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Research Article

Convergence Analysis of Parallel S-Iteration Process for System of Generalized Variational Inequalities

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We consider a new system of generalized variational inequalities (SGVI) defined on two closed convex subsets of a real Hilbert space. To find the solution of considered SGVI, a parallel Mann iteration process and a parallel S-iteration process have been proposed and the strong convergence of the sequences generated by these parallel iteration processes is discussed. Numerical example illustrates that the proposed parallel S-iteration process has an advantage over parallel Mann iteration process in computing altering points of some mappings.

1. Introduction

Variational inequalities are the most interesting and important mathematical problems and have been studied intensively in the past years. The variational inequality problem was first introduced and studied by Stampacchia [1] in 1964, which is defined as follows.

Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be a nonlinear mapping. Then the classical variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0 \quad \forall x \in C. \tag{1}$$

The problem (1) is denoted by $VI(C, T)$ and the set of solutions of (1) is denoted and defined by $\Omega[VI(C,T)] = \{x^* \in C : \langle Tx^*, x - x^* \rangle \geq 0 \text{ for all } x \in C\}$. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T. It is well known that the variational inequality problem (1) is equivalent to the following fixed point problem:

$$\text{find } x^* \in C \text{ such that } x^* = P_C(I - \mu T)x^*, \tag{2}$$

where P_C is the metric projection from H onto C, $\mu > 0$ is a constant, and I is the identity mapping from H into itself. It is well known that if the mapping T is k-Lipschitzian and η-strongly monotone, then the operator $P_C(I - \mu T)$ is a contraction on C provided that $0 < \mu < 2\eta/k^2$. In this case, the Banach contraction principle guarantees that $VI(C, T)$ has a unique solution x^* and the sequence of Picard iteration method given by

$$x_{n+1} = P_C(I - \mu T)x_n \quad \forall n \in \mathbb{N} \tag{3}$$

converges strongly to x^*. This method is called the projected gradient method [2]. This method has been widely used in many practical problems, due partially to its fast convergence.

In 2007, Agarwal et al. [3] posed the following query.

Question 1. Is it possible to develop an iterative method whose rate of convergence is faster than the Picard iteration method for contraction mappings?

They introduced the following iteration process known as S-iteration process as an answer to Question 1: let C be a nonempty convex subset of a normed linear space X, and let

$T : C \to C$ be an operator. Then, for arbitrary $x_1 \in C$, the S-iteration process is defined by

$$x_{n+1} = (1-\alpha_n) Tx_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n) x_n + \beta_n T x_n \quad \forall n \in \mathbb{N}, \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying some suitable conditions. In [4], Sahu proved that the rate of convergence of S-iteration process for contraction mappings is faster than that of Picard [5] and Mann [6] iteration processes by providing a numerical example. The S-iteration process is more applicable than the Picard [5], Mann [6], and Ishikawa [7] iteration processes because it converges faster than these iteration processes for contraction mappings and also works for nonexpansive mappings. Due to the super rate of convergence of above iteration process, Agarwal et al. [3] called it the S-iteration process. Due to its fastness, in recent years, the S-iteration process attracted many researchers as an alternate iteration process and is used for solving fixed point problems, common fixed point problems, convex minimization problems, the problem of solving nonlinear operator equations, and other allied areas (see [8–10]). Moreover, the idea of S-iteration process is applied by Cholamjiak et al. [11] for finding a minimizer of a convex function and fixed points of nonexpansive mappings in CAT(0) space setting. Sahu [4] also introduced the notion of S-operator of a mapping T generated by $\alpha \in (0,1)$ and T and normal S-iteration process in the following way: let C be a nonempty convex subset of a normed linear space X and let $T : C \to C$ be an operator. Then, for arbitrary $x_1 \in C$, the normal S-iteration process is defined by

$$x_{n+1} = T\left[(1-\alpha_n) x_n + \alpha_n T(x_n)\right] \quad \forall n \in \mathbb{N}, \tag{5}$$

where $\{\alpha_n\}$ is a sequence of real numbers in $(0,1)$. In 2017, Verma and Shukla [12] designed some new algorithms based on S-iteration processes and named them as S-iteration-based forward-backward algorithm (SFBA) and normal S-iteration-based forward-backward algorithm (NSFBA) and performed the nice experiments of the high-dimensional real datasets for SFBA, NSFBA, and others.

On the other hand, in Hilbert spaces, projection type methods have played a very crucial role in the numerical resolution of variational inequalities depending on their convergence analysis. By virtue of the projection, in 2011, Ceng et al. [13] proposed the following iterative method:

$$x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T x_n\right] \quad \forall n \in \mathbb{N}, \tag{6}$$

where $F : C \to H$ is k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$, $V : C \to H$ is an L-Lipschitzian mapping with $L \ge 0$, $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$, $\{\alpha_n\} \subset (0,1)$, and $x_1 \in C$ an arbitrary initial point. They proved that the sequence $\{x_n\}$ generated by the iterative method (6) converges strongly to a fixed point x^* of T which solves the following variational inequality problem:

$$\left\langle (\mu F - \gamma V) x^*, x - x^* \right\rangle \ge 0 \quad \forall x \in C := \mathrm{Fix}(T). \tag{7}$$

In 2001, Verma [14] generalized the concept of variational inequalities to a system of nonlinear variational inequalities (SNVI) in the following way: find $x^*, y^* \in C$ such that

$$\left\langle \rho T(y^*) + x^* - y^*, x - x^* \right\rangle \ge 0, \quad \forall x \in C,$$
$$\left\langle \eta T(x^*) + y^* - x^*, x - y^* \right\rangle \ge 0, \quad \forall x \in C, \tag{8}$$

where $T : C \to H$ is any mapping and $\eta > 0$ and $\rho > 0$ are constants. To solve (8), he introduced the following iterative method:

$$y_n = P_C\left[x_n - \eta T(x_n)\right],$$
$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n P_C\left[y_n - \rho T(y_n)\right], \tag{9}$$

and he proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by (9) converge to x^* and y^*, respectively. In 2005, Verma [15] also introduced the general model for two-step projection methods for applying the approximation solvability of SNVI in Hilbert space setting as follows: let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be a nonlinear mapping. For arbitrary chosen initial point $x_1 \in C$, let $\{x_n\}$ and $\{y_n\}$ be the sequences in C defined by

$$y_n = (1-\beta_n) x_n + \beta_n P_C\left[x_n - \eta T(x_n)\right],$$
$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n P_C\left[y_n - \rho T(y_n)\right], \tag{10}$$

where $\eta > 0$, $\rho > 0$ and $0 \le \alpha_n, \beta_n \le 1$. Further, problem (8) is equivalent to the following projection formulas:

$$x^* = P_C(I - \rho T) y^*,$$
$$y^* = P_C(I - \eta T) x^*, \tag{11}$$

for a monotone mapping $T : C \to H$. The problem of finding the solutions of (11) by using iterative methods has been studied by many authors (see [15–22]). A more general case has been studied in [23].

Parallel iteration processes have their own advantages. A variety of problems have been dealt with in these iteration processes (see [24, 25] and the references therein). Recently, Sahu [26] introduced the notion of altering points of nonlinear mappings and following the idea of S-operator and normal S-iteration process, he [26] introduced a parallel S-iteration process for finding altering points of nonlinear mappings as follows.

Let C_1 and C_2 be two nonempty closed convex subsets of a Banach space X and let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two mappings. Then, for $\alpha \in (0,1)$ and arbitrary $(x_1, y_1) \in C_1 \times C_2$, the parallel normal S-iteration process is defined by

$$x_{n+1} = T_2\left[(1-\alpha) y_n + \alpha T_1(x_n)\right],$$
$$y_{n+1} = T_1\left[(1-\alpha) x_n + \alpha T_2(y_n)\right] \quad \forall n \in \mathbb{N}. \tag{12}$$

The following convergence result is given in [26].

Theorem 1 (see [26]). *Let C_1 and C_2 be two nonempty closed convex subsets of a Banach space X. Let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two Lipschitz continuous mappings with Lipschitz* *constants $k_1 < 1$ and $k_2 < 1$, respectively. Then the sequence $\{(x_n, y_n)\}$ in $C_1 \times C_2$ generated by the parallel S-iteration process (12) converges strongly to a unique point $(x^*, y^*) \in C_1 \times C_2$ such that x^* and y^* are altering points of mappings T_1 and T_2.*

In this paper, motivated by the work of Ceng et al. [13], Verma [14, 15], Hao et al. [23], and Sahu [26], we consider a new SGVI defined on two closed convex subsets of a real Hilbert space and propose a parallel Mann and a more general parallel S-iteration process for solving considered SGVI in the context of altering points and study the strong convergence of the sequences generated by the proposed algorithms to altering points of some nonlinear mappings.

2. Preliminaries

Throughout this paper, the symbol $\mathbb{N}$ stands for the set of all natural numbers.

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. A mapping $T : C \to H$ is called

(1) *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in C, \tag{13}$$

(2) *η-strongly monotone* if there exists a positive real number η such that

$$\langle Tx - Ty, x - y\rangle \geq \eta \|x - y\|^2 \quad \forall x, y \in C, \tag{14}$$

(3) *k-Lipschitzian* if there exists a constant $k \geq 0$ such that

$$\|Tx - Ty\| \leq k \|x - y\| \quad \forall x, y \in C, \tag{15}$$

(4) *θ-contraction* if there exists a constant $\theta \in [0, 1)$ such that

$$\|Tx - Ty\| \leq \theta \|x - y\| \quad \forall x, y \in C, \tag{16}$$

(5) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \tag{17}$$

Definition 2 (see [26]). Let C_1 and C_2 be two nonempty subsets of a metric space X. Then $x^* \in C_1$ and $y^* \in C_2$ are altering points of mappings $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ if

$$\begin{aligned} T_1(x^*) &= y^*, \\ T_2(y^*) &= x^*. \end{aligned} \tag{18}$$

The set of altering points of mappings T_1 and T_2 is denoted and defined by

$$\begin{aligned} \operatorname{Alt}(T_1, T_2) = \{&(x^*, y^*) \in C_1 \times C_2 : T_1(x^*) \\ &= y^*,\ T_2(y^*) = x^*\}. \end{aligned} \tag{19}$$

We now give some numerical examples in support of the definition of altering points of some nonlinear mappings as follows.

Example 3 (see [26]). Let $X = [0, 1]$, $C_1 = [0, 1/2]$, and $C_2 = [1/2, 1]$. Define $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ by $T_i(x) = 1 - x$ for $i = 1, 2$. Note that $T_2T_1 : C_1 \to C_1$ is defined by $T_2T_1x = T_2(1 - x) = x$. Thus, each point of C_1 is a fixed point of T_2T_1. Then altering points $x^* \in C_1$ and $y^* \in C_2$ of T_1 and T_2 are given by the relation $x^* + y^* = 1$. Indeed,

$$\operatorname{Alt}(T_1, T_2) = \{(x^*, y^*) \in C_1 \times C_2 : x^* + y^* = 1\}. \tag{20}$$

Example 4. Let $X = \mathbb{R}^2$, $C_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$, and $C_2 = \{(x, y) \in \mathbb{R}^2 : x \leq 0,\ y \leq 0\}$. Let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two mappings defined, respectively, by

$$\begin{aligned} T_1(x, y) &= (-x, -y - 1) \quad \forall (x, y) \in C_1, \\ T_2(x, y) &= \left(\frac{1 - x}{2}, \frac{1 - y}{2}\right) \quad \forall (x, y) \in C_2. \end{aligned} \tag{21}$$

Note that $T_2T_1 : C_1 \to C_1$ is defined by $T_2T_1(x, y) = T_2(-x, -y-1) = ((x+1)/2, (y+2)/2)$. Clearly $(1, 2) \in C_1$ and $(-1, -3) \in C_2$ are fixed points of T_2T_1 and T_1T_2, respectively. Therefore, $x^* = (1, 2)$ and $y^* = (-1, -3)$ are altering points of mappings T_1 and T_2.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point $P_C(x)$ of C such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C. \tag{22}$$

The mapping P_C is called the metric projection [27] from H onto C. It is remarkable that the metric projection mapping P_C is nonexpansive from H onto C (see Agarwal et al. [28]).

