (37)$$

The result is sharp.

Theorem 8. *Let $\xi > 0$, $z(L_a^t(\alpha+1,\beta)f(z))'/L_a^t(\alpha,\beta)f(z) \neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{1+\left(\frac{L_a^t(\alpha,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right)^{\xi} \times\left(\frac{(\alpha+1)L_a^t(\alpha+2,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right) - \alpha\left(\frac{L_a^t(\alpha,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right)^{\xi} - 1\right\} < M, \quad (38)$$

where

$$1 < M \leq 1+\frac{n}{2\xi\log 2}. \quad (39)$$

Then

$$\Re\left(\frac{L_a^t(\alpha,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right)^{\xi} > 1-\frac{2\xi(M-1)}{n}\log 2 \quad (z \in U^*). \quad (40)$$

The bound in (40) *is the best possible.*

Proof. Define the function $q(z)$ by

$$q(z) = \left(\frac{L_a^t(\alpha+1,\beta)f(z)}{L_a^t(\alpha,\beta)f(z)}\right)^{\xi}. \quad (41)$$

Then, clearly $q(z) = 1+q_nz^n+q_{n+1}z^{n+1}+\cdots$ analytic function in U^* with $q(z) \neq 0$ for $z \in U^*$. Also by a simple computation and by making use of the familiar identity (11), we find from (41) that

$$1+\frac{1}{\xi}\frac{zq'(z)}{q^2(z)} = 1+\left(\frac{L_a^t(\alpha,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right)^{\xi} \times\left(\frac{(\alpha+1)L_a^t(\alpha+2,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)} -\alpha\left(\frac{L_a^t(\alpha,\beta)f(z)}{L_a^t(\alpha+1,\beta)f(z)}\right)^{\xi} - 1\right). \quad (42)$$

Applying Lemma 1, with $a = 1/\xi$, we get the required result. □

Letting $\alpha = \beta = 1$ in Theorem 8, we have the following.

Corollary 9. *Let $\xi > 0$, $z(G_{t,a}(z))'/G_{t,a}(z) \neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{1+\left(\frac{G_{t,a}(z)}{z(G_{t,a}(z))'}\right)^{\xi} \times\left(1+\frac{z(G_{t,a}(z))''}{(G_{t,a}(z))}-\left(\frac{G_{t,a}(z)}{z(G_{t,a}(z))'}\right)^{\xi}\right)\right\} < M, \quad (43)$$

where

$$1 < M \leq 1+\frac{n}{2\xi\log 2}. \quad (44)$$

Then

$$\Re\left(\frac{G_{t,a}(z)}{z(G_{t,a}(z))'}\right)^{\xi} > 1-\frac{2\xi(M-1)}{n}\log 2, \quad (z \in U). \quad (45)$$

The bound in (45) *is the best possible.*

Letting $\xi = 1$, $M = 1+n/2\log 2$, and $t = 0$ in Corollary 9, we have the following.

Corollary 10. *Let $zf'(z)/f(z) \neq 0$ for $z \in U^*$ and suppose that*

$$\Re\left\{1+\left(\frac{f(z)}{zf'(z)}\right)^{\xi}\left(1+\frac{zf''(z)}{f'(z)}-\left(\frac{f(z)}{zf'(z)}\right)^{\xi}\right)\right\} < 1+\frac{n}{2\log 2}. \quad (46)$$

Then

$$\Re\left(\frac{f(z)}{zf'(z)}\right) > 0, \quad (z \in U^*). \quad (47)$$

The result is sharp.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Both authors read and approved the final paper.

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References

[1] S. G. Krantz, "Meromorphic functions and singularities at infinity," in *Handbook of Complex Variables*, pp. 63–68, Birkhäauser, Boston, Mass, USA, 1999.

[2] R. K. Pandey, *Applied Complex Analysis*, Discovery Publishing House, Grand Rapids, Mich, USA, 2008.

[3] H. M. Srivastava and A. A. Attiya, "An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination," *Integral Transforms and Special Functions*, vol. 18, no. 3, pp. 207–216, 2007.

[4] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic, Boston, Mass, USA, 2001.

[5] H. M. Srivastava, D. Jankov, T. K. Pogány, and R. K. Saxena, "Two-sided inequalities for the extended Hurwitz-Lerch Zeta function," *Computers and Mathematics with Applications*, vol. 62, no. 1, pp. 516–522, 2011.

[6] H. M. Srivastava, R. K. Saxena, T. K. Pogany, and R. Saxena, "Integral transforms and special functions," *Applied Mathematics and Computation*, vol. 22, no. 7, pp. 487–506, 2011.

[7] J. Dziok and H. M. Srivastava, "Certain subclasses of analytic functions associated with the generalized hypergeometric function," *Integral Transforms and Special Functions*, vol. 14, no. 1, pp. 7–18, 2003.

[8] F. Ghanim and M. Darus, "A new class of meromorphically analytic functions with applications to the generalized hypergeometric functions," *Abstract and Applied Analysis*, vol. 2011, Article ID 159405, 10 pages, 2011.

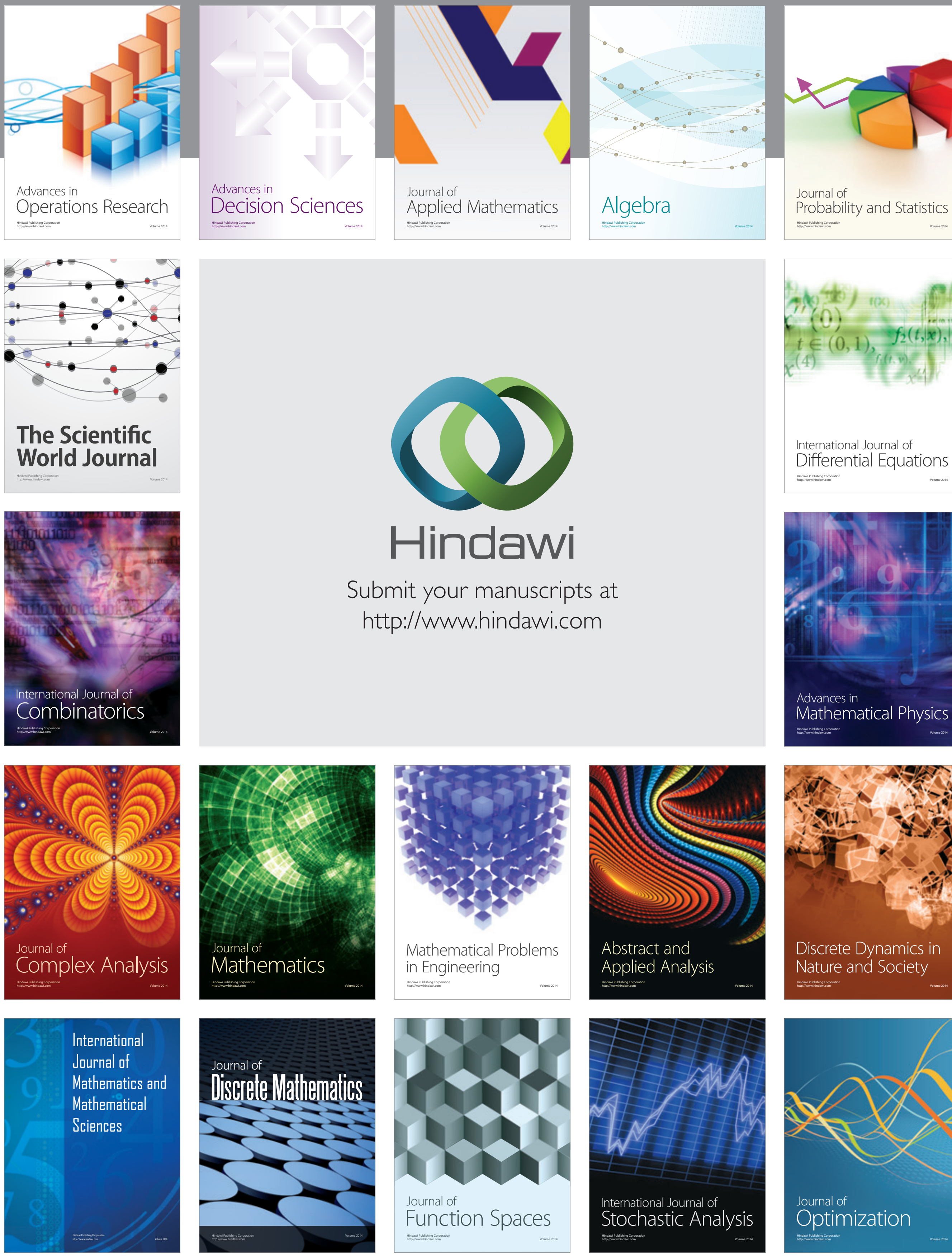
[9] F. Ghanim and M. Darus, "Some properties of certain subclass of meromorphically multivalent functions defined by linear operator," *Journal of Mathematics and Statistics*, vol. 6, no. 1, pp. 34–41, 2010.

[10] F. Ghanim and M. Darus, "Some properties on a certain class of meromorphic functions related to Cho-Kwon-Srivastava operator," *Asian-European Journal of Mathematics*, vol. 5, no. 4, Article ID 1250052, pp. 1–9, 2012.

[11] J.-L. Liu and H. M. Srivastava, "Certain properties of the Dziok-Srivastava operator," *Applied Mathematics and Computation*, vol. 159, no. 2, pp. 485–493, 2004.

[12] J.-L. Liu and H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hypergeometric function," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.

[13] D. Yang, "Some criteria for multivalent starlikeness," *Southeast Asian Bulletin of Mathematics*, vol. 24, no. 3, pp. 491–497, 2000.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