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A FINITE ELEMENT METHOD FOR EXTERIOR INTERFACE PROBLEMS

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ABSTRACT. A procedure is given for the approximate solution of a class of two-dimensional diffraction problems. Here the usual inner boundary conditions are replaced by an inner region together with interface conditions. The interface problem is treated by a variational procedure into which the infinite region behavior is incorporated by the use of a non-local boundary condition over an auxiliary curve. The variational problem is formulated and existence of a solution established. Then a corresponding approximate variational problem is given and optimal convergence results established. Numerical results are presented which confirm the convergence rates.

KEY WORDS AND PHRASES. Approximation Property, Approximate Variational Problem, Convergence, Convergence Rate, Elliptic, Finite Elements, Galerkin, Helmholtz Equation, Integral Equation, Optimality, Potential Theory, Regularity

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1. INTRODUCTION.+

In [9] a method was presented for the numerical solution of some diffraction problems. We believe this method, a combination of variational procedures and integral equations, to be of quite wide applicability. To illustrate the method we discuss here an exterior interface problem for the Helmholtz equation. The main idea is to use integral equations to reduce diffraction problems in infinite regions to variational problems over finite domains but with non-local boundary conditions. In section four we indicate how the general method can be adapted to other situations.

Let Γ be a simple smooth closed curve dividing $\mathbb{R}^2$ into two open sets, a bounded region Ω_1, and an exterior region Ω_2. (i.e., we assume that $\Omega_2 \supset \{\underset{\sim}{x} \in \mathbb{R}^2 \mid |\underset{\sim}{x}| > R\}$ for some $R > 0$.) We begin with the problem

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Find u such that

$$\text{(P)}\qquad \begin{aligned} &\Delta u + k_1^2 u = f \quad \text{in } \Omega_1 \\ &\Delta u + k_2^2 u = 0 \quad \text{in } \Omega_2 \\ &u(\underset{\sim}{x}_o)^- = u(\underset{\sim}{x}_o)^+, \quad \underset{\sim}{x}_o \in \Gamma \\ &\alpha_1 \frac{\partial u}{\partial n}(\underset{\sim}{x}_o)^- = \alpha_2 \frac{\partial u}{\partial n}(\underset{\sim}{x}_o)^+, \quad \underset{\sim}{x}_o \in \Gamma \\ &\lim_{r\to\infty} r^{1/2}\left|\frac{\partial u}{\partial r} - ik_2 u\right| = 0. \end{aligned}$$

Here $\alpha_1, \alpha_2 > 0$ are constants, $\frac{\partial}{\partial n}$ denotes differentiation in the direction of $\hat{n}$, the unit outward normal to Γ and, for $\underset{\sim}{x}_o \in \Gamma$,

$$u(\underset{\sim}{x}_o)^- \equiv \lim_{\substack{\underset{\sim}{x}\to \underset{\sim}{x}_o \\ (\underset{\sim}{x}\in\Omega_1)}} u(\underset{\sim}{x})$$

$$u(\underset{\sim}{x}_o)^+ \equiv \lim_{\substack{\underset{\sim}{x}\to \underset{\sim}{x}_o \\ (\underset{\sim}{x}\in\Omega_2)}} u(\underset{\sim}{x}).$$

We attach similar meaning to the notation $\frac{\partial u^{\pm}}{\partial n}(\underset{\sim}{x}_o)$. Also the numbers k_1^2 and k_2^2 are complex constants with k_2 satisfying:

$$\text{Im}(k_2) \geq 0 \quad \text{and} \quad \text{Re}(k_2) > 0 \quad \text{if} \quad \text{Im}(k_2) = 0.$$

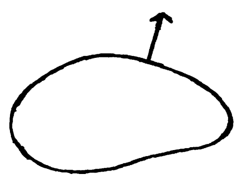

Figure 1.1

We refer the reader to [2] and [8] for existence and uniqueness results for problem P.

A physical realization of problem P is found in electromagnetic theory when the scattering of a time periodic incident wave from an infinite cylindrical conductor is considered. (See for example [4].)

The method that we present here is a mixed variational-integral equation technique for the interface problem. It is based on the introduction of a boundary condition which enables problem P to be reduced to a boundary value problem over a bounded subregion of $\mathbb{R}^2$. The boundary condition is described as follows.

Let Γ_∞ be a simple smooth closed curve and denote its exterior by A_∞. It is a classical result that the problem

$$(Q_k) \qquad \begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } A_\infty \\ u &= \varphi \quad \text{on } \Gamma_\infty \\ \lim_{r\to\infty} r^{1/2}\left|\frac{\partial u}{\partial r} - iku\right| &= 0 \end{aligned}$$

has a unique solution for any $k \in C$, $k \neq 0$ and for any function $\varphi : \Gamma \to C$. For given φ, k, and Γ_∞ we denote the solution of problem (Q_k) by $U_{\Gamma_\infty}(\underset{\sim}{x};k,\varphi)$ for $x \in A_\infty$ and define the operator

$$T_k[\varphi](\underset{\sim}{x}_o) \equiv \left\{\frac{\partial U_{\Gamma_\infty}}{\partial n}(\underset{\sim}{x}_o;k,\varphi)\right\}^+ \quad \text{for } \underset{\sim}{x}_o \in \Gamma_\infty . \tag{1.1}$$

We observe that the solution u of problem P is also a solution of the boundary value problem given below provided the curve Γ_∞ is chosen so that its interior contains $\Omega_1 \cup \Gamma$.

$$(\tilde{P}) \qquad \begin{aligned} \Delta u + k_1^2 u &= f \quad \text{in } \Omega_1 \\ \Delta u + k_2^2 u &= 0 \quad \text{in } \Omega_2^T \\ u(\underset{\sim}{x}_o)^- &= u(\underset{\sim}{x}_o)^+, \quad \underset{\sim}{x}_o \in \Gamma \\ \alpha_1 \frac{\partial u}{\partial n}(\underset{\sim}{x}_o)^- &= \alpha_2 \frac{\partial u}{\partial n}(\underset{\sim}{x}_o)^+, \quad \underset{\sim}{x}_o \in \Gamma \\ \frac{\partial u}{\partial x}(\underset{\sim}{x}_o)^- &= T_{k_2}[u|_{\Gamma_\infty}](\underset{\sim}{x}_o), \quad \underset{\sim}{x}_o \in \Gamma_\infty \end{aligned}$$

Here Ω_2^T denotes the annulus that lies between Γ and Γ_∞. This is shown in Figure 1.2.

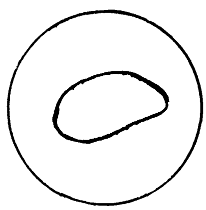

Figure 1.2

The validity of the boundary condition $\frac{\partial u}{\partial n}(\underset{\sim}{x}_o)^- = T_k[u|_{\Gamma_\infty}](\underset{\sim}{x}_o)$ for $\underset{\sim}{x}_o \in \Gamma_\infty$ for the solution of problem P follows from the fact that both u and $\frac{\partial u}{\partial n}$ are continuous across Γ_∞. Conversely, if u is a solution of problem $\tilde{P}$ we may extend it to a solution of problem P by defining $u = U_{\Gamma_\infty}(\underset{\sim}{x};k_2,u|_{\Gamma_\infty})$ for $\underset{\sim}{x} \in A_\infty$.

From here our plan is to approximate the solution of the problem P using the finite element method. The principle ingredient in the finite element method is a variational formulation of the problem which, here, we construct in a straightforward manner using Galerkin techniques. We begin this development by assuming that v is a trial function with

$$v \in C^{0}(\overline{\Omega_1 \cup \Omega_2^T}), \quad v|_{\Omega_1} \in C^{1}(\Omega), \quad v|_{\Omega_2^T} \in C^{1}(\Omega_2^T).$$

Then for u the solution of $\mathfrak{P}$ we have

$$\alpha_1 \int_{\Omega_1} (\Delta u + k_1^2 u)\bar{v}\, d\underset{\sim}{x} + \alpha_2 \int_{\Omega_2^T} (\Delta u + k_2^2 u)\bar{v}\, d\underset{\sim}{x} = \alpha_1 \int_{\Omega_1} f\bar{v}\, d\underset{\sim}{x}.$$

Integrating by parts and using the interface conditions together with the boundary condition on Γ_∞ we conclude that

$$\begin{aligned}\alpha_2 \int_{\Gamma_\infty} T_{k_2}[u|_{\Gamma_\infty}]\bar{v}\, ds &- \alpha_1 \int_{\Omega_1} (\nabla u \cdot \nabla\bar{v} - k_1^2 u\, \bar{v})\, d\underset{\sim}{x} \\ &- \alpha_2 \int_{\Omega_2^T} (\nabla u \cdot \nabla\bar{v} - k_2^2 u\, \bar{v})\, d\underset{\sim}{x} \qquad (1.2) \\ &= \alpha_1 \int_{\Omega_1} f\bar{v}\, d\underset{\sim}{x} \quad \text{for all such } v.\end{aligned}$$

We denote the left hand side of 1.2 by $a(u,v)$ and the right hand side by $F(v)$. Thus, u the solution of problem P satisfies the variational equation

$$a(u,v) = F(v) \tag{1.3}$$

for all $v \in C^{o}(\overline{\Omega_1 \cup \Omega_2^T})$, $v \in C^1(\Omega_1)$, $v \in C^1(\Omega_2^T)$.