We need the following technical lemmas.

Lemma 5 (see [28]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$, then $z = P_C(x)$ if and only if $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Lemma 6 (see [29]). *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0, 1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator. Define the mapping $T_\lambda : C \to H$ by*

$$T_\lambda(x) = (I - \lambda\mu F)(x) \quad \forall x \in C,\ \lambda \in (0, 1). \tag{23}$$

Then T_λ is a contraction provided $0 < \mu < 2\eta/k^2$. More precisely, for $\mu \in (0, 2\eta/k^2)$,

$$\|T_\lambda(x) - T_\lambda(y)\| \leq (1 - \lambda\tau) \|x - y\| \quad \forall x, y \in C, \tag{24}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1]$.

Lemma 7 (see [17]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + b_n + c_n \quad \forall n \geq n_0, \tag{25}$$

where n_0 is some nonnegative integer, $\alpha_n \in (0, 1)$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, $b_n = \circ(\alpha_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 8 (see [26, Theorem 3.1]). *Let C_1 and C_2 be two nonempty closed subsets of a complete metric space X. Suppose that $S_1 : C_1 \to C_2$ and $S_2 : C_2 \to C_1$ be two Lipschitz continuous mappings with Lipschitz constants k_1 and k_2, respectively, such that $k_1 k_2 < 1$. Then the following holds:*

(a) *There exists a unique point $(x^*, y^*) \in C_1 \times C_2$ such that x^* and y^* are altering points of mappings S_1 and S_2.*

(b) *For arbitrary $x_1 \in C_1$, a sequence $\{(x_n, y_n)\} \in C_1 \times C_2$ generated by*

$$\begin{aligned} y_n &= S_1 x_n, \\ x_{n+1} &= S_2 y_n \quad \forall n \in \mathbb{N} \end{aligned} \tag{26}$$

converges to (x^*, y^*).

3. Main Results

In this section, we introduce a new system of generalized variational inequalities and new iterative algorithms for solving the proposed system of generalized variational inequalities in the framework of real Hilbert spaces.

Let C_1 and C_2 be nonempty closed convex subsets of a real Hilbert space H and let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be some mappings. Let $g_1, g_2 : H \to H$ be mappings. Consider a general system of generalized variational inequalities (SGVI) defined on C_1 and C_2 as follows.

Find $(x^*, y^*) \in C_1 \times C_2$ such that

$$\begin{aligned} &\langle sT_1(x^*) + y^* - g_1(x^*), g_1(y) - y^* \rangle \geq 0, \\ &\qquad\qquad \forall y \in C_2, \\ &\langle tT_2(y^*) + x^* - g_2(y^*), g_2(x) - x^* \rangle \geq 0, \\ &\qquad\qquad \forall x \in C_1, \end{aligned} \tag{27}$$

where $s > 0$ and $t > 0$ are constants.

Remark 9. If $T_1 = T_2 = T$, $g_1 = g_2 = I$, and $C_1 = C_2 = C$, then the system of generalized variational inequalities (SGVI) (27) reduces to SNVI (8) studied by Verma [14].

The system of generalized variational inequalities (27) is more general in nature. One can find various systems of generalized variational inequalities from SGVI (27).

We now discuss some special cases of (27) as follows.

Let $g_i : H \to H$ be single-valued δ_i-strongly monotone, η_i-Lipschitz continuous, let $F_i : C_i \to H$ be k_i-Lipschitzian and ξ_i-strongly monotone operator with constants $k_i, \xi_i > 0$, and let $V_i : C_i \to H$ be L_i-Lipschitzian mapping with constant $L_i \geq 0$ for $i \in \{1,2\}$. Suppose that $0 < \mu_i < 2\xi_i/k_i^2$ and $0 \leq \gamma_i < \tau_i/L_i$, where $\tau_i = 1 - \sqrt{1 - \mu_i(2\xi_i - \mu_i k_i^2)}$ for $i \in \{1,2\}$.

If $T_i = \mu_i F_i - \gamma_i V_i$ for $i = 1,2$, then the system of generalized variational inequalities (27) reduces to the following system of generalized variational inequalities (SGVI).

Find $(x^*, y^*) \in C_1 \times C_2$ such that

$$\begin{aligned} &\langle s(\mu_1 F_1 - \gamma_1 V_1)(x^*) + y^* - g_1(x^*), g_1(y) - y^* \rangle \\ &\quad \geq 0, \quad \forall y \in C_2, \\ &\langle t(\mu_2 F_2 - \gamma_2 V_2)(y^*) + x^* - g_2(y^*), g_2(x) - x^* \rangle \\ &\quad \geq 0, \quad \forall x \in C_1. \end{aligned} \tag{28}$$

Define the mappings $S_1 : C_1 \to C_2$ and $S_2 : C_2 \to C_1$ by

$$S_1 := P_{C_2}[g_1 - s(\mu_1 F_1 - \gamma_1 V_1)] \tag{29}$$

$$S_2 := P_{C_1}[g_2 - t(\mu_2 F_2 - \gamma_2 V_2)], \tag{30}$$

where s and t are some constants in $(0,1]$. Using Lemma 5, one can easily observe that the SGVI (28) is equivalent to the following altering point formulation:

$$\text{to find } (x^*, y^*) \in C_1 \times C_2 \text{ such that } \begin{cases} x^* = P_{C_1}[g_2 - t(\mu_2 F_2 - \gamma_2 V_2)](y^*), \\ y^* = P_{C_2}[g_1 - s(\mu_1 F_1 - \gamma_1 V_1)](x^*). \end{cases} \tag{31}$$

First we introduce parallel Mann iteration process to solve system of generalized variational inequalities (28) as follows.

Algorithm 10. For any given $(x_1, y_1) \in C_1 \times C_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $C_1 \times C_2$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n S_2(y_n), \\ y_{n+1} &= (1 - \alpha_n) y_n + \alpha_n S_1(x_n), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{32}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and S_1 and S_2 are defined by (29) and (30), respectively.

Motivated by Sahu [26] and equivalent formulation (31), we now propose a more general parallel S-iteration process to solve SGVI (28) as follows.

