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Research Article

Global Well-Posedness and Stability for a Viscoelastic Plate Equation with a Time Delay

Baowei Feng

College of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Correspondence should be addressed to Baowei Feng; bwfeng@swufe.edu.cn

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A plate equation with a memory term and a time delay term in the internal feedback is investigated. Under suitable assumptions, we establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. Moreover, by using energy perturbation method, we prove a general decay result of the energy provided that the weight of the delay is less than the weight of the damping.

1. Introduction

In this paper, we are concerned with the following plate equation with a memory term and a time delay term in the internal feedback:

$$\begin{aligned} &u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u \\ &\quad + \int_0^t g(t-s)\,\Delta u(s)\,ds + \mu_1 u_t(x,t) \\ &\quad + \mu_2 u_t(x,t-\tau) + f(u) = 0, \end{aligned} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here $M(\cdot)$ is a function satisfying suitable conditions (see below), μ_1, μ_2 are positive constants, and $\tau > 0$ represents the time delay.

Equation (1) with the memory term $\int_0^t g(t-s)\Delta u(s)ds$, where the function g is called kernel, can be regarded as a fourth order viscoelastic plate equation with a lower order perturbation, and it can be also regarded as an elastoplastic flow equation with some kind of memory effect.

In this paper, we consider the following initial conditions:

$$\begin{aligned} &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ &u_t(x,t-\tau) = h_0(x,t-\tau), \quad x \in \Omega,\ t \in (0,\tau) \end{aligned} \tag{2}$$

and the following boundary conditions:

$$u = \Delta u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{3}$$

Fourth order equations with lower order perturbation are related to models of elastoplastic microstructure flows. For the single plate equation without delay, that is, $\mu_2 = 0$, as considered by Woinowsky-Krieger [1], the author first proposed the one-dimensional nonlinear equation of vibration of beams, which is given by

$$u_{tt} + \alpha u_{xxxx} - \left(\beta + \gamma \int_0^L |u_x|^2\,dx\right) u_{xx} = 0, \tag{4}$$

where L is the length of the beam and α, β, γ are positive physical constants. The nonlinear part of (4) represents for the extensible effect for the beam whose ends are restrained to remain in a fixed distance apart in its transverse vibrations. A more general equation of (4) reads

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \Delta^2 u + g(u_t) + f(u) = h(x), \tag{5}$$

where $M(\cdot)$ is a function satisfying some conditions. There are so many existing results concerning global existence, stability, and long-time dynamics for (5); we would like to refer the reader to de Brito [2], Cavalcanti et al. [3, 4], Ma [5], Ma and Narciso [6], de Lacerda Oliveira and de Lima [7], J. Y. Park and S. H. Park [8], Patcheu [9], Rivera [10, 11], Tusnal [12], Vasconcellos and Teixeira [13], Yang [14, 15], and the references therein. Very recently, Andrade et al. [16] investigated a viscoelastic plate equation with p-Laplacian and memory terms with strong damping

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t + f(u) = 0, \tag{6}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. Under suitable assumptions on the memory kernel g and a forcing term f, the authors proved the existence of weak solutions by using Faedo-Galerkin approximations, the uniqueness of strong solutions, and the exponential stability of solutions to (6) with initial and boundary value problem. For more results on viscoelastic equations, we can refer to Berrimi and Messaoudi [17], Messaoudi [18], Messaoudi and Tartar [19, 20], and the references therein.

In recent years, many mathematical workers studied some systems with time delay effects. Datko et al. [21] studied the following system:

$$\begin{gathered} u_{tt} - \Delta u = 0, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \Gamma_0,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau), \quad x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t-\tau) = g_0(x,t-\tau), \quad x \in \Omega,\ t \in [0,\tau]. \end{gathered} \tag{7}$$

By using an observability inequality, they proved the exponential stability for the energy when $\mu_2 < \mu_1$. Subsequently, Xu et al. [22] obtained the same result as in [21] for the one space dimension by using the spectral analysis approach. Later on, Kirane and Said-Houari [23] considered a viscoelastic wave equation with a delay term in internal feedback with initial conditions and boundary value conditions of Dirichlet type. Under suitable assumptions on the relaxation function and some restriction on the parameters μ_1 and μ_2, they established the global well-posedness of the system. Moreover, under the assumption $\mu_2 \le \mu_1$ between the weight of the delay term in the feedback and the weight of the term without delay, the authors proved a general decay of the total energy of the system. For more some results concerning the different boundary conditions under an appropriate assumption between μ_1 and μ_2, one can refer to Nicaise and Pignotti [24], Nicaise et al. [25], Nicaise and Valein [26], and the references therein.

Equation (1) is a plate equation with a memory term and a time delay term in the internal feedback. Noting that $\mu_1 \neq 0$, we know that it is a plate equation with weak damping. For viscoelastic plate equations, it is well known that one considered a memory of the form $\int_0^t g(t-s)\Delta^2 u(s)ds$ (see, e.g., [10, 27, 28]). However, because the main dissipation of the system (1)–(3) is given by a weak damping u_t, here we consider a weaker memory, acting only on Δu. To the best of our knowledge, the global well-posedness and energy decay for system (1)–(3) were not previously considered. So the objective of this work is to establish the global well-posedness and stability of initial boundary value problem (1)–(3). The main dissipation of the system (1)–(3) is given by a weak damping u_t, which makes the analysis in this work different from [16], because the authors considered the case of a strong damping $-\Delta u_t$ in [16].

The outline of this paper is as follows. In Section 2, we give some preparations for our consideration and our main results. In Section 3, we establish the global posedness of the system by using the Faedo-Galerkin approximations and some energy estimates. In Section 4, we will show a general decay result of the energy by using energy perturbation method provided that the weight of the delay is less than the weight of the damping.

The notation in this paper will be as follows: L^q, $1 \le q \le +\infty$, $W^{m,q}$, $m \in \mathbb{N}$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on Ω. In addition, $\|\cdot\|_B$ denotes the norm in the space B, and we also put $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

2. Preliminaries and Main Results

In this section, we give some preparations for our consideration and our main results.

(i) We assume that $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$\begin{gathered} zM(z) \ge \widehat{M}(z), \\ \widehat{M}(z) = \int_0^z M(s)\,ds \end{gathered} \tag{8}$$

(if $M(z)$ is monotone nondecreasing).

(ii) For the memory kernel $g(t)$, we assume that

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying

$$\begin{gathered} g(t) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad g(0) > 0, \\ 1 - \lambda \int_0^\infty g(s)\,ds = l > 0, \end{gathered} \tag{9}$$

where $\lambda > 0$ is the embedding constant for $\|\nabla u\|^2 \le \lambda \|\Delta u\|^2$.

(G2) There exists a positive nonincreasing differentiable function $\mu(t)$ such that

$$g'(t) \le -\mu(t)g(t), \quad \forall t \ge 0, \tag{10}$$

$$\int_0^\infty \mu(t)\,dt = \infty. \tag{11}$$

(iii) The nonlinear term $f(u)$ satisfies

$$f(0) = 0, \quad |f(u) - f(v)| \le c\left(1 + |u|^\rho + |v|^\rho\right)|u - v|, \quad \forall u, v \in \mathbb{R}, \tag{12}$$

where $c > 0$ is a constant, and ρ satisfies

$$0 < \rho < \frac{4}{n-4} \quad \text{if} \geq 5, \qquad \rho > 0 \quad \text{if } 1 \leq n \leq 4. \tag{13}$$

We denote $\widehat{f}(z) = \int_0^z f(s)ds$ and assume that

$$0 \leq \widehat{f}(u) \leq f(u)u, \quad \forall u \in \mathbb{R}. \tag{14}$$

In order to deal with the delay feedback term, motivated by [24, 26], we introduce the following new dependent variable:

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0. \tag{15}$$

Then it is easy to verify

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,\infty). \tag{16}$$

Thus, problem (1)–(3) is transformed into

$$\begin{gathered} u_{tt}(x,t) + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u \\ + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 z(x,1,t) = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \end{gathered} \tag{17}$$

with $x \in \Omega$, $\rho \in (0,1)$ and $t > 0$, and the initial and boundary conditions are

$$\begin{gathered} u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad x \in \Omega, \\ z(x,\rho,0) = h_0(x,t-\tau), \quad (x,t) \in \Omega \times (0,\tau), \\ u = \Delta u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(x,0,t) = u_t(x,t) \quad x \in \Omega,\ t > 0. \end{gathered} \tag{18}$$

Let ξ be a positive constant satisfying

$$\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2). \tag{19}$$

Now we define the weak solutions of (1)–(3): for given initial data $(u_0,u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$, we say that a function $z = (u,u_t) \in C(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega))$ is a weak solution to the problem (1)–(3) if $z(0) = (u_0,u_1)$ and

$$\begin{gathered} (u_{tt},\omega) + (\Delta u, \Delta\omega) + \left(M\left(\|\nabla u\|^2\right)\nabla u, \nabla\omega\right) \\ - \int_0^t g(t-s)(\nabla u(s), \nabla\omega)\,ds \\ + \mu_1(u_t,\omega) + \mu_2(u_t(t-\tau),\omega) + (f(u),\omega) = 0, \end{gathered} \tag{20}$$

for all $\omega \in H^2(\Omega) \cap H_0^1(\Omega)$.

Next we state the global well-posedness of problem (17)-(18) given in the following theorem.

Theorem 1. *Let $\mu_2 \leq \mu_1$ hold and assume the assumptions (8)–(14) hold.*

(i) *If the initial data $(u_0,u_1) \in (H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega))$, $h_0 \in L^2(\Omega \times (0,1))$, then problem (17)-(18) has a weak solution such that*

$$\begin{gathered} u \in C\left(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}^+; L^2(\Omega)\right), \\ u_t \in L^2\left(\mathbb{R}^+; L^2(\Omega)\right). \end{gathered} \tag{21}$$

(ii) *If the initial data $(u_0,u_1) \in H^3_{\partial\Omega}(\Omega) \times H_0^1(\Omega)$, $h_0 \in H^1(\Omega \times (0,1))$, where*

$$H^3_{\partial\Omega}(\Omega) = \left\{u \in H^3(\Omega) \mid u = \Delta u = 0 \text{ on } \partial\Omega\right\}, \tag{22}$$

then the above weak solution has higher regularity

$$\begin{gathered} u \in L^\infty\left(\mathbb{R}^+, H^3(\Omega)\right), \\ u_t \in L^\infty\left(\mathbb{R}^+, H_0^1(\Omega)\right) \cap L^2\left(\mathbb{R}^+, H_0^1(\Omega)\right). \end{gathered} \tag{23}$$

(iii) *In both cases, we have that the solution (u,u_t) depends continuously on the initial data in $H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$. In particular, problem (17)-(18) has a unique weak solution.*

We define the energy of problem (17)-(18) by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\widehat{M}\left(\|\nabla u(t)\|^2\right) \\ & + \int_\Omega \widehat{f}(u(t))\,dx + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx. \end{aligned} \tag{24}$$

Finally, we give the energy decay of problem (17)-(18).

Theorem 2. *Let $\mu_2 < \mu_1$ hold and assume the assumptions (8)–(14) hold. In both cases (i) and (ii), there exist two constants $\alpha > 0$ and $\beta > 0$ such that the energy $E(t)$ defined by (24) satisfies*

$$E(t) \leq \alpha \exp\left(-\beta\int_0^t \mu(s)\,ds\right), \quad \forall t \geq 0. \tag{25}$$

3. The Global Well-Posedness

In this section, we will prove the global existence and the uniqueness of the solution of problem (17)-(18) by using the classical Faedo-Galerkin approximations along with some priori estimates. We only prove the existence of solution in (i). For the existence of stronger solution in (ii), we can use the same method as in (i) and one can refer to Andrade e al. [16] and Jorge Silva and Ma [28].

3.1. Approximate Problem. Let $\{w_j\}$ be the Galerkin basis given by the eigenfunctions of Δ^2 with boundary condition $u = \Delta u = 0$ on $\partial\Omega$. For any $m \geq 1$, let $W_m = \text{span}\{w_1, w_2, \ldots, w_n\}$.

We define for $1 \leq j \leq m$ the sequence $\phi_j(x,\rho)$ by

$$\phi_j(x,0) = w_j(x). \tag{26}$$

Then we can extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2(\Omega\times(0,1))$ and denote $V_m = \text{span}\{\phi_1,\phi_2,\dots,\phi_n\}$.

Given initial data $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, and $h_0 \in L^2(\Omega\times(0,1))$, we define the approximations

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j(x),$$
$$z_m(x,\rho,t) = \sum_{j=1}^{m} h_{jm}(t)\,\phi_j(x,\rho), \tag{27}$$

which satisfy the following approximate problem:

$$\begin{aligned}&\left(u_{mtt}(t), w_j\right) + \left(\Delta u_m(t), \Delta w_j\right) + \left(-M\left(\|\nabla u_m\|^2\right)\Delta u_m, w_j\right)\\ &\quad + \left(f(u_m(t)), w_j\right) + \int_0^t g(t-s)\left(\Delta u_m(s), w_j\right) ds\\ &\quad + \left(\mu_1 u_{mt}(t), w_j\right) + \left(\mu_2 z_m(x,1,t), w_j\right) = 0,\\ &\left(\tau z_{mt}(x,\rho,t), \phi_j\right) + \left(z_{m\rho}(x,\rho,t), \phi_j\right) = 0,\end{aligned} \tag{28}$$

with initial conditions

$$u_m(0) = u_0^m, \qquad u_{mt}(0) = u_1^m \qquad z_m(x,\rho,0) = z_0^m, \tag{29}$$

which satisfies

$$\begin{aligned}u_0^m &\longrightarrow u_0 \quad \text{strongly in } H^2(\Omega)\cap H_0^1(\Omega),\\ u_1^m &\longrightarrow u_1 \quad \text{strongly in } L^2(\Omega),\\ z_0^m &\longrightarrow h_0 \quad \text{strongly in } L^2(\Omega\times(0,1)).\end{aligned} \tag{30}$$

By using standard ordinary differential equations theory, the problem (28)-(29) has a solution $(g_{jm}, h_{jm})_{j=1,m}$ defined on $[0,t_m)$. The following estimate will give the local solution being extended to $[0,T]$, for any given $T>0$.

3.2. A Priori Estimate. Now multiplying the first approximate equation of (28) by g'_{jm}, we see that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|u_{mt}(t)\|^2 + \frac{1}{2}\|\Delta u_m(t)\|^2 + \frac{1}{2}\widehat{M}\left(\|\nabla u_m(t)\|^2\right)\right.\\ &\quad\left. + \int_\Omega \widehat{f}(u_m(t))\right) + \mu_1\|u_{nt}(t)\|^2\\ &\quad + \mu_2\int_\Omega z_m(x,1,t)\,u_{mt}(t)\,dx\\ &\quad - \int_0^t g(t-s)\left(\nabla u_m(s), \nabla u_{mt}(t)\right) ds = 0.\end{aligned} \tag{31}$$

Noting the following fact:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left((g\circ\nabla u)(t) - \int_0^t g(s)\,ds\cdot\|\nabla u(t)\|^2\right)\\ &\quad + \int_0^t\int_0^t g(t-s)\left(\nabla u(s), \nabla u_t(t)\right) ds\\ &= \frac{1}{2}\left(g'\circ\nabla u\right)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2,\end{aligned} \tag{32}$$

where

$$(g\circ\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds, \tag{33}$$

we know that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u_{mt}(t)\|^2 + \|\Delta u_m(t)\|^2 + \widehat{M}\left(\|\nabla u_m(t)\|^2\right)\right.\\ &\quad - \left(\int_0^t g(s)\,ds\right)\|\nabla u_m(t)\|^2\\ &\quad \left. + 2\int_\Omega \widehat{f}(u_m(t))\,dx + (g\circ\nabla u_m)(t)\right)\\ &\quad + \mu_1\|u_{mt}(t)\|^2 + \mu_2\int_\Omega z_m(x,1,t)\,u_{mt}(t)\,dx\\ &= \frac{1}{2}\left(g'\circ\nabla u_m\right)(t) - \frac{1}{2}g(t)\|\nabla u_m(t)\|^2.\end{aligned} \tag{34}$$

