

ANYONS ON A TORUS

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Abstract

We prove the equivalence between anyon quantum mechanics on a torus and Chern-Simons gauge theory. It is also shown that the Hamiltonian and total momenta commute among themselves only in the physical Hilbert space.

INTRODUCTION

A few years ago Einarsson¹ gave braid group analysis of quantum mechanics of q anyons (with the statistics phase θ_s) on a torus. He showed that Schrödinger wavefunctions must have M -components, and that q , M , and θ_s must satisfy

$$e^{2iq\theta_s} = 1 = e^{2iM\theta_s} . \quad (1)$$

In particular, for $\theta_s = \pi/N$ (N : an integer), q and M must be multiples of N .

On the other hand it is known² that anyon quantum mechanics on a plane is equivalent to Chern-Simons gauge theory coupled to non-relativistic matter fields. Does the equivalence remain valid on a torus? If it does, how does the constraint (1) result from Chern-Simons gauge theory?

In this paper we show^{3,4} that the equivalence is exact and everything follows from Chern-Simons gauge theory.

CHERN-SIMONS THEORY

The Lagrangian is given by

$$\mathcal{L} = \frac{\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + i\psi^\dagger D_0 \psi - \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi) , \quad (2)$$

where $D_0 = \partial_0 + ia_0$ and $D_k = \partial_k - ia^k$. The Chern-Simons coefficient κ is related to θ_s by $\theta_s = \pi/\kappa$. To be definite, $\psi(x)$ is taken to be a fermion field.

On a torus ($0 < x_j < L_j$, $j=1,2$) there are two non-integrable phases of Wilson line integrals along non-contractible loops:

$$\exp \left(i \int_{C_j} d\mathbf{x} \cdot \mathbf{a} \right) \rightarrow W_j = e^{i\theta_j} . \quad (3)$$

They form a conjugate pair⁵:

$$[\theta_1, \theta_2] = \frac{2\pi i}{\kappa} . \quad (4)$$

Fields are not single-valued in general:

$$\begin{aligned} a_\mu [T_j x] &= a_\mu [x] + \partial_\mu \beta_j(x) \\ \psi [T_j x] &= e^{-i\beta_j(x)} \psi [x] \end{aligned} \quad (5)$$

where $T_1\mathbf{x} = (x_1 + L_1, x_2)$ etc. Field operators must be smooth on the covering space, and therefore $\psi[T_1T_2x] = \psi[T_2T_1x]$, from which the flux quantization condition follows:

$$\Phi = - \int d\mathbf{x} f_{12} = 2\pi m \quad (m : \text{integer}). \quad (6)$$

By solving Chern-Simons field equations $(\kappa/4\pi)\varepsilon^{\mu\nu\rho}f_{\nu\rho} = j^\mu$, $a_\mu(x)$ can be expressed in terms of $\theta_j(t)$ and $\psi(x)$. The resulting Hamiltonian is

$$H = \frac{1}{2m} \int d\mathbf{x} (D_k\psi)^\dagger (D_k\psi) ,$$

$$a^j(x) = \frac{\theta_j(t)}{L_j} - \frac{\Phi}{2L_1L_2} \epsilon^{jk} x_k +$$

$$\int d\mathbf{y} \epsilon^{jk} \partial_k^x G(\mathbf{x} - \mathbf{y}) \left(\frac{2\pi}{\kappa} \psi^\dagger \psi(y) + \frac{\Phi}{L_1L_2} \right) \quad (7)$$

where $G(\mathbf{r})$ is the periodic Green's function on a torus satisfying $\Delta G(\mathbf{r}) = \delta(\mathbf{r}) - (1/L_1L_2)$. Furthermore one has to impose a constraint on physical states,

$$Q + \frac{\kappa}{2\pi} \Phi \approx 0 \quad \left(Q = \int d\mathbf{x} \psi^\dagger \psi \right) \quad (8)$$

as (8), despite being a part of the original Chern-Simons field equations, does not follow from the Hamiltonian in (7).