Algorithm 11. For any given $(x_1, y_1) \in C_1 \times C_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $C_1 \times C_2$ defined by

$$\begin{aligned} x_{n+1} &= S_2[(1 - \alpha_n) y_n + \alpha_n S_1(x_n)], \\ y_{n+1} &= S_1[(1 - \alpha_n) x_n + \alpha_n S_2(y_n)], \quad \forall n \in \mathbb{N}, \end{aligned} \tag{33}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and S_1 and S_2 are defined by (29) and (30), respectively.

Before proving our main results, we will prove the following proposition which will be used in sequel.

Proposition 12. *Let C_1 and C_2 be nonempty closed convex subsets of a real Hilbert space H. Let $g_i : H \to H$ be single-valued δ_i-strongly monotone, η_i-Lipschitz continuous, let $F_i : C_i \to H$ be k_i-Lipschitzian and ξ_i-strongly monotone operator with constants $k_i, \xi_i > 0$, and let $V_i : C_i \to H$ be L_i-Lipschitzian mapping with constant $L_i \geq 0$ for $i \in \{1,2\}$. Suppose that $0 < \mu_i < 2\xi_i/k_i^2$ and $0 \leq \gamma_i < \tau_i/L_i$, where $\tau_i = 1 - \sqrt{1 - \mu_i(2\xi_i - \mu_i k_i^2)}$ for $i \in \{1,2\}$. Let $s, t \in (0,1]$ and let θ_1 and θ_2 be real constants defined by*

$$\theta_i = \sqrt{1 - 2\delta_i + \eta_i^2} \quad \textit{for } i = 1,2. \tag{34}$$

Then the mappings S_1 and S_2 defined by (29) and (30) are Lipschitz continuous with Lipschitz constants $[\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))]$ and $[\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))]$, respectively.

Proof. Let $x, y \in C_1$. Then, we have

$$\begin{aligned}
&\|S_1(x) - S_1(y)\| = \|P_{C_2}[g_1 - s(\mu_1 F_1 - \gamma_1 V_1)](x) \\
&\quad - P_{C_2}[g_1 - s(\mu_1 F_1 - \gamma_1 V_1)](y)\| \le \|g_1(x) \\
&\quad - s(\mu_1 F_1 - \gamma_1 V_1)(x) - g_1(y) \\
&\quad + s(\mu_1 F_1 - \gamma_1 V_1)(y)\| = \|g_1(x) - g_1(y) \\
&\quad - (x - y) + (x - y) - s(\mu_1 F_1 - \gamma_1 V_1)(x - y)\| \\
&\quad \le \|g_1(x) - g_1(y) - (x - y)\| + \|(x - y) \\
&\quad - s(\mu_1 F_1 - \gamma_1 V_1)(x - y)\| \le \|x - y \\
&\quad - (g_1(x) - g_1(y))\| + \|x - s\mu_1 F_1(x) - y \\
&\quad + s\mu_1 F_1(y)\| + \|s\gamma_1 V_1(x) - s\gamma_1 V_1(y)\| \le \|x - y \\
&\quad - (g_1(x) - g_1(y))\| + \|(I - s\mu_1 F_1)(x) \\
&\quad - (I - s\mu_1 F_1)(y)\| + s\gamma_1 \|V_1(x) - V_1(y)\| \le \|x \\
&\quad - y - (g_1(x) - g_1(y))\| + (1 - s\tau_1)\|x - y\| \\
&\quad + s\gamma_1 L_1 \|x - y\| = \|x - y - (g_1(x) - g_1(y))\| \\
&\quad + (1 - s(\tau_1 - \gamma_1 L_1))\|x - y\|, \\
&\|x - y - (g_1(x) - g_1(y))\|^2 = \|x - y\|^2 - 2\langle x \\
&\quad - y, g_1(x) - g_1(y)\rangle + \|g_1(x) - g_1(y)\|^2 \le \|x \\
&\quad - y\|^2 - 2\delta_1 \|x - y\|^2 + \eta_1^2 \|x - y\|^2 = (1 - 2\delta_1 \\
&\quad + \eta_1^2)\|x - y\|^2 = \theta_1^2 \|x - y\|^2.
\end{aligned} \tag{35}$$

From (35), we have

$$\begin{aligned}
&\|S_1(x) - S_1(y)\| \\
&\quad \le \theta_1 \|x - y\| + (1 - s(\tau_1 - \gamma_1 L_1))\|x - y\| \\
&\quad = [\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))]\|x - y\|.
\end{aligned} \tag{36}$$

Thus S_1 is $[\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))]$-Lipschitz continuous.

Similarly, we can show that S_2 is $[\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))]$-Lipschitz continuous. □

Now we are ready to present our main results. First we establish the convergence analysis of Algorithm 10 for solving SGVI (28).

Theorem 13. *Let C_1 and C_2 be nonempty closed convex subsets of a real Hilbert space H. Let $g_i : H \to H$ be single-valued δ_i-strongly monotone, η_i-Lipschitz continuous, let $F_i : C_i \to H$ be k_i-Lipschitzian and ξ_i-strongly monotone operator with constants $k_i, \xi_i > 0$, and let $V_i : C_i \to H$ be L_i-Lipschitzian mapping with constant $L_i \ge 0$ for $i \in \{1, 2\}$. Suppose that $0 < \mu_i < 2\xi_i/k_i^2$ and $0 \le \gamma_i < \tau_i/L_i$, where $\tau_i = 1 - \sqrt{1 - \mu_i(2\xi_i - \mu_i k_i^2)}$ for $i \in \{1, 2\}$. Let $s, t \in (0, 1]$ and let θ_1 and θ_2 be real constants defined by (34). Let S_1 and S_2 be defined by (29) and (30), respectively. For given initial point $(x_1, y_1) \in C_1 \times C_2$, let $\{(x_n, y_n)\}$ be an iterative sequence defined by parallel Mann iteration process (32), where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume that the following condition is satisfied:*

$$\begin{aligned}
\frac{\theta_1}{s} &< (\tau_1 - \gamma_1 L_1), \\
\frac{\theta_2}{t} &< (\tau_2 - \gamma_2 L_2).
\end{aligned} \tag{37}$$

Then we have the following:

(i) *There exists a unique point $(x^*, y^*) \in C_1 \times C_2$, which solves SGVI (28).*

(ii) *The sequence $\{(x_n, y_n)\}$ generated by parallel Mann iteration process (32) converges strongly to the point (x^*, y^*).*

Proof. (i) It follows from Lemma 8 and (31).

(ii) By (31), (32), and Proposition 12, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|(1 - \alpha_n) x_n + \alpha_n S_2(y_n) - (1 - \alpha_n) x^* - \alpha_n x^*\| \\
&\quad \le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|S_2(y_n) - x^*\| \\
&\quad = (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|S_2(y_n) - S_2(y^*)\| \\
&\quad \le (1 - \alpha_n)\|x_n - x^*\| \\
&\qquad + \alpha_n [\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))]\|y_n - y^*\|.
\end{aligned} \tag{38}$$

Again, by Proposition 12 that $S_1 : C_1 \to C_2$ is $[\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))]$-Lipschitz continuous and using (31) and (32), we have

$$\begin{aligned}
&\|y_{n+1} - y^*\| \\
&\quad \le (1 - \alpha_n)\|y_n - y^*\| \\
&\qquad + \alpha_n [\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))]\|x_n - x^*\|.
\end{aligned} \tag{39}$$

Set

$$\begin{aligned}
\theta = \max\{&\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1)), \theta_2 \\
&+ (1 - t(\tau_2 - \gamma_2 L_2))\}.
\end{aligned} \tag{40}$$

From (38) and (39), we get

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\quad \le (1 - \alpha_n)\|x_n - x^*\| \\
&\qquad + \alpha_n [\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))]\|y_n - y^*\|
\end{aligned}$$

$$+ (1-\alpha_n)\|y_n - y^*\| + \alpha_n[\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))]\|x_n - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - y^*\| + (1-\alpha_n)\|y_n - y^*\| + \alpha_n\theta\|x_n - x^*\| = (1 - \alpha_n(1-\theta))(\|x_n - x^*\| + \|y_n - y^*\|). \quad (41)$$

Now, we define the norm $\|\cdot\|_1$ on $H\times H$ by $\|(x,y)\|_1 = \|x\| + \|y\|$ for all $(x,y) \in H\times H$. Therefore, using (41), we have

$$\begin{aligned}\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_1 &= \|(x_{n+1} - x^*, y_{n+1} - y^*)\|_1 \\ &= \|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\ &\le (1 - \alpha_n(1-\theta))(\|x_n - x^*\| + \|y_n - y^*\|) \\ &= (1 - \alpha_n(1-\theta))\|(x_n, y_n) - (x^*, y^*)\|_1.\end{aligned} \quad (42)$$

Noticing that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\theta \in (0,1)$. Therefore, from Lemma 7, we have $\lim_{n\to\infty}\|(x_n, y_n) - (x^*, y^*)\|_1 = 0$. Thus, we get $\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\|y_n - y^*\| = 0$ and hence $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, respectively. □

Corollary 14. *Let C_1 and C_2 be nonempty closed convex subsets of a real Hilbert space H. Let $F_i : C_i \to H$ be k_i-Lipschitzian and ξ_i-strongly monotone operator with constants $k_i, \xi_i > 0$ and let $V_i : C_i \to H$ be L_i-Lipschitzian mapping with constant $L_i \ge 0$ for $i \in \{1,2\}$. Suppose that $0 < \mu_i < 2\xi_i/k_i^2$ and $0 \le \gamma_i < \tau_i/L_i$, where $\tau_i = 1 - \sqrt{1 - \mu_i(2\xi_i - \mu_i k_i^2)}$ for $i \in \{1,2\}$. Let $s,t \in (0,1]$. Define mappings $S_1 : C_1 \to C_2$ and $S_2 : C_2 \to C_1$ by*

$$\begin{aligned}S_1 &:= P_{C_2}[I - s(\mu_1 F_1 - \gamma_1 V_1)], \\ S_2 &:= P_{C_1}[I - t(\mu_2 F_2 - \gamma_2 V_2)].\end{aligned} \quad (43)$$

For given initial point $(x_1, y_1) \in C_1 \times C_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $C_1 \times C_2$ defined by

$$\begin{aligned}x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S_2(y_n), \\ y_{n+1} &= (1-\alpha_n)y_n + \alpha_n S_1(x_n), \quad \forall n \in \mathbb{N},\end{aligned} \quad (44)$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Assume that condition (37) of Theorem 13 is satisfied. Then the sequence $\{(x_n, y_n)\}$ generated by (44) converges strongly to the unique point (x^*, y^*), which solves system of generalized variational inequalities

$$\begin{aligned}&\langle s(\mu_1 F_1 - \gamma_1 V_1)(x^*) + y^* - x^*, y - y^*\rangle \ge 0, \quad \forall y \in C_2, \\ &\langle t(\mu_2 F_2 - \gamma_2 V_2)(y^*) + x^* - y^*, x - x^*\rangle \ge 0, \quad \forall x \in C_1.\end{aligned} \quad (45)$$

Proof. The proof follows from Theorem 13 by taking $g_1 = g_2 = I$. □

Now we study the convergence analysis of Algorithm 11, that is, the parallel S-iteration process defined by (33) for solving SGVI (28).

Theorem 15. *Let C_1 and C_2 be nonempty closed convex subsets of a real Hilbert space H. Let $g_i : H \to H$ be single-valued δ_i-strongly monotone, η_i-Lipschitz continuous, let $F_i : C_i \to H$ be k_i-Lipschitzian and ξ_i-strongly monotone operator with constants $k_i, \xi_i > 0$, and let $V_i : C_i \to H$ be L_i-Lipschitzian mapping with constant $L_i \ge 0$ for $i \in \{1,2\}$. Suppose that $0 < \mu_i < 2\xi_i/k_i^2$ and $0 \le \gamma_i < \tau_i/L_i$, where $\tau_i = 1 - \sqrt{1 - \mu_i(2\xi_i - \mu_i k_i^2)}$ for $i \in \{1,2\}$. Let $s,t \in (0,1]$ and let θ_1 and θ_2 be real constants defined by (34). Let S_1 and S_2 be defined by (29) and (30), respectively. For given initial point $(x_1, y_1) \in C_1 \times C_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $C_1 \times C_2$ defined by parallel S-iteration process (33), where $\{\alpha_n\}$ is a sequence in $(0,1)$. Assume that condition (37) of Theorem 13 is satisfied. Then we have the following:*

(i) *There exists a unique point $(x^*, y^*) \in C_1 \times C_2$, which solves SGVI (28).*

(ii) *The sequence $\{(x_n, y_n)\}$ generated by parallel S-iteration process (33) converges strongly to the point (x^*, y^*).*

Proof. (i) It follows from Lemma 8 and (31).

(ii) From (31), (33), and Proposition 12, we have

$$\|y_{n+1} - y^*\| = \|S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] - S_1(x^*)\| \le (\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))) \cdot \|(1-\alpha_n)x_n + \alpha_n S_2(y_n) - x^*\| \le (\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1)))\{(1-\alpha_n)\|x_n - x^*\| + \alpha_n\|S_2(y_n) - x^*\|\} = (\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))) \cdot \{(1-\alpha_n)\|x_n - x^*\| + \alpha_n\|S_2(y_n) - S_2(y^*)\|\} \le (\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1)))\{(1-\alpha_n)\|x_n - x^*\| + \alpha_n(\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2)))\|y_n - y^*\|\}. \quad (46)$$

Similarly

$$\|x_{n+1} - x^*\| \le (\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))) \cdot \{(1-\alpha_n)\|y_n - y^*\| + \alpha_n(\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1)))\|x_n - x^*\|\}. \quad (47)$$

Set

$$\theta = \max\{\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1)), \theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))\}. \quad (48)$$

From (46) and (47), we get

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \le (\theta_2 \\
&\quad + (1 - t(\tau_2 - \gamma_2 L_2))) \{(1 - \alpha_n) \|y_n - y^*\| \\
&\quad + \alpha_n (\theta_1 + (1 - s(\tau_1 - \gamma_1 L_1))) \|x_n - x^*\|\} + (\theta_1 \\
&\quad + (1 - s(\tau_1 - \gamma_1 L_1))) \{(1 - \alpha_n) \|x_n - x^*\| \\
&\quad + \alpha_n (\theta_2 + (1 - t(\tau_2 - \gamma_2 L_2))) \|y_n - y^*\|\} \\
&\quad \le \theta \{(1 - \alpha_n) \|y_n - y^*\| + \alpha_n \theta \|x_n - x^*\|\} \\
&\quad + \theta \{(1 - \alpha_n) \|x_n - x^*\| + \alpha_n \theta \|y_n - y^*\|\} = \theta (1 \\
&\quad - \alpha_n (1 - \theta)) \|x_n - x^*\| + \theta (1 - \alpha_n (1 - \theta)) \|y_n \\
&\quad - y^*\| = \theta (1 - \alpha_n (1 - \theta)) [\|x_n - x^*\| + \|y_n - y^*\|] .
\end{aligned} \tag{49}$$

Now, we define the norm $\|\cdot\|_1$ on $H \times H$ by $\|(x, y)\|_1 = \|x\| + \|y\|$ for all $(x, y) \in H \times H$. Therefore, using (49), we have

$$\begin{aligned}
&\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_1 \\
&\quad = \|(x_{n+1} - x^*, y_{n+1} - y^*)\|_1 \\
&\quad = \|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\quad \le \theta (1 - \alpha_n (1 - \theta)) [\|x_n - x^*\| + \|y_n - y^*\|] \\
&\quad = \theta (1 - \alpha_n (1 - \theta)) \|(x_n, y_n) - (x^*, y^*)\|_1 .
\end{aligned} \tag{50}$$

Since $\theta(1 - \alpha_n(1 - \theta)) \le \theta < 1$, we obtain that $\lim_{n\to\infty} \|(x_n, y_n) - (x^*, y^*)\|_1 = 0$. Thus, we get $\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0$ and hence $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, respectively. □

Corollary 16. *Let C_1 and C_2 be nonempty closed convex subsets of a real Hilbert space H. Let $F_i : C_i \to H$ be k_i-Lipschitzian and ξ_i-strongly monotone operator with constants $k_i, \xi_i > 0$ and let $V_i : C_i \to H$ be L_i-Lipschitzian mapping with constant $L_i \ge 0$ for $i \in \{1, 2\}$. Suppose that $0 < \mu_i < 2\xi_i/k_i^2$ and $0 \le \gamma_i < \tau_i/L_i$, where $\tau_i = 1 - \sqrt{1 - \mu_i(2\xi_i - \mu_i k_i^2)}$ for $i \in \{1, 2\}$. Let $s, t \in (0, 1]$ and let S_1 and S_2 be defined by (43). For given initial point $(x_1, y_1) \in C_1 \times C_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $C_1 \times C_2$ defined by*

$$\begin{aligned}
x_{n+1} &= S_2 [(1 - \alpha_n) y_n + \alpha_n S_1 (x_n)] , \\
y_{n+1} &= S_1 [(1 - \alpha_n) x_n + \alpha_n S_2 (y_n)] , \quad \forall n \in \mathbb{N},
\end{aligned} \tag{51}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$. Assume that condition (37) of Theorem 13 is satisfied. Then the sequence $\{(x_n, y_n)\}$ generated by (51) converges strongly to the unique point (x^*, y^*), which solves system of generalized variational inequalities

$$\begin{aligned}
&\langle s (\mu_1 F_1 - \gamma_1 V_1) (x^*) + y^* - x^*, y - y^* \rangle \ge 0, \\
&\qquad\qquad \forall y \in C_2, \\
&\langle t (\mu_2 F_2 - \gamma_2 V_2) (y^*) + x^* - y^*, x - x^* \rangle \ge 0, \\
&\qquad\qquad \forall x \in C_1.
\end{aligned} \tag{52}$$

Proof. The proof follows from Theorem 15 by taking $g_1 = g_2 = I$. □

4. Numerical Example

In this section, we discuss an example which leads to Theorems 13 and 15. The graphs are also presented for showing how the sequences $\{x_n\}$ and $\{y_n\}$ generated by both the algorithms, Algorithms 10 and 11, converge to the solutions of SGVI (28).

Example 17. Let $H = \mathbb{R}$, $C_1 = (-\infty, 0]$, and $C_2 = [0, \infty)$. Let g_1 and g_2 be two mappings from H onto itself defined by $g_1(x) = (2x - 3)/3$ for all $x \in H$ and $g_2(x) = (5x - 10)/6$ for all $x \in H$, respectively. Let $F_1 : C_1 \to H$ and $F_2 : C_2 \to H$ be two mappings defined by $F_1(x) = 2x - 3$ for all $x \in C_1$ and $F_2(x) = 3x - 2$ for all $x \in C_2$, respectively. Let $V_1 : C_1 \to H$ and $V_2 : C_2 \to H$ be two mappings defined by $V_1(x) = 1 - 4x$ for all $x \in C_1$ and $V_2(x) = 6 - 6x$ for all $x \in C_2$, respectively. Then g_i is δ_i-strongly monotone and η_i-Lipschitzian mapping for $i \in \{1, 2\}$. We have $\delta_1 = 2/3 = \eta_1$ and $\delta_2 = 5/6 = \eta_2$. Also F_i is ξ_i-strongly monotone and k_i-Lipschitzian mapping for $i \in \{1, 2\}$. We have $\xi_1 = 2 = k_1$ and $\xi_2 = 3 = k_2$. Moreover V_i is L_i-Lipschitzian mapping for $i \in \{1, 2\}$. We have $L_1 = 4$ and $L_2 = 6$. We take $\mu_1 = 1/2$, $\mu_2 = 1/3$, $\tau_1 = 1 = \tau_2$ and $\gamma_1 = 1/12$, $\gamma_2 = 5/72$. Define $\{\alpha_n\}$ in $[0, 1]$ by $\alpha_n = n/(n + 1)$, $T_1 = \mu_1 F_1 - \gamma_1 V_1$, and $T_2 = \mu_2 F_2 - \gamma_2 V_2$. Then $T_1(x) = (16x - 19)/12$ and $T_2(x) = (51x - 39)/36$.

Therefore S_1 and S_2 can be expressed as

$$\begin{aligned}
S_1 &:= P_{C_2} [g_1 - sT_1] , \\
S_2 &:= P_{C_1} [g_2 - tT_2] ,
\end{aligned} \tag{53}$$

where $s > 0$ and $t > 0$ are constants.

Let $s = 1$ and $t = 1$. Then,

$$\begin{aligned}
S_1 (x) &= P_{C_2} \left[\frac{(-8x + 7)}{12} \right] = \frac{(7 - 8x)}{12} \quad \forall x \in C_1, \\
S_2 (x) &= P_{C_1} \left[\frac{(-7x - 7)}{12} \right] = -\frac{(7x + 7)}{12} \quad \forall x \in C_2.
\end{aligned} \tag{54}$$

It can be easily seen that $S_1 : C_1 \to C_2$ is $(2/3)$-Lipschitzian and $S_2 : C_2 \to C_1$ is $(7/12)$-Lipschitzian. Also

$$\begin{aligned}
\frac{\theta_1}{s} - (\tau_1 - \gamma_1 L_1) &= \sqrt{1 - 2\delta_1 + \eta_1^2} - (\tau_1 - \gamma_1 L_1) \\
&= -\frac{1}{3} < 0, \\
\frac{\theta_2}{t} - (\tau_2 - \gamma_2 L_2) &= \sqrt{1 - 2\delta_2 + \eta_2^2} - (\tau_2 - \gamma_2 L_2) \\
&= -\frac{5}{12} < 0.
\end{aligned} \tag{55}$$

One can observe that all the conditions of Theorems 13 and 15 are satisfied.

Table 1: Numerical values of x_n and y_n.

n	Parallel Mann iteration process		Parallel S-iteration process	
	x_n	y_n	x_n	y_n
1	−4.000000000000000	4.000000000000000	−4.000000000000000	4.000000000000000
7	−1.847585060258671	1.950379984812855	−1.531137975937710	1.612050811863993
14	−1.529511272915085	1.610309552955496	−1.511402781884560	1.590950939614087
21	−1.512204443955688	1.591807954179574	−1.511363702883595	1.590909162022088
28	−1.511400117495092	1.590948090842850	−1.511363636469492	1.590909091022255
35	−1.511365162111111	1.590910722002483	−1.511363636363799	1.590909090909265
42	−1.511363698691403	1.590909157540259	−1.511363636363637	1.590909090909091
49	−1.511363638868444	1.590909093586843	−1.511363636363637	1.590909090909091
56	−1.511363636463094	1.590909091015415	−1.511363636363636	1.590909090909091
63	−1.511363636367549	1.590909090913274	−1.511363636363637	1.590909090909091
70	−1.511363636363790	1.590909090909254	−1.511363636363637	1.590909090909091
77	−1.511363636363642	1.590909090909097	−1.511363636363637	1.590909090909091
84	−1.511363636363637	1.590909090909091	−1.511363636363636	1.590909090909091
91	−1.511363636363637	1.590909090909091	−1.511363636363637	1.590909090909091
98	−1.511363636363637	1.590909090909091	−1.511363636363637	1.590909090909091

Now we will find the general term of the sequences $\{x_n\}$ and $\{y_n\}$ generated by the iteration process (32). For arbitrary $x_1 \in C_1$ and $y_1 \in C_2$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n) x_n + \alpha_n S_2(y_n) \\ &= \frac{x_n}{n+1} - \frac{n(7y_n+7)}{12(n+1)} \\ &= \frac{1}{12(n+1)}[12x_n - 7ny_n - 7n], \\ y_{n+1} &= (1-\alpha_n) y_n + \alpha_n S_1(x_n) \\ &= \frac{y_n}{n+1} + \frac{n(7-8x_n)}{12(n+1)} \\ &= \frac{1}{12(n+1)}[12y_n - 8nx_n + 7n]. \end{aligned} \tag{56}$$

Hence

$$\begin{aligned} x_{n+1} &= \frac{1}{12(n+1)}[12x_n - 7ny_n - 7n], \\ y_{n+1} &= \frac{1}{12(n+1)}[12y_n - 8nx_n + 7n] \quad \forall n \in \mathbb{N}. \end{aligned} \tag{57}$$

Also, we will find the general term of the sequences $\{x_n\}$ and $\{y_n\}$ generated by the iteration process (33). For arbitrary $x_1 \in C_1$ and $y_1 \in C_2$,

$$\begin{aligned} x_{n+1} &= S_2[(1-\alpha_n) y_n + \alpha_n S_1(x_n)] \\ &= S_2\left[\frac{y_n}{n+1} + \frac{n(7-8x_n)}{12(n+1)}\right] \\ &= \frac{1}{144(n+1)}[56nx_n - 84y_n - 133n - 84], \\ y_{n+1} &= S_1[(1-\alpha_n) x_n + \alpha_n S_2(y_n)] \\ &= S_1\left[\frac{x_n}{n+1} - \frac{n(7y_n+7)}{12(n+1)}\right] \\ &= \frac{1}{144(n+1)}[56ny_n - 96x_n + 140n + 84]. \end{aligned} \tag{58}$$

Hence

$$\begin{aligned} x_{n+1} &= \frac{1}{144(n+1)}[56nx_n - 84y_n - 133n - 84], \\ y_{n+1} &= \frac{1}{144(n+1)}[56ny_n - 96x_n + 140n + 84] \\ & \qquad \forall n \in \mathbb{N}. \end{aligned} \tag{59}$$

It is clear from (57) and (59) that the sequences $\{x_n\}$ and $\{y_n\}$ generated by the proposed iterative algorithms converge to the altering points $x^* \in C_1$ and $y^* \in C_2$ of the mappings S_1 and S_2, where $x^* = -1.511363636363637$ and $y^* = 1.590909090909091$. The numerical values of $\{x_n\}$ and $\{y_n\}$ have been calculated for different starting values of x_1 and y_1 in Tables 1 and 2, respectively, and the convergence of both the sequences is shown in Figures 1 and 2, respectively.

5. Conclusions

In this paper, we have considered a new system of generalized variational inequalities (SGVI) defined on closed convex subsets of a real Hilbert space. It has been shown that the considered SGVI is equivalent to altering points problem of some nonlinear mappings. We have proposed two algorithms, Algorithms 10 and 11, for solving considered SGVI. An example is given in support of our main results. We observed that the sequence generated by Algorithm 11

TABLE 2: Numerical values of x_n and y_n.

n	Parallel Mann iteration process		Parallel S-iteration process	
	x_n	y_n	x_n	y_n
1	−5.000000000000000	10.000000000000000	−5.000000000000000	10.000000000000000
7	−2.316760667744464	2.451261819028022	−1.558731808819676	1.641509397106758
14	−1.554814798532934	1.637363641212413	−1.511457370603911	1.591009289899863
21	−1.513376896656526	1.593061316754131	−1.511363795641447	1.590909261181073
28	−1.511450986841399	1.591002473172358	−1.511363636617098	1.590909091180052
35	−1.511367289637843	1.590912996409872	−1.511363636364025	1.590909090909507
42	−1.511363785601782	1.590909250451687	−1.511363636363637	1.590909090909092
49	−1.511363642361179	1.590909097320727	−1.511363636363637	1.590909090909091
56	−1.511363636601778	1.590909091163675	−1.511363636363636	1.590909090909091
63	−1.511363636373005	1.590909090919107	−1.511363636363637	1.590909090909091
70	−1.511363636364002	1.590909090909482	−1.511363636363637	1.590909090909091
77	−1.511363636363651	1.590909090909106	−1.511363636363637	1.590909090909091
84	−1.511363636363637	1.590909090909092	−1.511363636363636	1.590909090909091
91	−1.511363636363637	1.590909090909091	−1.511363636363637	1.590909090909091
98	−1.511363636363637	1.590909090909091	−1.511363636363637	1.590909090909091

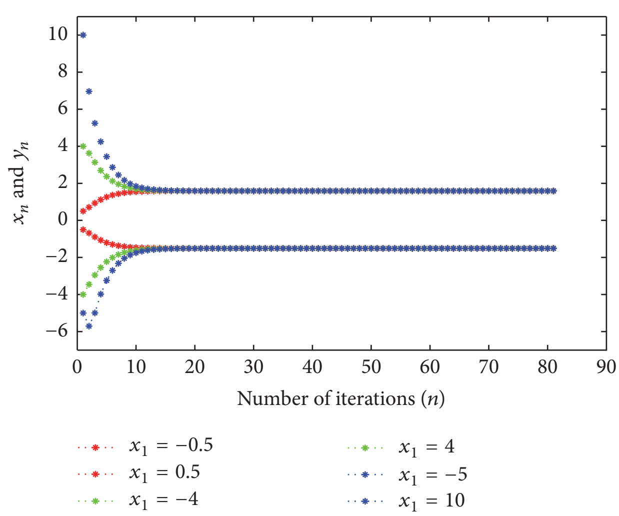

FIGURE 1: Convergence of sequences $\{x_n\}$ and $\{y_n\}$ generated by parallel Mann iteration process (32).

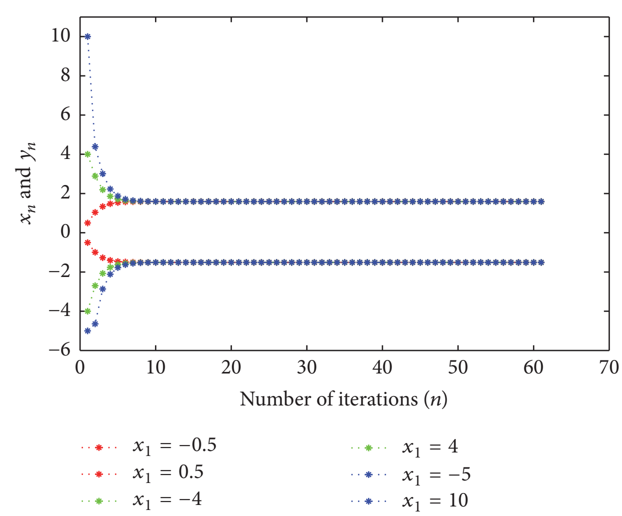

FIGURE 2: Convergence of sequences $\{x_n\}$ and $\{y_n\}$ generated by parallel S-iteration process (33).

converges faster than Algorithm 10 to altering points of some nonlinear mappings S_1 and S_2.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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