Multiplying the second approximate equation of (28) by $(\xi/\tau)h'_{jm}$ and then integrating over $(0,t)\times(0,1)$, we obtain

$$\begin{aligned}&\frac{\xi}{2}\int_\Omega\int_0^1 z_m^2(x,\rho,t)\,d\rho\,dx\\ &\quad + \frac{\xi}{\tau}\int_0^t\int_\Omega\int_0^1 z_{m\rho}z_m(x,\rho,s)\,d\rho\,dx\,ds\\ &= \frac{\xi}{2}\int_\Omega\int_0^1 z_0^m(x,\rho)\,d\rho\,dx.\end{aligned} \tag{35}$$

A straightforward calculation gives

$$\begin{aligned}&\int_0^t\int_\Omega\int_0^1 z_{m\rho}z_m(x,\rho,s)\,d\rho\,dx\,ds\\ &= \frac{1}{2}\int_0^t\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z_m^2(x,\rho,s)\,d\rho\,dx\,ds\\ &= \frac{1}{2}\int_0^t\int_\Omega\left(z_m^2(x,1,s) - z_m^2(x,0,s)\right) dx\,ds.\end{aligned} \tag{36}$$

Now integrating (34) and using (35)-(36) and $z_m^2(x,0,s) = u_{mt}^2(s)$, we infer that

$$\begin{aligned}&\mathscr{E}_m(t) + \left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^t\|u_{mt}(t)\|^2 ds\\ &\quad + \frac{\xi}{2\tau}\int_0^t\int_\Omega z_m^2(x,1,s)\,dx\,ds\\ &\quad + \mu_2\int_0^t\int_\Omega z_m(x,1,s)\,u_{mt}(s)\,dx\,ds\\ &\quad + \frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2 ds\\ &\quad - \frac{1}{2}\int_0^t\left(g'\circ\nabla u_m\right)(s)\,ds = \mathscr{E}_m(0),\end{aligned} \tag{37}$$

with

$$\mathscr{E}_m(t)=\frac{1}{2}\Big(\|u_{mt}(t)\|^2+\|\Delta u_m(t)\|^2+\widehat{M}\left(\|\nabla u_m(t)\|^2\right)-\left(\int_0^t g(s)\,ds\right)\|\nabla u_m(t)\|^2+2\int_\Omega \widehat{f}(u_m(t))\,dx+(g\circ\nabla u_m)(t)\Big)+\frac{\xi}{2}\int_\Omega\int_0^1 z_m^2(x,\rho,t)\,d\rho\,dx. \tag{38}$$

Then we have the following cases.

(i) Consider $\mu_2<\mu_1$. Using Young's inequality, we have

$$\mu_2\int_0^t\int_\Omega z_m(x,1,s)\,u_{mt}(s)\,dx\,ds \ge -\frac{\mu_2}{2}\int_0^t\int_\Omega z_m^2(x,1,s)\,dx\,ds-\frac{\mu_2}{2}\int_0^t\|u_{mt}^2(s)\|\,ds, \tag{39}$$

which, together with (37), yields

$$\mathscr{E}_m(t)+\left(\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^t\|u_{mt}(t)\|^2\,ds+\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^t\int_\Omega z_m^2(x,1,s)\,dx\,ds+\frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2\,ds-\frac{1}{2}\int_0^t\left(g'\circ\nabla u_m\right)(s)\,ds\le\mathscr{E}_m(0). \tag{40}$$

It follows from (19) that there exist two constants $c_1>0$ and $c_2>0$ such that

$$\mathscr{E}_m(t)+c_1\int_0^t\|u_{mt}(t)\|^2\,ds+c_2\int_0^t\int_\Omega z_m^2(x,1,s)\,dx\,ds+\frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2\,ds-\frac{1}{2}\int_0^t\left(g'\circ\nabla u_m\right)(s)\,ds\le\mathscr{E}_m(0). \tag{41}$$

(ii) Consider $\mu_1=\mu_2$. Taking $\xi=\tau\mu_1=\tau\mu_2$ and using (37), we know that

$$\mathscr{E}_m(t)+\frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|^2\,ds-\frac{1}{2}\int_0^t\left(g'\circ\nabla u_m\right)(s)\,ds\le\mathscr{E}_m(0). \tag{42}$$

Then, in both cases, we infer that there exists a positive constant C independent on m such that

$$\mathscr{E}_m(t)\le C,\quad t\ge 0. \tag{43}$$

It follows from (9), (14), and (43) that

$$\|u_{mt}(t)\|^2+\|\Delta u_m(t)\|^2+\widehat{M}\left(\|\nabla u_m(t)\|^2\right)+\int_\Omega\int_0^1 z_m^2(x,\rho,t)\,d\rho\,dx\le C. \tag{44}$$

Thus we can obtain $t_m=T$, for all $T>0$.

