Internat. J. Math. & Math. Sci.
VOL. 17 NO. 4 (1994) 817-820



# SEVERAL IDENTITIES IN STATISTICAL MECHANICS

**M. D. HIRSCHHORN**

Department of Pure Mathematics
University of New South Wales
P.O. Box 1, Kensington
New South Wales 2033, Australia

(Received March 16, 1993)

**ABSTRACT.** In an earlier paper concerning a solvable model in statistical mechanics, Miwa and Jimbo state a theta-function identity which they have checked to the 200th power, but of which they do not have a proof. The main objective of this note is to provide such a proof.

**KEY WORDS AND PHRASES.** $q$-series identity, Jacobi's triple product, sums and products.
**1991 AMS SUBJECT CLASSIFICATION CODE(S).** 83B23.

## 1. INTRODUCTION.

In their paper [1], Miwa and Jimbo state a theta function identity, which they have checked to the 2000th power, but of which they do not have a proof. They also state, and prove, two other theta function identities. The object of this note is to provide a proof of the first identity, and to make the observation that the other two identities are special cases of an identity of mine that generalizes the quintuple-product identity.

## 2. MAIN RESULTS.

Our only tool in proving the first identity is Jacobi's triple product identity,

$$(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty(q^2;q^2)_\infty = \sum a^n q^{n^2}. \tag{2.1}$$

[Here, $(a;q)_\infty = \prod_{n\geq 1}(1-aq^{n-1})$, and throughout this paper all sums are taken from $-\infty$ to $\infty$.]

The identity to be proved ([1], (B.13)) is

$$\begin{aligned}&\Big(\sum q^{(3n^2-3n)/2}\Big)\Big(\sum(-1)^n q^{6n^2-4n+1}\Big)+\Big(\sum q^{(3n^2-n)/2}\Big)\Big(\sum(-1)^n q^{6n^2}\Big)\\ &=\Big(\sum q^{n^2}\Big)\Big(\sum(-1)^n q^{6n^2-n}+\sum(-1)^n q^{6n^2-5n+1}\Big)\end{aligned} \tag{2.2}$$

Our first step is to make use of (2.1) to express all the sums in (2.2) as products.

Thus

$$\sum q^{(3n^2-3n)/2} = 2(-q^3;q^3)^2_\infty(q^3;q^3)_\infty$$

$$\sum(-1)^n q^{6n^2-4n} = (q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty$$

$$\sum q^{(3n^2-n)/2} = (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty$$

$$\sum(-1)^n q^{6n^2} = (q^6;q^{12})^2_\infty(q^{12};q^{12})_\infty$$

$$\sum q^{n^2} = (-q;q^2)^2_\infty (q^2;q^2)_\infty$$

and

$$\begin{aligned}\sum(-1)^n q^{6n^2-n} + \sum(-1)^n q^{6n^2-5n+1} &= \sum(-1)^n(-q)^{(3n^2-n)/2}\\ &= \prod_{n\ge 1}(1-(-q)^n)\\ &= (-q;q^2)_\infty(q^2;q^2)_\infty.\end{aligned}$$

So (2.2) is equivalent to

$$\begin{aligned}&(-q;q^2)^3_\infty(q^2;q^2)^2_\infty\\ &=(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty(q^6;q^{12})^2_\infty(q^{12};q^{12})_\infty\\ &+2q(-q^3;q^3)^2_\infty(q^3;q^3)_\infty(q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty\\ &=(-q;q^6)_\infty(-q^2;q^6)_\infty(q^3;q^6)_\infty(-q^4;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty\cdot\\ &\qquad\cdot(q^6;q^{12})^2_\infty(q^{12};q^{12})_\infty\\ &+2q(-q^3;q^6)^2_\infty(q^3;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty\cdot\\ &\qquad\cdot(q^2;q^{12})_\infty(q^{10};q^{12})(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty.\end{aligned} \tag{2.3}$$

If we put $-q$ for $q$, we see that (2.3) is equivalent to

$$\begin{aligned}&(q;q^2)^3_\infty(q^2;q^2)^2_\infty\\ &=(q;q^6)_\infty(-q^2;q^6)_\infty(-q^3;q^6)_\infty(-q^4;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty\\ &\qquad\cdot(q^3;q^6)^2_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty(-q^6;q^6)_\infty\\ &-2q(q^3;q^6)^2_\infty(-q^3;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty\cdot\\ &\qquad\cdot(q;q^6)_\infty(-q;q^6)_\infty(q^5;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty(-q^6;q^6)_\infty\end{aligned} \tag{2.4}$$

Next divide by

$$(q;q^2)_\infty = (q;q^6)_\infty(q^3;q^6)_\infty(q^5;q^6)_\infty$$

and use the fact that

$$(q;q^2)_\infty(-q;q^2)_\infty(-q^2;q^2)_\infty = (q;q^2)_\infty(-q;q)_\infty = (q;q^2)_\infty\frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{(q;q)_\infty}{(q;q)_\infty} = 1$$

with $q^3$ for $q$, and we see that (2.4) is equivalent to

$$\begin{aligned}(q;q)^2_\infty &= (-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty\\ &-2q(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty.\end{aligned} \tag{2.5}$$

We now prove (2.5)

$$(q;q)^2_\infty = \sum(-1)^{r+s}q^{(3r^2-r+3s^2-s)/2}.$$

Split this sum into two, according to whether $r+s$ is even or odd. If $r+s$ is even, let $r=m+n, s=m-n$; if $r+s$ is odd, let $r=m+n+1, s=m-n$.

