
The Inverse Variational Problem in Classical Mechanics

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The Inverse Variational Problem in Classical Mechanics

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To my friends physicists.

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Foreword

This book arose from my informal discussions with P.C. Stichel and J. Cisko in which we tried to probe into the fascinating problem, the inverse variational problem in classical mechanics. It contains mostly our own reflections and ideas developed during studying the vast literature of the subject. The ideas – I am afraid – are not all new and are probably well known to a small circle of experts, as the problem under consideration is over hundred years old.

My notes do not intend to present all results obtained in this domain of research. Their aim is to give a concise picture of present state of affairs, imbued with our own, personal flavor. No advanced methods of contemporary differential geometry were used. All function we are going to use in this lecture notes are assumed to be piecewise continuous and sufficiently many times differentiable. This is meant to make things more simple.

The contents of this book is well suited to be used as lecture notes in a university course for physicists.

Acknowledgement.

I am grateful to Dr. J. Cisko for a careful reading of these notes, for his important critical comments as well as for his deep and fruitful remarks.

I express my warm thanks to Dr. C. Juszczak for the painstaking typing of my manuscript.

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