VACUUM

The field theory defined by (7) and (8) with commutation relations for θ_j and $\psi(x)$ is invariant under large gauge transformations:

$$\theta_j \rightarrow \theta_j + 2\pi n_j ,$$

$$\psi(x) \rightarrow e^{2\pi i(n_1 x_1/L_1 + n_2 x_2/L_2)} \psi(x) \quad (9)$$

where $(n_1, n_2) = (1, 0)$ and $(0, 1)$. The associated unitary operators are given by

$$U_j = \exp \left\{ i\epsilon^{jk} \kappa \theta_k - 2\pi i \int d\mathbf{x} \frac{x_j}{L_j} \psi^\dagger \psi(x) \right\} . \quad (10)$$

U_j 's and W_j 's satisfy

$$U_1 U_2 = U_2 U_1 e^{-2\pi i \kappa} ,$$

$$W_1 W_2 = W_2 W_1 e^{-2\pi i / \kappa} . \quad (11)$$

Two gauge transformations U_1 and U_2 do not commute with each other in general.^{5,6} Consistent quantum theory is possible only if the coefficient κ is a rational number.⁷ Two cases are important, an integer κ in the anyon superconductivity and an inverse integer κ in the fractional quantum Hall effect.

Let us concentrate on the integer $\kappa = N$ case, in which U_1 and U_2 commute. As a consequence of (11) there are N degenerate vacua. Choosing $U_j|0_a\rangle = e^{i\alpha_j}|0_a\rangle$, one finds

$$W_1|0_a\rangle = e^{-i\alpha_2/N}|0_a\rangle ,$$

$$W_2|0_a\rangle = e^{+i\alpha_1/N}|0_{a-1}\rangle . \quad (12)$$

WAVEFUNCTIONS

A q -particle Schrödinger wavefunction in quantum mechanics is a matrix element of q field operators $\psi(x)$ between the vacuum and corresponding q -particle state $|\Psi_q\rangle$. One elaboration is necessary on a torus. It is given by

$$\phi_a^f(t; \mathbf{x}_1, \dots, \mathbf{x}_q) = \langle 0_a | \Omega \psi(1) \cdots \psi(q) | \Psi_q \rangle ,$$

$$\Omega = \exp \left\{ -i \sum_{p=1}^q \left(\frac{x_1^p}{L_1} \theta_1 + \frac{x_2^p}{L_2} \theta_2 \right) \right\} . \quad (13)$$

A couple of things should be noted. There are N degenerate vacua so that the wavefunction must have N -components: ϕ_a^f ($a = 1, \dots, N$). Secondly, the operator Ω is necessary in the definition of ϕ_a^f to make it invariant under large gauge transformation (9).

Recalling $\theta_s = \pi/\kappa$, we see that the constraint (1) is satisfied. We have just shown that $M=N$, and (6) and (8) imply that $q=mN$.

BRAID GROUP

The wavefunction ϕ^f in (13) satisfies the braid group algebra on a torus. There are three sets of operations on a torus: (a) σ_j : the (counterclock-wise) interchange of the j^{th} and $(j+1)^{\text{th}}$ particles, (b) τ_j : the loop transport of the j^{th} particle in the x_1 -direction, (c) ρ_j : the corresponding transport in the x_2 -direction. These three, σ_j , τ_j , and ρ_j satisfy the braid group algebra, from which Einarsson derived the aforementioned constraint.¹

Action of these operators on ϕ^f is simple:

$$\begin{aligned}\sigma_j &: \mathbf{x}_j \leftrightarrow \mathbf{x}_{j+1} \\ \tau_j &: \mathbf{x}_j \rightarrow T_1 \mathbf{x}_j \\ \rho_j &: \mathbf{x}_j \rightarrow T_2 \mathbf{x}_j\end{aligned}\quad (14)$$

In particular, since ψ is a fermion,

$$\sigma_j \phi^f = -\phi^f . \quad (15)$$

ϕ^f is the wavefunction in the fermion representation.

As a consequence of (14), the braid group algebra is trivially satisfied. However, ϕ^f is transformed quite nontrivially under the action of τ_j and ρ_j :

$$\begin{aligned}(\tau_j \phi^f)_a &= \exp \left[-i\beta_1(x^j) + \frac{i\pi}{N} \sum_p \frac{x_2^p}{L_2} + \frac{2\pi ia}{N} \right] \phi_a^f \\ (\rho_j \phi^f)_a &= \exp \left[-i\beta_2(x^j) - \frac{i\pi}{N} \sum_p \frac{x_1^p}{L_1} \right] \phi_{a-1}^f\end{aligned}\quad (16)$$

Notice that ϕ^f is a regular function of $\{\mathbf{x}^p\}$, without any singularity. Under τ_j and ρ_j , ϕ^f

picks up $\{\mathbf{x}^p\}$ dependent phases, natural in gauge theory. In Einarsson's analysis it was implicitly assumed that phases must be constant, which demands multi-valued wavefunctions.

SINGULAR TRANSFORMATION

Einarsson's wavefunction, ϕ^E , is related to ϕ^f by a singular gauge transformation.^{3,8,9}

$$\begin{aligned}\phi_a^E &= \Omega_{\text{sing}} \phi_a^f \\ \Omega_{\text{sing}} &= \prod_{j < k} \left[\frac{\vartheta_1(w_{jk})}{\vartheta_1(\bar{w}_{jk})} \right]^{\frac{1}{2N}} \cdot e^{i\pi x_1^{jk} x_2^{jk} / L_1 L_2}\end{aligned}\quad (17)$$

where $\vartheta_1(w)$ is Jacobi's theta function, $x_1^{jk} = x_1^j - x_1^k$, $w = (x_1 + ix_2)/L_1$, etc..

It is straightforward to see that

$$\sigma_j \phi^E = -e^{-i\pi/N} \phi^E . \quad (18)$$

The action of τ_j and ρ_j is somewhat simplified. The result generalizes Einarsson's to arbitrary particle configurations. All topological information is contained in Ω_{sing} .

SCHRÖDINGER EQUATION

The Schrödinger equation for ϕ^f

$$\begin{aligned}i \frac{\partial}{\partial t} \phi_a^f(t; \mathbf{x}_1, \dots, \mathbf{x}_q) &= \hat{H} \phi_a^f \\ &= (q!)^{-1/2} \langle 0_a | \Omega \psi(1) \dots \psi(q) H | \Psi_q \rangle\end{aligned}\quad (19)$$

is obtained by permuting H (defined in(7)) to the left of Ω and $\psi(j)$'s. \hat{H} is given by

$$\begin{aligned}\hat{H} &= -\frac{1}{2m} \sum_j (\nabla^{(j)} - i\mathbf{A}^{(j)})^2 \\ \mathbf{A}^{(j)k} &= \epsilon^{kl} \frac{2\pi}{N} \sum_{p \neq j} \left(\frac{x_l^j - x_l^p}{2L_1 L_2} \right. \\ &\quad \left. + \nabla_l^{(j)} G(\mathbf{x}^j - \mathbf{x}^p) \right).\end{aligned}\quad (20)$$

The equation for ϕ^E is obtained by inserting (17) into (20). The result is very simple:

$$i \frac{\partial}{\partial t} \phi_a^E = -\frac{1}{2m} \sum_j (\nabla_k^{(j)})^2 \phi^E . \quad (21)$$

It is a “free” equation. The anyon interaction is hidden in the boundary condition (18).

TRANSLATION INVARIANCE

The total momentum operator in the second quantized theory is given by

$$P^k = -i \int d\mathbf{x} \psi^\dagger D_k \psi . \quad (22)$$

The corresponding operator in quantum mechanics is found to be

$$\hat{P}^k = -i \sum_j \nabla_k^{(j)} , \quad (23)$$

where $\hat{P}^k \phi_a^f = (q!)^{-1/2} \langle 0_a | \dots P^k | \Psi_q \rangle$.

P^k and H in (7) form an algebra:

$$\begin{aligned} [P^j, P^k] &= i\epsilon^{jk} \frac{2\pi}{\kappa L_1 L_2} Q \left(Q + \frac{\kappa}{2\pi} \Phi \right) , \\ [P^j, H] &= i\epsilon^{jk} \frac{2\pi}{\kappa L_1 L_2} J^k \left(Q + \frac{\kappa}{2\pi} \Phi \right) , \end{aligned} \quad (24)$$

where $J^k = \int d\mathbf{x} j^k$ and $=P^k/m$ in the nonrelativistic theory. They do not commute among themselves as operators, but do commute in the physical Hilbert space defined by the constraint (8).

Hence the translation invariance is maintained in the Hilbert space. Obviously the corresponding operators in quantum mechanics, \hat{P}^j and \hat{H} , commute with each other.¹⁰

Another important set of commutators are

$$\begin{aligned} [W_j, P^k] &= \epsilon^{jk} \frac{\pi}{\kappa L_1 L_2} \{Q, W_j\} , \\ [W_j, H] &= \epsilon^{jk} \frac{\pi}{\kappa L_1 L_2} \{J^k, W_j\} . \end{aligned} \quad (25)$$

It turns out that the relations (24) and (25) remain valid even for relativistic theory with Dirac fields. They are universal relations in Chern-Simons theory.

SUMMARY

Chern-Simons gauge theory was born many years ago. It is simple, beautiful, and rich. It is important in the fractional quantum Hall effect and superconductivity. It is the most powerful and fruitful way of describing anyon physics, embodied with a unique algebraic structure. Much is hidden to be discovered in future.

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