With a variational formulation of P over a bounded region available, the next step is to introduce approximate variational problems. To do this we select a finite number of trial functions $\varphi_1^h, \varphi_2^h, \ldots, \varphi_N^h$ and set $S^h = \text{span}\{\varphi_1^h, \ldots, \varphi_N^h\}$. We then attempt to find a function $u^h \in S^h$ which satisfies

$$a(u^h, v_h) = F(v_h) \quad \text{for all} \quad v_h \in S^h. \tag{1.4}$$

The solution u^h of (1.4) is taken as an approximation to u.

The complication in the above procedure is the determination of the operator T_k. This can be done by integral equations and in particular by using integral representations for the solution. $U_{\Gamma_\infty}(\underset{\sim}{x};k,\varphi)$ of problem Q_k. It is shown in [6] that one can obtain $U_{\Gamma_\infty}(\underset{\sim}{x};k,\varphi)$ in the form

$$U_{\Gamma_\infty}(\underset{\sim}{x};k,\varphi) = \int_{\Gamma_\infty} \sigma(\underset{\sim}{y}) G_k(\underset{\sim}{x},\underset{\sim}{y})\, ds_{\underset{\sim}{y}} \tag{1.5}$$

where $G_k(\underset{\sim}{x},\underset{\sim}{y}) = -\, i/4\, H_o^{(1)}(k|\underset{\sim}{x}-\underset{\sim}{y}|)$, $H_o^{(1)}$ is the Hankel function of the first kind of order zero and σ is determined by the equation

$$K_k[\sigma](\underset{\sim}{x}_o) \equiv \int_{\Gamma_\infty} \sigma(\underset{\sim}{y}) G_k(\underset{\sim}{x},\underset{\sim}{y})\, ds_{\underset{\sim}{y}} = \varphi(\underset{\sim}{x}_o), \quad \underset{\sim}{x}_o \in \Gamma_\infty. \tag{1.6}$$

Equation (1.6) is a Fredholm integral equation of the first kind. It is shown in [6] that this equation is uniquely solvable and we set $\sigma = K_k^{-1}[\varphi]$. From the representation (1.5) and a standard result in potential theory one has

$$\begin{aligned} \frac{\partial U_{\Gamma_\infty}}{\partial n}(\underset{\sim}{x}_o;k,\varphi)^{+} &= \frac{1}{2}\sigma(\underset{\sim}{x}_o) + \int_{\Gamma_\infty} \sigma(\underset{\sim}{y}) \frac{\partial}{\partial n} G_k(\underset{\sim}{x}_o;\underset{\sim}{y})\, ds_{\underset{\sim}{y}} \\ &\equiv (\tfrac{1}{2} I + M_k)[\sigma](\underset{\sim}{x}_o) \quad \text{for} \quad \underset{\sim}{x}_o \in \Gamma_\infty . \end{aligned} \tag{1.7}$$

Thus we have the following characterization of T_k

$$T_k[\varphi] = (\tfrac{1}{2} I + M_k) K_k^{-1}[\varphi] . \tag{1.8}$$

The remainder of the paper proceeds as follows. In section two we describe the variational procedure precisely, and we state the convergence results. The proof of these is reduced to two coercivity inequalities. The verification of these is extremely technical and postponed to section five and the appendix.

In section three we discuss the implementation of the method including an approximate treatment of the operator T_k. We report on some numerical experiments which confirm our estimates for convergence rates.

Section four contains a brief discussion of other problems to which the method applies.

The authors wish to express their appreciation to Professor G. J. Fix for his help in the development of this paper.

2. VARIATIONAL FORMULATION

For any region Z we denote by $H^k(Z)$ the space of complex valued functions on Z with square integrable derivatives of order $\leq k$ and we write

$$\|f\|^2_{k,Z} = \sum_{|\alpha|\leq k} \int_Z |D^\alpha f|^2 d\underset{\sim}{x} \quad \text{for} \quad f\in H^k(Z). \tag{2.1}$$

For closed curves γ we also need the boundary spaces $H^r(\gamma)$, $r \in \mathbb{R}$. It is known (see [1]) that if ∂Z is smooth then there are continuous mappings

$$u \to u|_{\partial Z} : H^k(Z) \to H^{k-1/2}(\partial Z). \tag{2.2}$$

For our variational formulation we will need a space H_E which we define as follows

$$H_E = \{v|v|_{\Omega_1}\in H^1(\Omega_1), v|_{\Omega_2^T}\in H^1(\Omega_2^T), v^-(\underset{\sim}{x}_o) = v^+(\underset{\sim}{x}_o), \underset{\sim}{x}_o\in\Gamma\}.$$

We also define the norms $|||\cdot|||_j$ on H_E by

$$|||v|||^2_j = \|v\|^2_{j,\Omega_1} + \|v\|^2_{j,\Omega_2^T}$$

and note that H_E is a Hilbert space under the norm $|||\cdot|||_1$.

To proceed with the variational formulation we need the following properties of T_k, which are proved in the appendix.

LEMMA 1. T_k is a bounded linear operator from $H^r(\Gamma_\infty)$ to $H^{r-1}(\Gamma_\infty)$ and satisfies

$$\int_{\Gamma_\infty} T_k(\varphi)\psi \, ds = \int_{\Gamma_\infty} \varphi T_k(\psi) ds \tag{2.3}$$

for all $\varphi, \psi \in H^{1/2}(\Gamma_\infty)$.

With this we observe that the bilinear form

$$\begin{aligned} a(u,v) \equiv \alpha_2 \int_{\Gamma_\infty} T_{k_2}(u|_{\Gamma_\infty})\bar{v} \, ds - \alpha_1 \int_{\Omega_1} (\nabla u \cdot \nabla \bar{v} - k_1^2 u\bar{v}) d\underset{\sim}{x} \\ - \alpha_2 \int_{\Omega_2^T} (\nabla u \cdot \nabla \bar{v} - k_2^2 u\bar{v}) d\underset{\sim}{x} \end{aligned} \tag{2.4}$$

is well defined on $H_E \times H_E$. By (2.2) $u|_{\Gamma_\infty}$ and $\bar{v}$ are in $H^{1/2}(\Gamma_\infty)$. By Lemma 1 then $T_k(u|_{\Gamma_\infty})$ is in $H^{-1/2}(\Gamma_\infty)$ with

$$\|T_{k_2}(u|_{\Gamma_\infty})\|_{-1/2,\Gamma_\infty} \leq C\|u\|_{1/2,\Gamma_\infty} .$$

Hence by the generalized Schwarz inequality and (2.2)

$$\left|\int_{\Gamma_\infty} T_{k_2}(u|_{\Gamma_\infty})\bar{v} \, ds\right| \leq C\|u\|_{1/2,\Gamma_\infty} \|v\|_{1/2,\Gamma_\infty} \leq C' |||u|||_1 \, |||v|||_1 . \tag{2.5}$$

Thus $a(u,v)$ is well defined and

$$|a(u,v)| \leq C |||u|||_1 \, |||v|||_1 \tag{2.6}$$

showing that $a(\cdot,\cdot) : H_E \times H_E \to C$ is bounded. We may also comment here that if $f \in L^2(\Omega_1) \equiv H^o(\Omega_1)$ then

$$F(v) = \int_{\Omega_1} f\bar{v} \, d\underset{\sim}{x}$$

is a bounded linear functional on H_E.

The variational form of problem $\tilde{\mathcal{P}}$ that we use is stated as follows

$$\text{Find } u \in H_E \text{ such that}$$

$$(\mathrm{V}\tilde{\mathcal{P}}) \qquad a(u,v) = F(v)$$

$$\text{for all } v \in H_E.$$

Next we state the approximate problems. We suppose that $\{S_E^h\}_{0<h<1}$ is a family of finite dimensional subspaces of H_E which satisfy the following:

APPROXIMATION PROPERTY. There exists an integer $t \geq 2$ and positive constants C_o and C_1 such that for any $u \in H_E$ with $|||u|||_\ell < \infty$, $\ell \leq t$, there exists a function $u^* \in S_E^h$ which satisfies

$$|||u-u^*|||_j \leq C_j h^{\ell-j} |||u|||_\ell \quad j = 0,1 \tag{2.7}$$

(the constants C_o, C_1 are independent of h and u).
With such a family $\{S_E^h\}$ we pose the approximate problems:

$$\text{Find } u^h \in S_E^h \text{ such that}$$

$$(\mathrm{AV}\tilde{\mathcal{P}}) \qquad a(u^h,v_h) = F(v_h)$$

$$\text{for all } v_h \in S_E^h .$$

Our main results concerning problems $\mathrm{V}\tilde{\mathcal{P}}$ and $\mathrm{AV}\tilde{\mathcal{P}}$ are:

THEOREM 1. There exists a unique solution u of problem $V\tilde{P}$ and there exists an $h_o > 0$ such that problem $AV\tilde{P}$ has a unique solution u^h whenever $h < h_o$.

THEOREM 2. There exists constants C_o and $C_1 > 0$ such that, for $h < h_o$

$$|||u-u^h|||_1 \leq C_1 |||u-w_h|||_1 \quad \text{for all} \quad w_h \in S_E^h \tag{2.8}$$

and

$$|||u-u^h|||_o \leq C_o h |||u-u^h|||_1. \tag{2.9}$$

The constants C_o and C_1 are independent of u and $h < h_o$.

Theorem 2 is an optimality result and is typical for finite element methods applied to elliptic problems. If we use the approximation property of $\{S_E^h\}$, together with the regularity of the solution u, we obtain

COROLLARY 1. Suppose that $f \in H^{\ell-2}(\Omega_1)$ with $2 \leq \ell \leq t$ then for $h < h_o$ there are constants C_j, $j = 0,1$, independent of $h < h_o$ such that

$$|||u-u^h|||_j \leq C_j h^{\ell-j}. \tag{2.10}$$

PROOF OF COROLLARY 1. Regularity results for problem P show that if $f \in H^{\ell-2}(\Omega_1)$ then $u|_{\Omega_1} \in H^{\ell}(\Omega_1)$ and $u|_{\Omega_2^T} \in H^{\ell}(\Omega_2^T)$ thus $|||u|||_\ell < \infty$. By (2.7) we have that

$$\inf_{w_h \in S_E^h} |||u-w_h|||_1 \leq C_1 h^{\ell-1} |||u|||_\ell.$$

Applying this in (2.8) we find that

$$|||u-u^h|||_1 \leq C_1 h^{\ell-1} |||u|||_\ell . \tag{2.11}$$

From (2.9) and (2.11) we obtain (2.10) for $j = 0$, i.e.,

$$|||u-u^h|||_o \leq C_o h^{\ell} |||u|||_\ell .$$

The proofs of Theorems 1 and 2 are complicated by the fact that the variational problem is not positive definite. Hence we need the following rather technical result which is treated in section five.

THEOREM 3. There exists constants $h_o, C_a > 0$ such that

$$\sup_{0\neq v\in H_E} \frac{|a(u,v)|}{|||v|||_1} \geq C_a |||u|||_1 \quad \text{for all} \quad u\in H_E \tag{2.12}$$

$$\sup_{0\neq v_h\in S_E^h} \frac{|a(u_h,v_h)|}{|||v_h|||_1} \geq C_a |||u_h|||_1 \quad \text{for all} \quad u_h\in S_E^h . \tag{2.13}$$

Once (2.12) and (2.13) are established we may use these estimates in a standard way to prove the existence and uniqueness results stated in Theorem 1 (see [1]). Before proving (2.12) and (2.13) we will first show that 2.13 yields 2.8 of Theorem 2. (The result 2.9 will be treated in section five.)

From the formulation of problems $V\mathcal{P}$ and $AV\mathcal{P}$ we have $a(u,v_h) = F(v_h) = a(u^h,v_h)$ for all $v_h \in S_E^h$. Thus, for any $w_h \in S_E^h$

$$\frac{|a(u^h-w_h,v_h)|}{|||v_h|||_1} = \frac{|a(u-w_h,v_h)|}{|||v_h|||_1}. \tag{2.14}$$

The right hand side of (2.14) is bounded above (using (2.6)) by $C|||u-w_h|||_1$. By taking the supremum over $v_h \in S_E^h$, $v_h \neq 0$ and applying (2.13) we obtain

$$|||u^h-w_h|||_1 \leq C'|||u-w_h|||_1. \tag{2.15}$$

The triangle inequality applied to $|||u-u^h|||_1$ gives

$$|||u-u^h|||_1 \leq |||u-w_h|||_1 + |||u^h-w_h|||_1.$$

Using tnis and (2.15) we obtain

$$|||u-u^h|||_1 \leq (1 + C')|||u-w_h|||_1.$$

Thus (2.13) implies (2.8).

3. IMPLEMENTATION OF THE METHOD.

The approximate problem

Find $u^h \in S_E^h$ such that

(AV$\tilde{P}$)
$$a(u^h,v_h) = F(v_h)$$
for all $v_h \in S_E^h$

is seen to be equivalent to a matrix problem by selecting a basis $\{\varphi_1,\varphi_2,\ldots,\varphi_{N_H}\}$ for S_E^h.

We find a function u^h given by

$$u^h = \sum_{j=1}^{N_h} q_j\varphi_j \tag{3.1}$$

which satisfies

$$a(u^h,\varphi_i) = F(\varphi_i), \quad i = i,\ldots,N_h. \tag{3.2}$$

The system of equations (3.2) is the matrix problem

$$K\underset{\sim}{q} = \underset{\sim}{f} \tag{3.3}$$

where $q = (q_1,\ldots,q_{N_H})^T$ is the vector of weights in (3.1),

$\underset{\sim}{f} = (F(\varphi_1),\ldots,F(\varphi_{N_H}))^T$ is the source term

and

$K = (K_{ij})$ is the stiffness matrix

with entries

$$K_{ij} = a(\varphi_j,\varphi_i) = \alpha_2 \int_{\Gamma_\infty} T(\varphi_j|_{\Gamma_\infty})\overline{\varphi}_i ds - \alpha_1 \int_{\Omega_1} (\nabla\varphi_j\cdot\nabla\overline{\varphi}_i - k_1^2\varphi_j\overline{\varphi}_i)d\underset{\sim}{x}$$
$$- \alpha_2 \int_{\Omega_2} (\nabla\varphi_j\cdot\nabla\overline{\varphi}_i - k_2^2\varphi_j\overline{\varphi}_i)d\underset{\sim}{x}. \tag{3.4}$$

To use the ideas presented so far in actual computation we must be able to impose the nonlocal boundary condition

$$\frac{\partial u}{\partial n} = T_{k_2}(u|\Gamma_\infty)$$

along the outer boundary Γ_∞. We see from 3.4 that, in the approximate variational problem, this amounts to computing the integrals

$$\int_{\Gamma_\infty} T_k(\varphi_i|_{\Gamma_\infty})\varphi_i ds \tag{3.5}$$

for the basis functions $\varphi_1, \varphi_2, \ldots, \varphi_{N_h}$ of the approximation space S_E^h. This computation may be carried out in a straightforward manner according to the definition of T_k by solving the integral equations

$$\int_{\Gamma_\infty} \sigma_i(\underset{\sim}{y}) G_k(\underset{\sim}{x},\underset{\sim}{y}) ds_{\underset{\sim}{y}} = \varphi_i(\underset{\sim}{x}), \quad \underset{\sim}{x} \in \Gamma_\infty, \quad i = 1, \ldots, N_h \tag{3.6}$$

for the densities σ_i, computing $T_k(\varphi_i)(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \Gamma_\infty$ from the formula

$$T_k[\varphi_i](\underset{\sim}{x}) = \frac{1}{2}\sigma_i(\underset{\sim}{x}) + \int_{\Gamma_\infty} \sigma_i(\underset{\sim}{y}) \frac{\partial}{\partial n} G_k(\underset{\sim}{x},\underset{\sim}{y}) ds_{\underset{\sim}{y}} \tag{3.7}$$

and finally computing the integrals 3.5 using a suitable quadrature rule. The execution of this procedure for general finite element spaces S_E^h is a lengthy process at best. Fortunately the matter of computing the integrals 3.5 can be greatly simplified by making special choices of the curve Γ_∞ and the approximation spaces S_E^h. In the following discussion we take Γ_∞ to be a circle and choose S_E^h so that the restrictions of the trial functions to Γ_∞ are piecewise linear functions of arclength corresponding to a uniform mesh along Γ_∞.

Figure 3.1 shows the region Ω_2^T when Γ_∞ is a circle of radius R. We assume that a finite element space $S_E^h = \text{span}\{\varphi_1,\ldots,\varphi_{N_h}\}$ has been chosen so that $\varphi_i|_{\Gamma_\infty}$ is that piecewise linear function of arclength along Γ_∞ which either vanishes identically along Γ_∞ or is equal to one at one node on Γ_∞ and vanishes at the remaining nodes.