We obtain

$$\begin{aligned}(q;q)^2_\infty &= \sum q^{(3(m+n)^2-(m+n)+3(m-n)^2-(m-n))/2}\\ &-\sum q^{(3(m+n+1)^2-(m+n+1)+3(m-n)^2-(m-n))/2}\end{aligned}$$

$$= \sum q^{3m^2 - m + 3n^2} - \sum q^{3m^2 + 2m + 3n^2 - 3n + 1}$$
$$= (-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty$$
$$- 2q(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty$$

as required.

The other two identities stated and proved by Miwa and Jimbo are ([1], (B.12))

$$\left(\frac{1}{2}\sum q^{(3n^2-3n)/2}\right)\left(\sum q^{12n^2}\right) + \left(\sum q^{(3n^2-n)/2}\right)\left(\sum q^{12n^2-8n+1}\right) = \left(\sum q^{2n^2-n}\right)\left(\sum q^{4n^2-2n}\right) \tag{2.6}$$

and

$$\left(\frac{1}{2}\sum q^{(3n^2-3n)/2}\right)\left(\sum q^{12n^2-12n+3}\right) + \left(\sum q^{(3n^2-n)/2}\right)\left(\sum q^{12n^2-4n}\right) = \left(\sum q^{2n^2-n}\right)\left(\sum q^{4n^2-2n}\right). \tag{2.7}$$

These can be written, respectively, as

$$\left(\sum q^{2n^2-n}\right)\left(\sum q^{4n^2-2n}\right) = \left(\sum q^{6n^2-3n}\right)\left(\sum q^{12n^2}\right) + \left(\sum q^{6n^2-n} + \sum q^{6n^2-5n+1}\right)\left(\sum q^{12n^2-8n+1}\right) \tag{2.8}$$

and

$$\left(\sum q^{2n^2-n}\right)\left(\sum q^{4n^2-2n}\right) = \left(\sum q^{6n^2-3n}\right)\left(\sum q^{12n^2-12n+3}\right) + \left(\sum q^{6n^2-n} + \sum q^{6n^2-5n+1}\right)\left(\sum q^{12n^2-4n}\right). \tag{2.9}$$

In product form, these become

$$\begin{aligned}
&(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty(-q^2;q^8)_\infty(-q^6;q^8)_\infty(q^8;q^8)_\infty \\
&= (-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^{12};q^{12})_\infty(-q^{12};q^{24})^2_\infty(q^{24};q^{24})_\infty \\
&+ q(-q^5;q^{12})_\infty(-q^7;q^{12})_\infty(q^{12};q^{12})_\infty(-q^4;q^{24})_\infty(-q^{20};q^{24})_\infty(q^{24};q^{24})_\infty \\
&+ q^2(-q;q^{12})_\infty(-q^{11};q^{12})_\infty(q^{12};q^{12})_\infty(-q^4;q^{24})_\infty(-q^{20};q^{24})_\infty(q^{24};q^{24})_\infty
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty(-q^2;q^8)_\infty(-q^6;q^8)_\infty(q^8;q^8)_\infty \\
&= (-q^5;q^{12})_\infty(-q^7;q^{12})_\infty(q^{12};q^{12})_\infty(-q^8;q^{24})_\infty(-q^{16};q^{24})_\infty(q^{24};q^{24})_\infty \\
&+ q(-q;q^{12})_\infty(-q^{11};q^{12})_\infty(q^{12};q^{12})_\infty(-q^8;q^{24})_\infty(-q^{16};q^{24})_\infty(q^{24};q^{24})_\infty \\
&+ 2q^3(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^{12};q^{12})_\infty(-q^{24};q^{24})^2_\infty(q^{24};q^{24})_\infty
\end{aligned} \tag{2.11}$$