3.3. Passage to Limit. From (44), we conclude that for any $m\in\mathbb{N}$,

$$u_m \text{ is bounded in } L^\infty\left(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega)\right), \tag{45}$$

$$u_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}^+;L^2(\Omega)\right), \tag{46}$$

$$z_m \text{ is bounded in } L^\infty\left(\mathbb{R}^+;L^2(\Omega\times(0,1))\right). \tag{47}$$

Thus we get

$$\begin{aligned}&u_m\rightharpoonup u \quad \text{weakly star in } L^\infty\left(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega)\right),\\&u_{mt}\rightharpoonup u_t \quad \text{weakly star in } L^2\left(\mathbb{R}^+;L^2(\Omega)\right),\\&z_m\rightharpoonup z \quad \text{weakly star in } L^2\left(\mathbb{R}^+;L^2(\Omega\times(0,1))\right).\end{aligned} \tag{48}$$

By (45)–(47), we can also deduce that u_m is bounded in $L^2(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega))$ and u_{mt} is bounded in $L^2(\mathbb{R}^+;L^2(\Omega))$. Then from Aubin-Lions theorem [29], we infer that for any $T>0$,

$$u_m\longrightarrow u \quad \text{strongly in } L^\infty\left(0,T;H_0^1(\Omega)\right). \tag{49}$$

We also obtain by Lemma 1.4 in Kim [30] that

$$u_m\longrightarrow u \quad \text{strongly in } C\left(0,T;H_0^1(\Omega)\right). \tag{50}$$

Then we can pass to limit the approximate problem (28)-(29) in order to get a weak solution of problem (17)-(18).

3.4. Continuous Dependence and Uniqueness. Firstly we prove the continuous dependence and uniqueness for stronger solutions of problem (17)-(18).

Let $(u(t),u_t(t),z_1(x,\rho,t))$ and $(v(t),v_t(t),z_2(x,\rho,t))$ be two global solutions of problem (17)-(18) with respect to initial data (u_0,u_1,h_{01}) and (v_0,v_1,h_{02}) respectively. Let $\omega(t)=u(t)-v(t)$, $\chi(x,\rho,t)=z_1(x,\rho,t)-z_2(x,\rho,t)$. Then $(\omega(t),\chi(x,\rho,t))$ verifies

$$\begin{aligned}&\omega_{tt}+\Delta^2\omega-\left(M\left(\|\nabla u\|^2\right)\Delta u-M\left(\|\nabla v\|^2\right)\Delta v\right)+\int_0^t g(t-s)\,\Delta\omega(s)\,ds+\mu_1\omega_t\\&\quad+\mu_2\chi(x,1,t)+f(u)-f(v)=0,\\&\tau\chi_t+\chi_\rho=0,\end{aligned} \tag{51}$$

with boundary conditions

$$\omega = \Delta\omega = 0, \quad \text{on } \partial\Omega \tag{52}$$

and initial data

$$\omega(x,0) = \omega_0, \qquad \omega_t(x,0) = \omega_1, \\ \chi(x,\rho,0) = \chi_0 = h_{01} - h_{02}. \tag{53}$$

Multiplying (47) by ω_t and integrating the result over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\omega_t(t)\|^2 + \|\Delta\omega(t)\|^2 \right.\\ &\qquad \left. - \left(\int_0^t g(s)\,ds\right)\|\nabla\omega(t)\|^2 + (g\circ\nabla\omega)(t)\right) \\ &\quad + \mu_1\|\omega_t(t)\|^2 \\ &\le \left|\left(M\left(\|\nabla u\|^2\right)\Delta u - M\left(\|\nabla v\|^2\right)\Delta v, \omega_t\right)\right| \\ &\quad + \mu_2\int_\Omega |\chi(x,1,t)|\,|\omega_t|\,dx \\ &\quad + \int_\Omega |f(u) - f(v)|\,|\omega_t|\,dx. \end{aligned} \tag{54}$$

By mean value theorem and Hölder's inequality, we derive

$$\begin{aligned} &\left|\left(M\left(\|\nabla u\|^2\right)\Delta u - M\left(\|\nabla v\|^2\right)\Delta v, \omega_t\right)\right| \\ &= \left|\int_\Omega \left(M\left(\|\nabla u\|^2\right)\Delta u - M\left(\|\nabla u\|^2\right)\Delta v \right.\right. \\ &\qquad \left.\left. + M\left(\|\nabla u\|^2\right)\Delta v - M\left(\|\nabla v\|^2\right)\Delta v\right)\omega_t\,dx\right| \\ &\le \int_\Omega M\left(\|\nabla u\|^2\right)|\Delta\omega|\,|\omega_t|\,dx \\ &\quad + \int_\Omega |M'(\eta)|\left|\left(\|\nabla u\|^2 - \|\nabla v\|^2\right)\right||\Delta v|\,|\omega_t|\,dx \\ &\le \int_\Omega M\left(\|\nabla u\|^2\right)|\Delta\omega|\,|\omega_t|\,dx \\ &\quad + C_1\int_\Omega (|\nabla u| + |\nabla v|)\,|\nabla\omega|\,|\Delta v|\,|\omega_t|\,dx \\ &\le C_1\left(\|\Delta\omega\|^2 + \|\omega_t\|^2\right) \\ &\quad + C_1\left(\|\nabla u\|_{L^\infty} + \|\nabla v\|_{L^\infty}\right)\|\nabla\omega\|_{L^\infty}\|\Delta v\|\,\|\omega_t\| \\ &\le C_1\left(\|\Delta\omega\|^2 + \|\omega_t\|^2\right). \end{aligned} \tag{55}$$

It follows from (12)-(13) and Hölder's inequality that

$$\begin{aligned} &\int_\Omega |f(u) - f(v)|\,|\omega_t|\,dx \\ &\quad \le C_1\int_\Omega \left(1 + |u|^\rho + |v|^\rho\right)|\omega|\,|\omega_t|\,dx \\ &\quad \le C_1\left(\|u\|_{2(\rho+1)}^\rho + \|v\|_{2(\rho+1)}^\rho\right)\|\omega\|_{2(\rho+1)}\|\omega_t\| \\ &\quad \le C_1\left(\|\Delta\omega\|^2 + \|\omega_t\|^2\right). \end{aligned} \tag{56}$$

Moreover,

$$\begin{aligned} &\mu_2\int_\Omega |\chi(x,1,t)|\,|\omega_t|\,dx \\ &\qquad \le \frac{\mu_2}{2}\int_\Omega \chi^2(x,1,t)\,dx + \frac{\mu_2}{2}\|\omega_t\|^2. \end{aligned} \tag{57}$$

Noting that (35)-(36) and combining (54)–(57), we conclude that

$$\begin{aligned} &\frac{d}{dt}\mathscr{E}(t) + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|\omega_t\|^2 \\ &\qquad + \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_\Omega \chi^2(x,1,t)\,dx \\ &\quad \le C_1\left(\|\Delta\omega\|^2 + \|\omega_t\|^2\right), \end{aligned} \tag{58}$$

where

$$\begin{aligned} \mathscr{E}(t) &= \|\omega_t(t)\|^2 + \|\Delta\omega(t)\|^2 - \left(\int_0^t g(s)\,ds\right)\|\nabla\omega(t)\|^2 \\ &\quad + (g\circ\nabla\omega)(t) + \frac{\xi}{2}\int_\Omega\int_0^1 \chi^2(x,\rho,t)\,d\rho\,dx. \end{aligned} \tag{59}$$

It follows (19) that

$$\mathscr{E}(t) \le \mathscr{E}(0) + C_1\int_0^t \left(\|\Delta\omega\|^2 + \|\omega_t\|^2\right)(s)\,ds, \tag{60}$$

which, along with (9), gives

$$\|\Delta\omega\|^2 + \|\omega_t\|^2 \le \mathscr{E}(0) + C_1\int_0^t \left(\|\Delta\omega\|^2 + \|\omega_t\|^2\right)(s)\,ds. \tag{61}$$

Applying Gronwall's inequality to (61), we get

$$\|\Delta\omega\|^2 + \|\omega_t\|^2 \le e^{C_1 t}\mathscr{E}(0). \tag{62}$$

This shows that solution of problem (17)-(18) depends continuously on the initial data. In particular, problem (17)-(18) has a unique stronger solution.

We can prove the continuous dependence and uniqueness for weak solutions by using density arguments (see, e.g., Cavalcanti et al. [27]) which also can be found in Lions [29] (Chapter 1, Theorem 1.2) by using a regularization method and in Pata and Zucchi [31] or Giorgi et al. [32] by using the mollifiers.

This ends the proof of Theorem 1.

4. General Decay

In this section, we will establish the decay property of the solution for problem (17)-(18) in the case $\mu_2 < \mu_1$. Motivated by [27, 33], we use a perturbed energy method and suitable Lyapunov functionals.

We first consider stronger solutions. Define the modified energy by

$$\begin{aligned} F(t) = {} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 \\ & + \frac{1}{2}\widehat{M}\left(\|\nabla u(t)\|^2\right) + \int_\Omega \widehat{f}(u(t))\,dx \\ & - \frac{1}{2}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ & + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho\,dx, \end{aligned} \tag{63}$$

where ξ is a positive constant satisfying (19).

It follows from (9) and (14) that

$$\begin{aligned} F(t) &= E(t) - \frac{1}{2}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ &\geq lE(t), \end{aligned} \tag{64}$$

that is,

$$E(t) \leq \frac{1}{l}F(t). \tag{65}$$

Lemma 3. *Under the assumptions in Theorem 2, the modified energy functional defined by (63) satisfies that there exists a constant $c > 0$ such that, for any $t \geq 0$,*

$$\begin{aligned} \frac{d}{dt}F(t) \leq {} & -c\int_\Omega u_t^2(t)\,dx - c\int_\Omega z^2(x,1,t)\,dx \\ & + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2. \end{aligned} \tag{66}$$

Proof. For the same argument as (41) in Section 3.2, we can easily get (66). Here we omit the detailed proof. □

Now we define the following functional:

$$\Phi(t) = \int_\Omega u_t(t)\,u(t)\,dx. \tag{67}$$

Then we have the following lemma.

Lemma 4. *Under the assumptions in Theorem 2, the functional $\Phi(t)$ defined in (67) satisfies that, for any $\eta > 0$,*

$$\begin{aligned} \frac{d}{dt}\Phi(t) \leq {} & \left(1 + \frac{\mu_1}{4\eta}\right)\|u_t(t)\|^2 \\ & - (l - \eta\lambda - 2\eta\lambda\lambda_1(\mu_1+\mu_2))\|\Delta u(t)\|^2 \\ & - \widehat{M}\left(\|\nabla u(t)\|^2\right) + \frac{\mu_2}{4\eta}\int_\Omega z^2(x,1,t)\,dx \\ & + \frac{1-l}{4\eta\lambda}(g\circ\nabla u)(t), \end{aligned} \tag{68}$$

where λ_1 is the Poincaré constant.

Proof. By taking a derivative of (67) and using the first equation of (17), we conclude that

$$\begin{aligned} \frac{d}{dt}\Phi(t) = {} & \int_\Omega u_{tt}(t)\,u(t)\,dx + \|u_t(t)\|^2 \\ = {} & \|u_t(t)\|^2 \\ & + \int_\Omega \Big(-\Delta^2 u(t) + M\left(\|\nabla u(t)\|^2\right)\Delta u(t) \\ & - \int_0^t g(t-s)\,\Delta u(s)\,ds \\ & - \mu_2 u_t(t) - \mu_2 z(x,1,t) - f(u)\Big) \\ & \cdot u(t)\,dx \\ \leq {} & \|u_t(t)\|^2 - \|\Delta u(t)\|^2 - \widehat{M}\left(\|\nabla u(t)\|^2\right) \\ & + \int_0^t g(t-s)(\nabla u(s),\nabla u(t))\,ds \\ & - \mu_1\int_\Omega u_t(t)\,u(t)\,dx - \mu_2\int_\Omega z(x,1,t)\,u(t)\,dx \\ & - \int_\Omega f(u(t))\,u(t)\,dx. \end{aligned} \tag{69}$$

Using Hölder's inequality, we know that, for any $\eta > 0$,

$$\begin{aligned} & \int_0^t g(t-s)(\nabla u(s),\nabla u(t))\,ds \\ & = \int_0^t g(t-s)\int_\Omega (\nabla u(s) - \nabla u(t) + \nabla u(t))\cdot\nabla u(t)\,dx\,ds \\ & \leq \int_0^t g(t-s)\int_\Omega |\nabla u(t) - \nabla u(s)|\,|\nabla u(t)|\,dx\,ds \\ & \quad + \int_0^t g(s)\,ds\cdot\|\nabla u(t)\|^2 \\ & \leq \|\nabla u(t)\|^2\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,ds \\ & \quad + \int_0^t g(s)\,ds\cdot\|\nabla u(t)\|^2 \\ & \leq \eta\|\nabla u(t)\|^2 + \frac{1}{4\eta}\|g(t)\|_{L^1(\mathbb{R}^+)}(g\circ\nabla u)(t) \\ & \quad + \int_0^t g(s)\,ds\cdot\|\nabla u(t)\|^2 \\ & \leq \eta\lambda\|\Delta u(t)\|^2 + \frac{1}{4\eta}\|g(t)\|_{L^1(\mathbb{R}^+)}(g\circ\nabla u)(t) \\ & \quad + \int_0^t g(s)\,ds\cdot\|\nabla u(t)\|^2 \end{aligned}$$

$$\le \left(1 - l + \eta\lambda\right) \|\Delta u(t)\|^2 + \frac{1}{4\eta} \|g(t)\|_{L^1(\mathbb{R}^+)} \left(g \circ \nabla u\right)(t). \quad (70)$$

By using Young's inequality and Poincaré's inequality and noting $\|\nabla u\|^2 \le \lambda \|\Delta u\|^2$, we infer that, for any $\eta > 0$,

$$\int_\Omega u_t(t) u(t)\, dx \le \eta\lambda\lambda_1 \|\Delta u(t)\|^2 + \frac{1}{4\eta} \|u_t(t)\|^2, \quad (71)$$

$$\int_\Omega z(x,1,t) u(t)\, dx \le \eta\lambda\lambda_1 \|\Delta u(t)\|^2 + \frac{1}{4\eta} \int_\Omega z^2(x,1,t)\, dx. \quad (72)$$

Combining (69)–(72) and noting (14), we complete the proof. □

In order to handle the term $z(x,\rho,t)$, we introduce the functional

$$\Psi(t) := \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho\, dx. \quad (73)$$

Then we have the following estimate.

Lemma 5. *Under the assumptions in Theorem 2, the functional $\Psi(t)$ defined in (73) satisfies that*

$$\frac{d}{dt}\Psi(t) \le -\rho\Psi(t) - \frac{c_1}{2\tau} \int_0^1 z^2(x,1,t)\, dx + \frac{1}{2\tau} \int_0^1 u_t^2(t)\, dx, \quad (74)$$

where c_1 is a positive constant.

Proof. Differentiating (73) with respect to t and using the second equation (17), we obtain

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= -\frac{1}{\tau} \int_\Omega \int_0^1 e^{-2\tau\rho} z(x,\rho,t) z_\rho(x,\rho,t)\, d\rho\, dx \\ &= -\int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho\, dx \\ &\quad - \frac{1}{2\tau} \int_\Omega \int_0^1 \frac{\partial}{\partial\rho} \left(e^{-2\tau\rho} z^2(x,\rho,t)\right) d\rho\, dx.\end{aligned} \quad (75)$$

Then it is easy to verify that there exists a constant $c_1 > 0$ satisfying (74). □

Proof of Theorem 2. We define the Lyapunov functional

$$\mathcal{G}(t) := F(t) + \epsilon\Phi(t) + \epsilon\Psi(t), \quad (76)$$

where $\epsilon > 0$ is a real number which will be taken later.

First, we claim that there exist two positive constants β_1 and β_2 such that, for any $t \ge 0$,

$$\beta_1 F(t) \le \mathcal{G}(t) \le \beta_2 F(t). \quad (77)$$

Indeed, it is easy to get

$$\begin{aligned}|\Phi(t) + \Psi(t)| &\le \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2\lambda'} \|\Delta u(t)\|^2 + \int_\Omega \int_0^1 z^2(x,\rho,t)\, d\rho\, dx \\ &\le \frac{1}{l} \max\left\{1, \frac{1}{\lambda'}, \xi\right\} F(t),\end{aligned} \quad (78)$$

where $\lambda' > 0$ is the first eigenvalue of $\Delta^2 u = \lambda u$ in Ω with $u = \Delta u = 0$ on $\partial\Omega$. Choosing $C_1 = (1/l)\max\{1, 1/\lambda', \xi\}$, we know that

$$|\mathcal{G}(t) - F(t)| = \epsilon |\Phi(t) + \Psi(t)| \le \epsilon C_1 F(t). \quad (79)$$

Now putting $\epsilon > 0$ small enough and choosing $\beta_1 := 1 - \epsilon C_1 > 0$ and $\beta_2 = 1 + \epsilon C_1 > 0$, we see that (77) holds.

Next, combining (66), (68), and (74), we arrive at

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) \le &-\left(c - \epsilon\left(1 + \frac{\mu_1}{4\eta}\right) - \frac{\epsilon}{2\tau}\right) \|u_t(t)\|^2 \\ &- \epsilon\widehat{M}\left(\|\nabla u(t)\|^2\right) - \rho\epsilon\Psi(t) \\ &- \epsilon\left(l - \eta\lambda - 2\eta\lambda\lambda_1\left(\mu_1 + \mu_2\right)\right) \|\Delta u(t)\|^2 \\ &- \left(c + \frac{\epsilon c_1}{2\tau} - \frac{\epsilon}{4\eta}\right) \int_\Omega z^2(x,1,t)\, dx \\ &+ \frac{1-l}{4\eta\epsilon} \left(g \circ \nabla u\right)(t) + \frac{1}{2}\left(g' \circ \nabla u\right)(t) \\ &- \frac{1}{2} g(t) \|\nabla u(t)\|^2.\end{aligned} \quad (80)$$

Now we choose $\eta > 0$ and $\epsilon > 0$ so small that we can take two positive constants α_1 and α_2 such that, for any $t \ge 0$,

$$\mathcal{G}'(t) \le -\alpha_1 F(t) + \alpha_2 \left(g \circ \nabla u\right)(t). \quad (81)$$

Multiplying (81) by $\mu(t)$, we have, for any $t \ge 0$,

$$\mu(t)\mathcal{G}'(t) \le -\alpha_1 \mu(t) F(t) + \alpha_2 \mu(t) \left(g \circ \nabla u\right)(t), \quad (82)$$

which, along with (10) and (66), implies

$$\begin{aligned}\mu(t)\mathcal{G}'(t) &\le -\alpha_1 \mu(t) F(t) - \alpha_2 \left(g' \circ \nabla u\right)(t) \\ &\le -\alpha_1 \mu(t) F(t) - 2\alpha_2 E'(t), \quad \forall t \ge 0,\end{aligned} \quad (83)$$

that is,

$$\begin{aligned}&\left(\mu(t)\mathcal{G}(t) + 2\alpha_2 F(t)\right)' - \mu'(t)\mathcal{G}(t) \\ &\quad \le -\alpha_1 \mu(t) F(t), \quad \forall t \ge 0.\end{aligned} \quad (84)$$

Denote $\mathcal{F}(t) = \mu(t)\mathcal{G}(t) + 2\alpha_2 F(t)$, and then $\mathcal{F}(t)$ is equivalent to $F(t)$; that is,

$$\mathcal{F}(t) \sim F(t). \quad (85)$$

Thus we conclude that, for any $t \geq 0$,

$$\mathscr{F}'(t) \leq -\alpha_2 \mu(t) F(t) \leq -\alpha_3 \mu(t) \mathscr{F}(t). \quad (86)$$

Integrating (86) over $(0, t)$, we will see the following:

$$\mathscr{F}(t) \leq \mathscr{F}(0) \exp\left(-\alpha_3 \int_0^t \mu(s)\, ds\right), \quad \forall t \geq 0, \quad (87)$$

which, together with (65), (77), and (85), gives (25).

This proves the general decay for regular solutions. We can extend the result to weak solutions by using a standard density argument; one can refer to Cavalcanti et al. [27]. The proof is hence complete. □

Remark 6. There are some open problems concerning our present work, and here we give some of them.

(1) It is obvious that the weak damping term $\mu_1 u_t$ plays a crucial role in our proofs. It is still an open problem when $\mu_1 = 0$.

(2) We only obtain the general decay for $\mu_1 > \mu_2$. Whether the stability property holds for $\mu_1 = \mu_2$ is still open.

(3) It is interesting to study that the weight of the delay is bigger than the weight of the damping; that is, $\mu_1 < \mu_2$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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