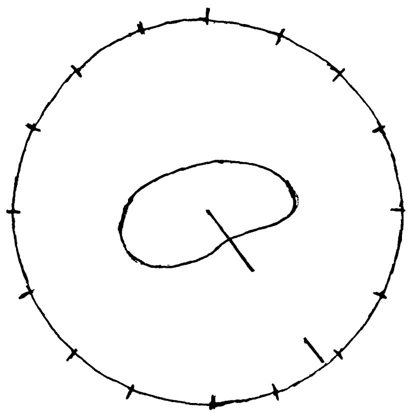

Figure 3.1

If we set $\theta_j = h_1 j$, $j = 1,2,\ldots,N_1$ where $h_1 = \frac{2\pi}{N_1}$ we may characterize each $\varphi_i|_{\Gamma_\infty}$ (which does not vanish identically on Γ_∞) in terms of the polar angle θ as a translation of a function $\psi_o(\theta)$ where ψ_o is the 2π-periodic extension of the function defined by

$$\psi_o(\theta) = \begin{cases} -\theta/h_1 + 1 & 0 \le \theta \le h_1 \\ \theta/h_1 + 1 & -h_1 \le \theta \le 0 \\ 0 & h_1 \le |\theta| \le \pi. \end{cases}$$

We have, for those φ_i's which do not vanish on Γ_∞

$$\varphi_i(R\cos\theta, R\sin\theta) = \psi_o(\theta - h_1 m_i) \tag{3.8}$$

for some integer m_i, $1 \leq m_i \leq N_1$. (By renumbering the φ_i's we may assume that $m_i = i$.) We note that, by solving elementary boundary value problems for the circle, one obtains the formulas,

$$T_k(\cos(n\theta + \alpha)) = k\,\frac{H_n^{(1)\nabla}(kR)}{H_n^{(1)}(kR)}\cos(n\theta + \alpha)$$

for $n = 0,1,2,\ldots$ where $H_n^{(1)}$ is the Hankel function of the first kind of order n. (The superscript "∇" denotes differentiation with respect to the argument.) Then, if one expands ψ_o in a Fourier series one obtains, after some algebraic rearrangement, see [9],

$$\int_{\Gamma_\infty} T_k[\varphi_j|_{\Gamma_\infty}]\varphi_i ds = R\Big[\frac{kH_o^{(1)\nabla}(kR)}{2\pi}h_1^2 + \frac{1}{\pi h_1^2}\Delta_{h_1}^4\{T_k[p](\theta)\}\big|_{\theta=(\ell-2)h_1}\Big] \tag{3.9}$$

where

$$\Delta_{h_1} f(\theta) = f(\theta + h_1) - f(\theta)$$

is the forward difference operator,

$$\ell = |i - j|$$

and

$$p(\theta) = \frac{\pi^4}{90} - \frac{\pi^2|\theta|^2}{12} + \frac{\pi|\theta|^3}{12} - \frac{|\theta|^4}{48}, \quad |\theta| \leq 2\pi. \tag{3.10}$$

From the formula 3.9 we see that to compute all of the integrals 3.5 in the special case under consideration we need only compute $T_k[p](\theta)$ at $\theta = \theta_j$, $j = 1,\ldots,N_1$ for the single function $p(\theta)$ defined by 3.10. This amounts to first solving the integral equation

$$-\frac{Ri}{4}\int_0^{2\pi} \sigma(t)H_o^{(1)}(2kR|\sin\frac{\theta-t}{2}|)dt = p(\theta) \quad 0 \leq \theta \leq 2\pi \quad (3.11)$$

for the density $\sigma(t)$. (We have specialized to the case when Γ_∞ is a circle of radius R and used the fact that

$$G_k(\underset{\sim}{x},\underset{\sim}{y}) = -\frac{i}{4}H_o^{(1)}(2kR|\sin\frac{\theta-t}{2}|)$$

when $\underset{\sim}{x} = (R\cos\theta, R\sin\theta)$, $\underset{\sim}{y} = (R\cos t, R\sin t)$ are points on Γ_∞.) Once $\sigma(t)$, $0 \leq t \leq 2\pi$ is determined $T_k(p)(\theta)$ is found from the formula: (again specialized to the case when $\Gamma_\infty = \{\underset{\sim}{x} \mid |\underset{\sim}{x}| = R\}$)

$$T_k(p)(\theta) = \frac{1}{2}\sigma(\theta) + \frac{Ri^k}{2}\int_0^{2\pi}\sigma(t)H_1^{(1)}(2kR|\sin\frac{\theta-t}{2}|)|\sin\frac{\theta-t}{2}|dt. \quad (3.12)$$

The kernel of the integral operator in 3.8 is obtained using the fact that

$$\frac{\partial}{\partial n_{\underset{\sim}{x}}} G_k(\underset{\sim}{x},\underset{\sim}{y}) = -\frac{ik}{4}H_o^{(1)\nabla}(2kR|\sin\frac{\theta-t}{2}|)|\sin\frac{\theta-t}{2}|$$

when $\underset{\sim}{x} = (R\cos\theta, R\sin\theta)$ and $\underset{\sim}{y} = (R\cos t, R\sin t)$ are on Γ_∞ together with the identity

$$H_o^{(1)\nabla}(\cdot) = -H_1^{(1)}(\cdot).$$

We may also observe that this kernel is continuous at $\theta = t$, in fact,

$$\lim_{\theta\to t} H_1^{(1)}(2kR|\sin\frac{\theta-t}{2}|)|\sin\frac{\theta-t}{2}| = -\frac{i}{\pi R}\ .$$

In the numerical examples that follow the equation, 3.11 was solved using numerical methods described in [6] with Simpson's rule replaced by the rectangular quadrature rule. With this modification the discretized form of 3.11 is a matrix problem

$$A\underset{\sim}{\sigma} = \underset{\sim}{p}$$

with A a circulant matrix. This feature enables the problem to be solved efficiently using well known inversion formulas for circulants (see [5] or [10]).

To verify the convergence rates predicted by the theory we consider the following example for various values of α_1, α_2, k_1, k_2.

Find u such that

$$\left\{\begin{array}{ll} \Delta u + k_1^2 u = 0 & 1 < |\underline{x}| < 2 \\ \Delta u + k_2^2 u = 0 & |\underline{x}| > 2 \\ u = 1 \quad \text{on} \quad |\underline{x}| = 1 & \\ u^+ = u^- \quad \text{on} \quad |\underline{x}| = 2 & \\ \alpha_2\left(\frac{\partial u}{\partial n}\right)^+ = \alpha_1\left(\frac{\partial u}{\partial n}\right)^- \quad \text{on} \quad |\underline{x}| = 2 & \\ \lim_{r\to\infty} r^{1/2}\left|\frac{\partial u}{\partial n} - ik_2 u\right| = 0. & \end{array}\right. \tag{3.14}$$

In this example the curve Γ_∞ is a circle with radius greater than two (we use $R = 3$). Following our procedures we construct the boundary value problem

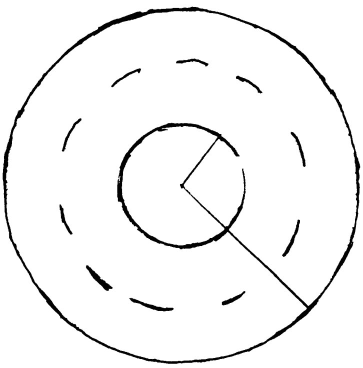

Figure 3.2

$$\Delta u + k_1^2 u = 0 \qquad 1 < |\underset{\sim}{x}| < 2$$

$$\Delta u + k_2^2 u = 0 \qquad 2 < |\underset{\sim}{x}| < 3$$

$$u = 1 \quad \text{on} \quad |\underset{\sim}{x}| = 1$$

$$u^+ = u^- \quad \text{on} \quad |\underset{\sim}{x}| = 2$$

$$\alpha_2 \left(\frac{\partial u}{\partial n}\right)^+ = \alpha_1 \left(\frac{\partial u}{\partial n}\right)^- \quad \text{on} \quad |\underset{\sim}{x}| = 2$$

$$\frac{\partial u}{\partial n} = T_{k_2}[u|_{|\underset{\sim}{x}|=R}] \quad \text{on} \quad |\underset{\sim}{x}| = 3.$$

The approximation spaces $\{S_E^h\}$ used here are sets of piecewise linear functions of the polar coordinates r, θ. They may be constructed by first mapping the region $\{\underset{\sim}{x} | 1 < |\underset{\sim}{x}| < R\}$ into the rectangle $[0, 2\pi] \times [1, R]$ in the r-θ coordinate system. We then construct piecewise linear finite element spaces (composed of 2π-periodic functions) corresponding to triangulations of the rectangle and transform back to rectangular coordinates. We thus obtain a distorted triangular grid with associated trial functions which are linear in r and θ. The resulting family of subspaces $\{S_E^h\}$ (n = maximum diameter of the triangles) satisfy the approximation property with $t = 2$. According to Corollary 1 we should observe that

$$|||u-u^h|||_o = O(h^2)$$

and

$$|||u-u^h|||_1 = O(h)$$

for this family $\{S_E^h\}$. Our examples are chosen so that the exact solutions are known and we measure convergence rates by computing $|||u^h-u^I|||_o$ and $|||u^h-u^I|||_1$ where u^I is the interpolant of the exact solution in S_E^h. From approximation theory we have that

$$|||u-u^I|||_o = O(h^2) \quad \text{and} \quad |||u-u^I|||_1 = O(h).$$

Using this and the triangle inequality we may show that the errors $|||u-u^h|||_o$ and $|||u-u^h|||_1$ will be optimal order ($O(h^2)$ and $O(h)$, respectively) if we observe in the calculations that

$$|||u^h-u^I|||_o \leq c_o h^2 \quad \text{and} \quad |||u^h-u^I|||_1 < c_1 h. \tag{3.15}$$

We display the results graphically in Figure 3.4 and Figure 3.5 by plotting

$$|||u^h-u^I|||_j \quad \text{vs} \quad 1/h \qquad j = 0,1$$

on a log-log scale. A slope of -2 (-1) indicates quadratic (linear) convergence. Eight trials were conducted. The values of α_1, α_2, k_1 and k_2 used to solve problem 3.14 in these cases are listed in Table 3.1.

TRIAL	α_1	α_2	k_1	k_2
1	1	4	1	2
2	1	2	1	2
3	1	4	1	4
4	1	2	1	4
5	1	4	1	10
6	2	1	1	4
7	4	1	1	4
8	4	1	1	10

Table 3.1

Figure 3.4 shows $|||u^h - u^I|||_o$ vs $1/h$ and we observe in every case, for sufficiently small h, that the convergence is quadratic. In Figure 3.5 $|||u^h - u^I|||_1$ is plotted against $1/h$. Here the slopes of the curves lie between -1 and -2 indicating that

$$|||u^h - u^I|||_1 \leq c_1 h.$$

Thus, from the remarks preceding 3.15, we observe that the convergence rates are optimal.

4. EXTENSIONS OF THE METHOD.

The particular problem studied here was chosen for illustrative purposes only. It demonstrates the power of variational methods to handle complicated situations on finite regions and the ability of integral equations to deal with infinite regions. We sketch a few more examples.

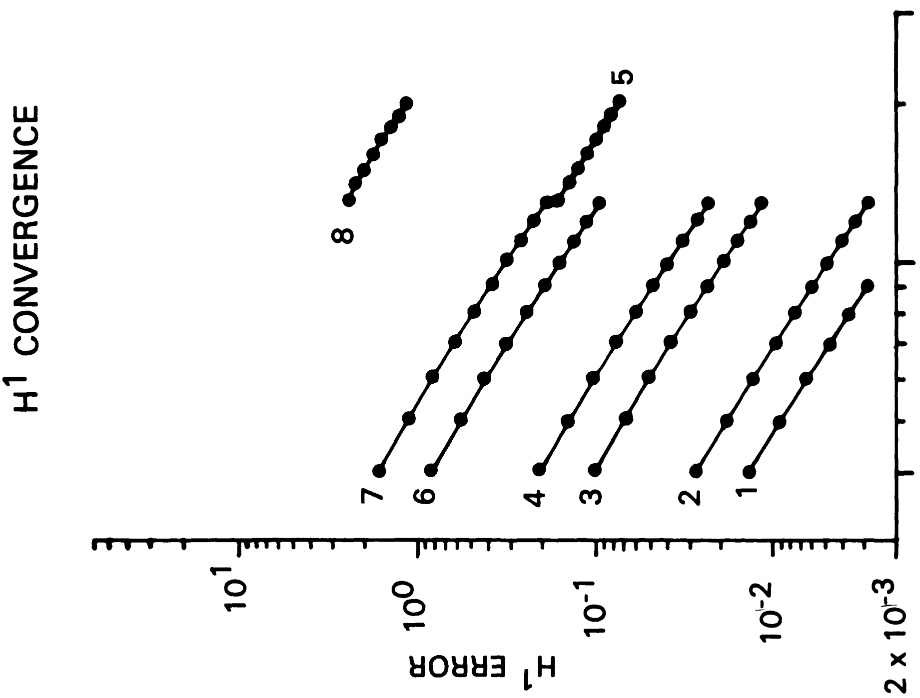

FIGURE 3.5

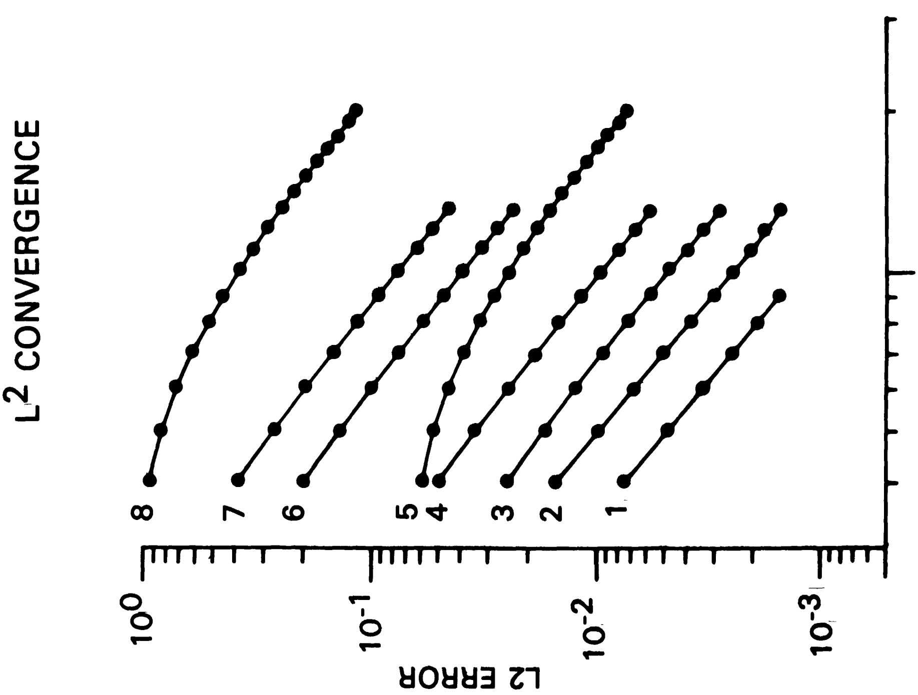

FIGURE 3.4

We note first that all the standard exterior problems for the Helmholtz equation can be treated. That is one can solve the problem $\Delta u + k^2 u = 0$ in Ω_2 with the radiation condition and any combination of Dirichlet, Neumann or mixed data given on Γ_2.

The next observation is that the exterior problem can also be treated in the case of variable coefficients. Consider the equation,

$$\operatorname{div}(A(\underset{\sim}{x})\nabla u) + \underset{\sim}{b}(\underset{\sim}{x})\cdot\nabla u + k^2(\underset{\sim}{x})u = 0 \quad \text{in } \Omega_2. \tag{4.1}$$

Suppose there is an R_o such that for $|\underset{\sim}{x}| > R_o$ we have

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underset{\sim}{b} = (0,0,0), \quad k^2(\underset{\sim}{x}) = k_2^2. \tag{4.2}$$

Then if one chooses Γ_∞ so that it contains the circle $|\underset{\sim}{x}| = R_o$ one can formulate boundary value problems for 4.1, with the radiation condition as variational problems over Ω_2^T with the condition $\frac{\partial u}{\partial n} = T_{k_2}(u|_{\Gamma_\infty})$ on Γ_∞.

An example of equation 4.1 occurs in [3] in the study of acoustic radiation from a cylinder when heating causes local spatial inhomogenities. The equation there is

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p + \frac{1}{\rho}\nabla\rho\cdot\nabla p = 0 \tag{4.3}$$

where $c = c(\underset{\sim}{x})$ is sound speed, $\rho = \rho(\underset{\sim}{x})$ is the density and $p(\underset{\sim}{x},t)$ is the acoustic pressure. If one seeks periodic solutions of the form $p(\underset{\sim}{x},t) = \operatorname{Re}(u(\underset{\sim}{x}))e^{i\omega t}$ then one arrives at 4.1 with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underset{\sim}{b} = -\frac{1}{\rho}\nabla\rho, \quad k = \frac{\omega}{c} .$$

Finally it can be seen that one can treat interface problems with the geometry of Figure 1.1, but with equations of the form 4.2 holding in Ω_1 and Ω_2 with associated $(A_1, \underset{\sim}{b}_1, k_1)$ and $(A_2, \underset{\sim}{b}_2, k_2)$. It is necessary only that 4.2 hold for $(A_2, \underset{\sim}{b}_2, k_2)$ for $|\underset{\sim}{x}| \geq R_o$ and that the second interface condition be replaced by one which is naturally associated with 4.1, that is,

$$\alpha_1 (A_1 \nabla u \cdot \hat{n})^- = \alpha_2 (A_2 \nabla u \cdot \hat{n})^+ .$$

5. PROOF OF THEOREM 3.

We begin by considering an auxiliary problem. This is:

Find u such that

$$Q_o \qquad \begin{aligned} \Delta u &= 0 \quad \text{in } A_\infty \\ u &= \varphi \quad \text{on } \Gamma_\infty \\ u &= O(1) \\ \nabla u &= O(r^{-2}) \end{aligned} \quad \text{as } r \to \infty$$

This problem has a unique solution which we denote by $U^o_{\Gamma_\infty}(\underset{\sim}{x};\varphi)$, and we may define the associated T operator T_o as in problem Q_k. That is

$$T_o[\varphi](\underset{\sim}{x}_o) = \left\{ \frac{\partial U^o_{\Gamma_\infty}(\underset{\sim}{x}_o;\varphi)^+}{\partial n} \right\}. \tag{5.1}$$

The following results concerning the operator T_o are established in the Appendix.

LEMMA 2. T_o is a bounded linear operator from $H^r(\Gamma_\infty)$ to $H^{r-1}(\Gamma_\infty)$ with the following properties.

(i) $\int_{\Gamma_\infty} T_o(\varphi)\psi ds = \int_{\Gamma_\infty} \varphi\, T_o(\psi) ds$ for all $\varphi,\psi \in H^{1/2}(\Gamma_\infty)$

(ii) $\int_{\Gamma_\infty} T_o(\varphi)\bar{\varphi} ds \leq 0$ for all $\varphi \in H^{1/2}(\Gamma_\infty)$

(iii) For all k, $T_k - T_o$ is a bounded linear operator from $H^r(\Gamma_\infty)$ into $H^{r+1}(\Gamma_\infty)$.

With Lemma 2 stated we can outline the proof of Theorem 3. We write $a(u,v)$ in the form

$$a(u,v) = a_1(u,v) + a_2(u,v) \tag{5.2}$$

where

$$a_1(u,v) = \int_{\Gamma_\infty} \alpha_2 T_o(u|_{\Gamma_\infty})\bar{v}\, ds - \int_{\Omega_1} \alpha_1 \nabla u\cdot\nabla\bar{v} d\underset{\sim}{x} - \int_{\Omega_2} \alpha_2 \nabla u\cdot\nabla\bar{v} d\underset{\sim}{x} \tag{5.3}$$

$$a_2(u,v) = \int_{\Gamma_\infty} \alpha_2 (T-T_o)[u|_{\Gamma_\infty}]\bar{v}\, ds + \alpha_1 k_1^2 \int_{\Omega_1} u\bar{v} d\underset{\sim}{x} + \alpha_2 k_2^2 \int_{\Omega_2^T} u\bar{v} d\underset{\sim}{x} \tag{5.4}$$

and look for a $v \in H_E$ of the form $v = u + w$, $w \in H_E$ for which the inequality 2.12 holds. For this function v we use the decomposition 5.2 and find

$$a(u,v) = \{a_1(u,u) - \alpha_1 \int_{\Omega_1} u\bar{u}d\underset{\sim}{x} - \alpha_2 \int_{\Omega_2^T} u\bar{u}d\underset{\sim}{x}\}$$
$$+ \{a_2(u,u) + a_1 \int_{\Omega_1} u\bar{u}d\underset{\sim}{x} + a_2 \int_{\Omega_2^T} u\bar{u}d\underset{\sim}{x}\} + a(u,w) \tag{5.5}$$

The first bracketed expression on the right side of 5.5 is negative and bounded above by $-C'|||u|||_1^2$. This follows from Lemma 2(ii) and the definition of $a_1(\cdot,\cdot)$. If w can be chosen so that

$$a(u,w) = -\{a_2(u,u) + \alpha_1 \int_{\Omega_1} u\bar{u}d\underset{\sim}{x} + \alpha_2 \int_{\Omega_2^T} u\bar{u}d\underset{\sim}{x}\} \tag{5.6}$$

then we would have for $v = u + w$,

$$|a(u,v)| \geq C'|||u|||_1^2. \tag{5.7}$$

If, in addition, w satisfies the estimate

$$|||w|||_1 \leq C'|||u|||_1 \tag{5.8}$$

then $|||v|||_1 \leq (1 + C')|||u|||_1$ and it would follow from 5.7 that

$$\frac{|a(u,v)|}{|||v|||_1} \geq \frac{C'}{1 + C} |||u|||_1$$

proving 2.12. Inequality 2.13 follows from 2.12 and the approximation property for $\{S_E^h\}$ if the estimate 5.8 can be strengthened to

$$|||w|||_2 \leq C'|||u|||_1. \tag{5.9}$$

To see this we set $u = u_h \in S_E^h$ and construct $w \in H_E$ so that 5.6 and 5.7 hold. Then

$$|a(u_h,v)| \geq C' |||u_h|||_1^2 \tag{5.10}$$

holds for $v = u_h + w$. By the approximation property 2.7 and the assumption 5.9 we may pick $w_h \in S_E^h$ such that

$$|||w-w_h|||_1 \leq C_1 h |||w|||_2 \leq C_1' \, h |||u_h|||_1 . \tag{5.11}$$

If we set $v_h = u_h + w_h$ then there exists $h_o > 0$ such that

$$|a(u_h,v_h)| \geq C'' |||u_h|||_1^2 \quad \text{for} \quad h < h_o .^{\dagger} \tag{5.12}$$

Moreover, using 5.9 and 5.11,

$$\begin{aligned} |||v_h|||_1 &\leq |||u_h|||_1 + |||w_h|||_1 \\ &\leq |||u_h|||_1 + |||w|||_2 + |||w-w_h|||_1 \\ &\leq (1 + C' + C_1'h) \, |||u_h|||_1 . \end{aligned}$$

Thus

$$|||v_h|||_1 \leq C''' \; |||u_h|||_1 . \tag{5.13}$$

Finally, we note that

$$\frac{|a(u_h,v_h)|}{|||v_h|||_1} \geq \frac{C''}{C'''} \; |||u_h|||_1 \quad \text{for} \quad h < h_o$$

follows from 5.12 and 5.13. This proves 2.13.

---

† This result is a simple consequence of 5.10, 5.11 and 2.6 using the estimate

$$|a(u_h,v_h)| \geq |a(u_h,v)| - |a(u_h,w-w_h)| .$$

To complete the proof of Theorem 3 we must show that for arbitrary $u \in H_E$ there exists $w \in H_E$ such that 5.6 and 5.9 hold. To do this we consider the variational problem

$$\text{Find } w \in H_E \text{ such tnat}$$

$$\mathrm{VP}^* \qquad a(\theta,w) = -\{a_2(\theta,u) + \alpha_1 \int_{\Omega_1} \theta\bar{u}d\underset{\sim}{x} + \alpha_2 \int_{\Omega_2^T} \theta\bar{u}d\underset{\sim}{x}\}$$

$$\text{for all } \theta \in H_E .$$

Our arguments to this point utilize what is known as Nitsche's Trick [11], and the problem VP* is the sort of adjoint problem that arises in these instances. The desired results 5.6 and 5.9 are immediate consequences of an existence and regularity result for VP*. This is stated in the following lemma which is discussed in the Appendix.

LEMMA 3. There exists $w \in H_E$ satisfying problem VP*. Moreover, $|||w|||_2 < \infty$ and satisfies

$$|||w|||_2 \leq C'|||u|||_1 .$$

Having completed the outline of the proof of Theorem 3 we return to the matter of proving the L^2 estimate 2.9 of Theorem 2. We again use Nitsche's Trick. Let $e_h = u - u^h$ and consider the problem

Find $w \in H_E$ such that

$$a(v,w) = \alpha_1 \int_{\Omega_1} v\bar{e}_h d\underset{\sim}{x} + \alpha_2 \int_{\Omega_2^T} v\bar{e}_h d\underset{\sim}{x} \tag{5.14}$$

for all $v \in H_E$.

We may show, using integration by parts and Lemma 1, that this is equivalent to the following boundary value problem

$$\begin{aligned}
\Delta\bar{w} + k_1^2\bar{w} &= \bar{e}_h \quad \text{in } \Omega_1 \\
\Delta\bar{w} + k_2^2\bar{w} &= \bar{e}_h \quad \text{in } \Omega_2^T \\
\bar{w}^- &= \bar{w}^+ \quad \text{on } \Gamma \\
\alpha_1\left(\frac{\bar{w}}{\partial n}\right)^- &= \alpha_2\left(\frac{\partial\bar{w}}{\partial n}\right)^+ \quad \text{on } \Gamma \\
T_{k_2}(\bar{w}|_{\Gamma_\infty}) - \left(\frac{\partial\bar{w}}{\partial n}\right)^- &= 0 \quad \text{on } \Gamma_\infty
\end{aligned}$$

This may in turn be recast as an exterior interface problem

$$\begin{aligned}
\Delta\bar{w} + k_1^2\bar{w} &= \bar{e}_h \quad \text{in } \Omega_1 \\
\Delta\bar{w} + k_1^2\bar{w} &= \begin{cases} \bar{e}_h & \text{in } \Omega_2^T \\ 0 & \text{in } A_\infty \end{cases} \\
\bar{w}^- &= \bar{w}^+ \quad \text{on } \Gamma \\
\alpha_1\left(\frac{\partial\bar{w}}{\partial n}\right)^- &= \alpha_2\left(\frac{\partial\bar{w}}{\partial n}\right)^+ \quad \text{on } \Gamma \\
\lim_{r\to\infty} r^{1/2}\left|\frac{\partial\bar{w}}{\partial r} - ik_2\bar{w}\right| &= 0
\end{aligned} \tag{5.15}$$

Problem 5.15 has a unique solution and, by arguments similar to those outlined in the discussion of the proof of Lemma 3, its solution satisfies the estimate

$$|||w|||_2 \leq C' |||e_h|||_o. \tag{5.16}$$

We put $v = e_h$ in 2.14 and obtain

$$a(e_h,w) = |||e_h|||_o^2. \tag{5.17}$$

Since $e_h = u - u^h$ and $a(u,w_h) = a(u^h,w_h) = F(w_h)$ for all $w_h \in S_E^h$ we have $a(e_h,w_h) = 0$ for all $w_h \in S_E^h$. We subtract this form 5.17 to obtain

$$a(e_h,w-w_h) = |||e_h|||_o^2 \quad \text{for all} \quad w_h \in S_E^h. \tag{5.18}$$

From 2.6 and 5.18 we have

$$|||e_h|||_o^2 \leq C|||e_h|||_1 |||w-w_h|||_1 \quad \text{for all} \quad w_h \in S_E^h$$

or

$$|||e_h|||_o^2 \leq C|||e_h|||_1 \inf_{w_h \in S_E^h} |||w-w_h|||_1. \tag{5.19}$$

The approximation property 2.7 implies that

$$\inf_{w_h \in S_E^h} |||w-w_h|||_1 \leq C_1 h |||w|||_2.$$

Using this and 5.16 gives

$$\inf_{w_h \in S_E^h} |||w-w_h|||_1 \leq C'h |||e_h|||_o. \tag{5.20}$$

Finally, 2.19 and 2.20 establish

$$|||e_h|||_o \le C''h|||e_h|||_1$$

which proves the L^2 estimate 2.9.

APPENDIX: PROOFS OF LEMMAS.

PROOF OF LEMMA 1. It is shown in [7] that K_k is a bounded linear map from $H^r(\Gamma_\infty)$ onto $H^{r-1}(\Gamma_\infty)$ with a bounded inverse. When Γ_∞ is a smooth curve it is known that the quantity $\frac{\partial}{\partial n} G_k$ in the definition of M_k is a smooth function and the first statement of Lemma 1 follows.

In order to establish the property 2.3 for T_k we use a Green's theorem argument. Suppose $\varphi, \psi \in H^{1/2}(\Gamma_\infty)$. Define U and V by $U = U_{\Gamma_\infty}(\underset{\sim}{x};k,\varphi)$, $V = U_{\Gamma_\infty}(\underset{\sim}{x};k,\psi)$. Then Green's theorem yields

$$\int_{\Gamma_\infty} \left(\frac{\partial U}{\partial n} V - \frac{\partial V}{\partial n} U\right) = \int_{\Gamma_R} \left(\frac{\partial U}{\partial n} V - U \frac{\partial V}{\partial n}\right) \qquad (A.1)$$

where Γ_R is a large circle (radius = R) containing Γ_∞. The radiation condition implies that the limit of the right-hand side as R tends to infinity is zero. Hence the left-hand side is zero and this is the result stated.

PROOF OF LEMMA 2. We first obtain a representation for the operator T. To do this we need to discuss the solution of problem Q_o. It is shown in [6] that the solution can be obtained in the form

$$U^o_{\Gamma_\infty}(\underset{\sim}{x},\varphi) = \frac{1}{2\pi}\int_{\Gamma_\infty} \sigma(\underset{\sim}{y}) \ln|\underset{\sim}{x}-\underset{\sim}{y}| ds_{\underset{\sim}{y}} - C \qquad (A.2)$$

where σ is determined by the equations

$$K_o[\sigma] \equiv \frac{1}{2\pi}\int_{\Gamma_\infty} \sigma(y)\ln|\underline{x}-\underline{y}|ds_{\underline{y}} = \varphi + C \tag{A.3}$$

$$\int_{\Gamma_\infty} \sigma(\underline{y})ds_{\underline{y}} = 0. \tag{A.4}$$

The equation $K_o[\sigma] = \chi$ can be solved for any χ. The condition A.4 determines the constant C and serves to make $U^o_{\Gamma_\infty}$ bounded at infinity. It is shown in [7] that K_o is a bounded map from $H^r(\Gamma_\infty)$ onto $H^{r+1}(\Gamma_\infty)$ with a bounded inverse. In order to establish the results in Lemma 2 we must look a little more closely at the solution procedure (A.2)-(A.4). From A.3 we have

$$\sigma = K_o^{-1}[\varphi] + CK_o^{-1}[1] \tag{A.5}$$

and then A.4 determines C by the formula

$$C = (-\int_{\Gamma_\infty} K_o^{-1}[\varphi]ds)/(\int_{\Gamma_\infty} K_o^{-1}[1]ds). \tag{A.6}$$

(It is shown in [6] that if Γ_∞ is chosen so that its mapping radius is not one then the denominator in A.6 does not vanish.) Since K_o^{-1} maps $H^r(\Gamma_\infty)$ into $H^{r-1}(\Gamma_\infty)$ we observe that A.6 defines C as a continuous linear functional on $H^r(\Gamma_\infty)$. Indeed by the generalized Schwarz inequality we have

$$|C| \equiv |C(\varphi)| \leq c_1\|K_o^{-1}(\varphi)\|_{r-1,\Gamma_\infty}\|1\|_{-(r-1),\Gamma_\infty} \leq c_2\|\varphi\|_{r,\Gamma_\infty}. \tag{A.7}$$

In order to determine T_o we observe that by A.2 and A.5

$$T_o[\varphi](\underset{\sim}{x}_o) = \{\frac{\partial U^o_{\Gamma_\infty}}{\partial n}(\underset{\sim}{x}_o;\varphi)\}^+ = \frac{1}{2}\sigma(\underset{\sim}{x}_o) + \frac{1}{2\pi}\int_{\Gamma_\infty}\sigma(\underline{y})\frac{\partial}{\partial n}\ln|\underline{x}-\underline{y}|ds_{\underset{\sim}{y}} \tag{A.8}$$

$$= (\tfrac{1}{2}I + M_o)K_o^{-1}[\varphi](\underset{\sim}{x}_o) + C(\varphi)(\tfrac{1}{2}I + M_o)K_o^{-1}[1](\underset{\sim}{x}_o),$$

$$\underset{\sim}{x}_o \in \Gamma_\infty .$$

It follows from this formula that T_o maps $H^r(\Gamma_\infty)$ into $H^{r-1}(\Gamma_\infty)$ continuously.

Property (i) of Lemma 2 follows by the same Green's theorem type result as in Lemma 1. The negativity result (ii) is another Green's theorem argument. We have (for $u = U^o_{\Gamma_\infty}(\underset{\sim}{x};\varphi)$),

$$\int_{\Gamma_\infty} T_o[\varphi]\overline{\varphi}ds = \int_{\Gamma_\infty}\frac{\partial u}{\partial n}\overline{u}\,ds = -\int_{\Omega_R}|\nabla u|^2 d\underline{x} + \int_{\Gamma_R}\frac{\partial u}{\partial n}\overline{u}. \tag{A.9}$$

Here Γ_R is as before and $\Omega_R = A_\infty \cap \text{int}(\Gamma_R)$. Once again the conditions at infinity imply that tne limit of the integral over Γ_R as R tends to infinity is zero hence we obtain (ii).

It remains to establisy (iii) of Lemma 2. We begin by observing that $G_k(\underline{x},\underline{y})$ and $\frac{1}{2\pi}\ln|\underline{x}-\underline{y}|$ have the same singularity. We have in fact,

$$G_k(x,y) = \frac{1}{2\pi}\ln|x-y| + R_k(|\underline{x}-\underline{y}|) \tag{A.10}$$

with

$$R_k(|\underset{\sim}{x}-\underset{\sim}{y}|) = \alpha_k + |\underset{\sim}{x}-\underset{\sim}{y}|^2\ln|\underset{\sim}{x}-\underset{\sim}{y}|\gamma_k(|\underset{\sim}{x}-\underset{\sim}{y}|) + \delta_k(|\underset{\sim}{x}-\underset{\sim}{y}|) \tag{A.11}$$

where α_k is a constant and γ_k and δ_k are analytic.
Thus we may write 1.6 in the form

$$\frac{1}{2\pi}\int_{\Gamma_\infty} \sigma(y)\ln|\underset{\sim}{x}_o-\underset{\sim}{y}|ds_{\underset{\sim}{y}} \equiv K_o[\sigma](\underset{\sim}{x}_o) = \varphi(\underset{\sim}{x}_o) - \int_{\Gamma_\infty} \sigma(\underset{\sim}{y})R_k(|\underset{\sim}{x}_o-\underset{\sim}{y}|)ds_{\underset{\sim}{y}} \tag{A.12}$$

or,

$$\sigma(\underset{\sim}{x}_o) = K_o^{-1}[\varphi](\underset{\sim}{x}_o) - K_o^{-1}\int_{\Gamma_\infty} \sigma(\underset{\sim}{y})R_k(|\underset{\sim}{x}_o-\underset{\sim}{y}|)ds_{\underset{\sim}{y}}. \tag{A.13}$$

It is shown in [7], on the basis of A.11, that the integral operator

$$\int_{\Gamma_\infty} \sigma(\underset{\sim}{y})R_k(|\underset{\sim}{x}-\underset{\sim}{y}|)ds_{\underset{\sim}{y}}$$

takes $H^r(\Gamma_\infty)$ into $H^{r+3}(\Gamma_\infty)$. Hence if we compare A.13 with A.5 we see that

$$U_{\Gamma_\infty}(\underset{\sim}{x};k,\varphi) = U^o_{\Gamma_\infty}(\underset{\sim}{x};\varphi) + \frac{C(\varphi)}{2\pi}\int_{\Gamma_\infty} K_o^{-1}[1]\ln|\underset{\sim}{x}-\underset{\sim}{y}|ds_{\underset{\sim}{y}}$$
$$+ \int_{\Gamma_\infty} G_k(\underset{\sim}{x},\underset{\sim}{y})\int_{\Gamma_\infty} \sigma(z)R_k(|\underset{\sim}{y}-z|)ds_z ds_{\underset{\sim}{y}}. \tag{A.14}$$

Now we obtain T_k by computing $\left(\frac{\partial U_{\Gamma_\infty}}{\partial n}\right)^+$. This introduces the operator M_k as in 1.7. We note, however, that A.12 implies that M_k differs from M_o by terms with more regularity (specifically $M_k-M_o : H^r(\Gamma_\infty) \to H^{r+2}(\Gamma_\infty)$). If one performs the calculations with A.13 and A.8 one finds that

$$T_k[\varphi] = T_o[\varphi] - C(\varphi)(\tfrac{1}{2}I + M_o)K_o^{-1}[1] + \mathcal{L}_k[\varphi]$$

where $\mathcal{L}_k$ is a continuous map from $H^r(\Gamma_\infty)$ into $H^{r+1}(\Gamma_\infty)$.

Now, by A.6 the functional is given by

$$C(\varphi) = \beta \int_{\Gamma_\infty} K_o^{-1}[\varphi]ds$$

where β is a constant. But K_o^{-1} is self-adjoint hence,

$$\int_{\Gamma_\infty} K_o^{-1}[\varphi]ds = \int_{\Gamma_\infty} \varphi\, K_o^{-1}[1]ds. \tag{A.16}$$

(The constant β also involves $K_o^{-1}[1]$.) Now if the curve Γ_∞ is smooth then $K_o^{-1}[1]$ would be a smooth function and then A.16 and the generalized Schwarz inequality yields

$$|C(\varphi)| \leq |\beta|\,\|\varphi\|_{r,\Gamma_\infty}\|K_o^{-1}[1]\|_{-r,\Gamma_\infty},$$

$$\begin{aligned}\|C(\varphi)(\tfrac{1}{2}I + \mathfrak{m}_o)K_o^{-1}[1]\|_{r+1,\Gamma_\infty} &\leq |C(\varphi)|\,\|(\tfrac{1}{2}I + \mathfrak{m}_o)K_o^{-1}[1]\|_{r+1,\Gamma_\infty}\\ &\leq C'\|\varphi\|_{r,\Gamma_\infty}\|K_o^{-1}[1]\|_{r+1,\Gamma_\infty}\\ &\leq C''\|\varphi\|_{r,\Gamma_\infty}.\end{aligned}$$

Thus A.15 yields (iii) of Lemma 2.

PROOF OF LEMMA 3. The definitions of $a(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ and integration by parts used in a standard way yield that the problem VP^* is equivalent to the boundary value problem

$$\begin{aligned}
&\Delta\bar{w} + k_1^2\bar{w} = -\bar{u} \quad \text{in} \quad \Omega_1 \\
&\Delta\bar{w} + k_2^2\bar{w} = -\bar{u} \quad \text{in} \quad \Omega_2^T \\
&\bar{w}^+ = \bar{w}^- \quad \text{on} \quad \Gamma \\
&\alpha_2\left(\frac{\partial\bar{w}}{\partial n}\right)^+ = \alpha_1\left(\frac{\partial\bar{w}}{\partial n}\right)^- \quad \text{on} \quad \Gamma \\
&T_{k_2}(\bar{w}|_{\Gamma_\infty}) - \left(\frac{\partial\bar{w}}{\partial n}\right)^- = -(T_{k_2}-T_0)(\bar{u}) \equiv h \quad \text{on} \quad \Gamma_\infty .
\end{aligned} \tag{A.17}$$

The result (iii) of Lemma 2 together with the fact that $\bar{u} \in H^{1/2}(\Gamma_\infty)$ gives the information that

$$h \in H^{3/2}(\Gamma_\infty) \quad \text{and} \quad \|h\|_{\frac{3}{2},\Gamma_\infty} \le C\|u\|_{\frac{1}{2},\Gamma_\infty} . \tag{A.18}$$

We may further note that problem A.17 is equivalent to the exterior interface problem

$$\begin{aligned}
&\Delta\bar{w} + k_1^2\bar{w} = -\bar{u} \quad \text{in} \quad \Omega_1 \\
&\Delta\bar{w} + k_2^2\bar{w} = \begin{cases} -\bar{u} & \text{in} \quad \Omega_2^T \\ 0 & \text{in} \quad A_\infty \end{cases} \\
&\bar{w}^+ = \bar{w}^- \quad \text{on} \quad \Gamma \\
&\alpha_2\left(\frac{\partial\bar{w}}{\partial n}\right)^+ = \alpha_1\left(\frac{\partial\bar{w}}{\partial n}\right)^- \quad \text{on} \quad \Gamma \\
&\bar{w}^+ = \bar{w}^- \quad \text{on} \quad \Gamma_\infty \\
&\left(\frac{\partial\bar{w}}{\partial n}\right)^+ - \left(\frac{\partial\bar{w}}{\partial n}\right)^- = h \quad \text{on} \quad \Gamma_\infty \\
&\lim_{r\to\infty} r^{1/2}\left|\frac{\partial\bar{w}}{\partial n} - ik_2\bar{w}\right| = 0.
\end{aligned} \tag{A.19}$$

The solution of A.19 can be obtained in the following form:

$$\bar{w}(\underset{\sim}{x}) = \begin{cases} \int_{\Gamma} \sigma_1(\underset{\sim}{y}) G_{k_1}(\underset{\sim}{x},\underset{\sim}{y}) ds_{\underset{\sim}{y}} + w_o(\underset{\sim}{x}) & \text{in } \Omega_1 \\ \int_{\Gamma} \sigma_2(\underset{\sim}{y}) G_{k_2}(\underset{\sim}{x},\underset{\sim}{y}) ds_{\underset{\sim}{y}} + w_1(\underset{\sim}{x}) + w_2(\underset{\sim}{x}) & \text{in } \Omega_2 \end{cases} \tag{A.20}$$

where

$$w_o(\underset{\sim}{x}) = -\int_{\Omega_1} \bar{u}(\underset{\sim}{y}) G_{k_1}(\underset{\sim}{x},\underset{\sim}{y}) d\underset{\sim}{y} \tag{A.21}$$

$$w_1(\underset{\sim}{x}) = -\int_{\Omega_2^T} \bar{u}(\underset{\sim}{y}) G_{k_2}(\underset{\sim}{x},\underset{\sim}{y}) d\underset{\sim}{y} \tag{A.22}$$

$$w_2(\underset{\sim}{x}) = \int_{\Gamma_\infty} h(\underset{\sim}{y}) G_{k_2}(\underset{\sim}{x},\underset{\sim}{y}) ds_{\underset{\sim}{y}}. \tag{A.23}$$

Standard potential theory arguments show that A.20 satisfies all the conditions of A.19 except the interface conditions on Γ. The imposition of these leads to the integral equations,

$$K_{k_2}[\sigma_2] + w_1 + w_2 = K_{k_1}[\sigma_1] + w_o \quad \text{on } \Gamma \tag{A.24}$$

$$\alpha_2\{(\tfrac{1}{2}I + M_{k_2})[\sigma_2] + \frac{\partial w_1}{\partial n} + \frac{\partial w_2}{\partial n}\} = \alpha_1\{(-\tfrac{1}{2}I + M_{k_1})[\sigma_1] + \frac{\partial w_o}{\partial n}\} \text{ on } \Gamma. \tag{A.25}$$

Here the integral operators are as in 1.6 and 1.7 but on Γ instead of Γ_∞.

It can be shown that the solution of A.19 is unique and the Fredholm alternative can be used to establish the existence of solutions of A.24 and A.25. The estimate in Lemma 3 can be established by tedious but fairly straightforward analysis of the mapping properties of the operators in A.20. We omit these details.

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