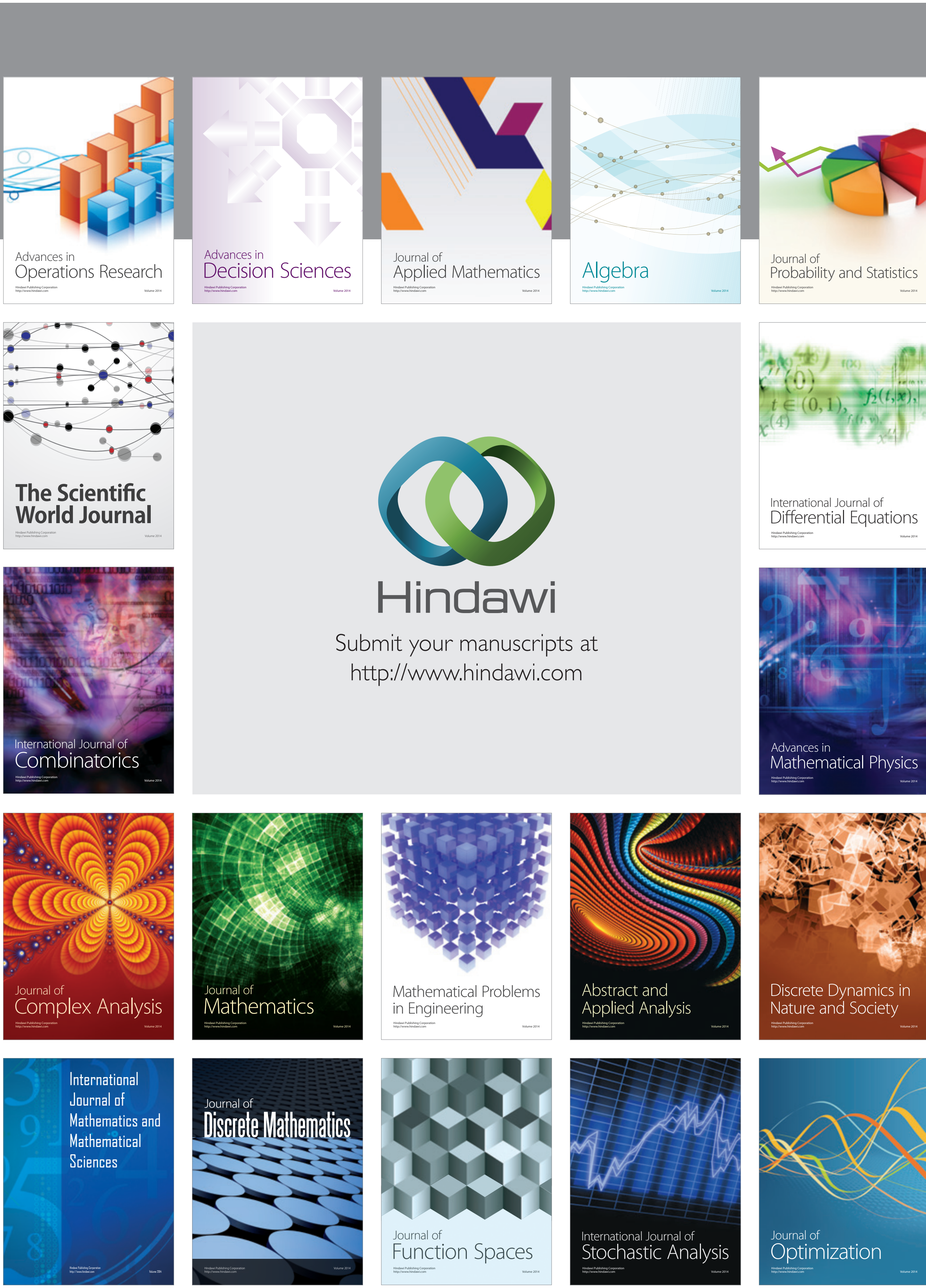
Now, consider the identity ([2], (2))

$$\begin{aligned}
&(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty(q^2;q^2)_\infty(-bq^2;q^4)_\infty(-b^{-1}q^2;q^4)_\infty(q^4;q^4)_\infty \\
&= (-ab^{-1}q^3;q^6)_\infty(-a^{-1}bq^3;q^6)_\infty(q^6;q^6)_\infty(-a^2bq^6;q^{12})_\infty(-a^{-2}b^{-1}q^6;q^{12})_\infty(q^{12};q^{12})_\infty \\
&+ aq(-ab^{-1}q^5;q^6)_\infty(-a^{-1}bq;q^6)_\infty(q^6;q^6)_\infty(-a^2bq^{10};q^{12})_\infty(-a^{-2}b^{-1}q^2;q^{12})_\infty(q^{12};q^{12})_\infty \\
&+ a^{-1}q(-ab^{-1}q;q^6)_\infty(-a^{-1}bq^5;q^6)_\infty(q^6;q^6)_\infty(-a^2bq^2;q^{12})_\infty(-a^{-2}b^{-1}q^{10};q^{12})_\infty(q^{12};q^{12})_\infty
\end{aligned} \tag{2.12}$$

If in (2.12), we set $q^2$ for $q$, then $a = q^{-1}, b = q^2$, we obtain (2.10), while if we set $q^2$ for $q$ then $a = q^{-1}, b = q^{-2}$, we obtain (2.11).

## REFERENCES

1. JIMBO, M. and MIWA, T., A solvable model and related Rogers-Ramanujan type identities, *Physica D* **15** (1985), 335-353.
2. HIRSCHHORN, M.D., A generalization of the quintuple product identity, *J. Austral. Math. Soc. (Series A)* **44** (1988), 42-45.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization