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Research Article

Existence of Positive Solutions for Multi-Point Boundary Value Problems on Infinite Intervals in Banach Spaces

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We investigate the positive solutions of a class of second-order nonlinear singular differential equations with multi-point boundary value conditions on an infinite interval in Banach spaces. The tools we used are the cone theory and Mönch fixed point theorem and a monotone iterative technique. An example is also given to demonstrate the applications of our results, which include and extend some existing results.

1. Introduction

Let E be a real Banach space and let P be a cone in E. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. P is said to be regular (fully regular) if $u_1 \le u_2 \le \cdots \le u_n \le y \cdots$, $\sup_n \|u_n\| < \infty$ implies that their exists $x \in E$ such that $\lim_{n\to\infty}\|u_n - x\| = 0$. Let $P_+ = P \setminus \{\theta\}$. In what follows, we always assume that $x_0^* \in P_+$. Let $P_{0\lambda} = \{u \in P : u \ge \lambda x_0^*\}$ $(\lambda > 0)$. Obviously, $P_{0\lambda} \subset P_+$ for any $\lambda > 0$. When $\lambda = 1$, we write $P_0 = P_{01}$, that is, $P_0 = \{u \in P : u \ge x_0^*\}$.

In this paper, we will consider the following boundary value problems (BVPs) for multipoint singular differential equations of mixed type on an unbounded domain in a real Banach space $(E, \|\cdot\|)$

$$\begin{gathered} u''(t) + f\left(t, u(t), u'(t), (Tu(t)), (Su(t))\right) = \theta, \quad t \in J_+, \\ au(0) - bu'(0) - \sum_{i=1}^{n} k_i u(\xi_i) = \theta, \qquad \lim_{t\to+\infty} u'(t) = y_\infty, \end{gathered} \tag{1.1}$$

where $J = [0,+\infty)$, $J_+ = (0,+\infty)$, $a > 0$, $b \geq 0$, $k_i \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$, $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P \times P, P]$, $\lambda > 0$, $y_\infty \geq x_0^*$,

$$(Tu)(t) = \int_0^t K(t,s)u(s)ds, \qquad (Su)(t) = \int_0^\infty H(t,s)u(s)ds, \tag{1.2}$$

$K \in C[D,J]$, $D = \{(t,s) \in J \times J : t \geq s\}$, $H \in C[J \times J, J]$. Here the nonlinear term f may be singular at $t = 0$ and $x_0, x_1, x_2, x_3 = \theta$. By singularity, we mean that $\|f(t,x_0,x_1,x_2,x_3)\| \to \infty$ as $t \to 0^+$ or $x_i \to \theta^+$ $(i = 0,1,2,3)$.

Second-order boundary value problems (BVPs) on infinite intervals, arising from the study of radially symmetric solutions of nonlinear elliptic equation and models of gas pressure in a semi-infinite porous medium, have received much attention. We can see papers [1, 2] and the references therein. In a recent paper, Liu [3] investigated the existence of solutions of the following second-order two-point boundary value problems on the half-line:

$$\begin{gathered} x''(t) + f(t,x(t)) = 0, \quad t \in (0,+\infty), \\ x(0) = 0, \qquad x'(\infty) = y_\infty > 0. \end{gathered} \tag{1.3}$$

Lian and Ge [4] studied the solvability of the three-point BVP

$$\begin{gathered} u''(t) + f\big(t,u(t),u'(t)\big) = 0, \quad t \in (0,+\infty), \\ u(0) = \alpha u(\eta), \qquad \lim_{t \to +\infty} u'(t) = 0, \end{gathered} \tag{1.4}$$

where $\alpha \neq 1$, $\eta \in (0,+\infty)$. With the help of the established Green function and the Leray-Schauder continuation theorem, suitable conditions imposed on f are presented for the existence of solutions. Yan et al. [5] established the results of existence and multiplicity of positive solutions to the BVP on the half-line

$$\begin{gathered} u''(t) + \Phi(t)f\big(t,u,u'\big) = 0, \quad t \in (0,+\infty), \\ au(0) - bu'(0) = u_0 \geq 0, \qquad \lim_{t \to +\infty} u'(t) = k > 0, \end{gathered} \tag{1.5}$$

by using the lower and upper solutions technique. Zhang [6] researched the problem

$$\begin{gathered} x''(t) + q(t)f(t,x(t)), \quad t \in J_+, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(\infty) = y_\infty, \end{gathered} \tag{1.6}$$

by using the fixed point theorem and the monotone iterative technique.

We note that these works [3–6] are all in real space. To the best of our knowledge, very few literatures are available for the computation of positive solutions for mutlipoint BVP on the half-line in Banach space. There are two papers we should present here. Liu [7] discussed the existence of solutions of the following second-order two-point BVP on infinite intervals in a Banach space E:

$$\begin{gathered} x''(t) = f\left(t, x(t), x'(t)\right), \quad t \in J, \\ x(0) = x_0, \qquad x'(\infty) = y_\infty, \end{gathered} \tag{1.7}$$

where $f \in C[J \times E \times E]$, $J = [0, +\infty)$, $x'(\infty) = \lim_{t\to\infty} x'(t)$. The main tool is the Sadovskii's fixed point theorem. Zhang [8] concerned the existence of solutions of the following singular problems

$$\begin{gathered} x''(t) + f\left(t, x(t), x'(t)\right) = \theta, \quad t \in J_+, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(\infty) = y_\infty, \end{gathered} \tag{1.8}$$

where $J_+ = (0, +\infty)$. The nonlinear term f may be singular at $t = 0$, $x = \theta$ and/or $x' = \theta$. They used the Mönch fixed point theorem.

Motivated by the above papers, we also use the Mönch fixed point theorem to give the existence of a positive solutions of the more general BVP (1.1) for integrodifferential equations on infinite intervals in a Banach spaces. The main features of the present paper are as follows. Firstly, comparing with [3–6], the space in this paper is Banach space. The equation we discussed here is more general than those of [3–8] because the function f of (1.1) has new terms Tu, Su and the boundary value conditions are more complicated. Moreover, the singularity of nonlinear term f in this paper is more complex than [2, 7, 9–11]. Furthermore, an iterative sequence for the solution under some normal type conditions is established which makes it very important and convenient in applications. [3–7, 9–11] did not obtain this kind of result.

The rest of this paper is organized as follows. In Section 2, we give several important Lemmas. The main theorems are formulated and proved in Section 3, followed by an example in Section 4 to demonstrate the application of our results.

2. The Preliminary and Several Lemmas

Let

$$\begin{gathered} FC[J, E] = \left\{ u \in C[J, E] : \sup_{t\in J} \frac{\|u(t)\|}{e^t} < \infty \right\}, \\ DC^1[J, E] = \left\{ u \in C^1[J, E] : \sup_{t\in J} \frac{\|u(t)\|}{e^t} < \infty,\ \sup_{t\in J} \frac{\|u'(t)\|}{e^t} < \infty \right\}. \end{gathered} \tag{2.1}$$

Evidently, $C^1[J, E] \subset C[J, E]$, $DC^1[J, E] \subset FC[J, E]$. It is easy to see that $FC[J, E]$ is a Banach space with norm $\|u\|_F = \sup_{t\in J} \|u(t)\|/e^t$ and $DC^1[J, E]$ is also a Banach space with norm $\|u\|_D = \max\{\|u\|_F, \|u'\|_C\}$, where $\|u'\|_C = \sup_{t\in J} \|u'(t)\|/e^t$.

Let

$$P(F) = \{u \in FC[J,E] : u(t) \geq \theta,\ \forall t \in J\},$$
$$P(D) = \left\{u \in DC^1[J,E] : u(t) \geq \theta,\ u'(t) \geq \theta,\ \forall t \in J\right\}. \tag{2.2}$$

It is clear that $P(F)$, $P(D)$ are cones in $FC[J,E]$ and $DC^1[J,E]$, respectively. A map $u \in DC^1[J,E] \bigcap C^2[J_+,E]$ is called a positive solutions of the BVP (1.1) if $u \in P(D)$ and satisfies (1.1).

Let α, α_D denote the Kuratowski measure of noncompactness in E and $DC^1[J,E]$, respectively. For details on the definition and properties of the measure of noncompactness, the reader is referred to [12, 13].

Some conditions to be used throughout the rest of the paper are listed below.

(H_1)

$$\lim_{t \to \infty}\left(e^{-t}\int_0^t K(t,s)e^s ds\right) = 0, \quad \int_0^\infty H(t,s)e^s ds < \infty, \quad \forall t \in J,$$
$$\lim_{t \to \infty}\left(e^{-t}\int_0^\infty H(t,s)e^s ds\right) = 0, \quad \lim_{t' \to t}\int_0^\infty \left|H(t',s) - H(t,s)\right| e^s ds = 0, \quad \forall t, t' \in J. \tag{2.3}$$

In this case, let

$$k^* = \sup_{t \in J}\left(e^{-t}\int_0^t K(t,s)e^s ds\right), \qquad h^* = \sup_{t \in J}\left(e^{-t}\int_0^\infty H(t,s)e^s ds\right). \tag{2.4}$$

(H_2) $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P \times P, P]$, for any $\lambda > 0$ and there exist $m, p, q \in L[J_+, J]$ and $g \in C[J_+ \times J_+ \times J \times J, J]$ such that

$$\|f(t,u_0,u_1,u_2,u_3)\| \leq m(t) + p(t)g(\|u_0\|, \|u_1\|, \|u_2\|, \|u_3\|), \quad \forall t \in J_+,\ u_0, u_1 \in P_0,\ u_2, u_3 \in P,$$
$$\frac{\|f(t,u_0,u_1,u_2,u_3)\|}{q(t)(\|u_0\| + \|u_1\| + \|u_2\| + \|u_3\|)} \longrightarrow 0$$
$$(u_0, u_1 \in P_0,\ u_2, u_3 \in P,\ \|u_0\| + \|u_1\| + \|u_2\| + \|u_3\| \longrightarrow \infty) \tag{2.5}$$

uniformly for $t \in J_+$, and

$$m^* = \int_0^\infty m(t)dt < \infty, \qquad p^* = \int_0^\infty p(t)dt < \infty, \qquad q^* = \int_0^\infty q(t)e^t dt < \infty. \tag{2.6}$$

(H_3) For any $t \in J_+$, $R > 0$ and countable sets $V_i \subset DC^1[J, P^*_{0R}]$ $(i = 0, 1), V_i \subset DC^1[J, P^*_R]$ $(i = 2, 3)$, there exist $L_i \in L[J, J]$ $(i = 0, 1, 2, 3)$ such that

$$\alpha\big(f(t, V_0(t), V_1(t), V_2(t), V_3(t))\big) \leq \sum_{i=0}^{3} L_i(t)\alpha(V_i(t)),$$
$$L^* = \int_0^\infty [L_0(s) + L_1(s) + L_2(s)k^* + L_3(s)h^*]e^s ds < \frac{1}{4}, \tag{2.7}$$

where $P^*_{0R} = \{u \in P : u \geq x_0^*,\ \|u\| \leq R\}$ $(i = 0, 1)$ and $P^*_R = \{u \in P : \|u\| \leq R\}$.

(H_4) $t \in J_+$, $x_0^* \leq u_i \leq \overline{u}_i$ $(i = 0, 1)$, $\theta \leq u_2 \leq \overline{u}_2$, $\theta \leq u_3 \leq \overline{u}_3$ imply

$$f(t, u_0, u_1, u_2, u_3) \leq f(t, \overline{u}_0, \overline{u}_1, \overline{u}_2, \overline{u}_3). \tag{2.8}$$

Lemma 2.1 (Lemma 1 see, [9]). *If condition* (H_1) *is satisfied, then the operators T and S defined by* (1.2) *are bounded linear operators from* $FC[J, E]$ *into* $FC[J, E]$ *and*

$$\|T\| \leq k^*, \quad \|S\| \leq h^*, \quad T(P(F)) \subset P(F), \quad S(P(F)) \subset P(F). \tag{2.9}$$

In what follows, we write

$$Q = \left\{u \in DC^1[J, P] : u^{(i)}(t) \geq x_0^*,\ \forall t \in J,\ i = 0, 1\right\}. \tag{2.10}$$

Evidently, Q is a closed convex set in $DC^1[J, E]$. We will reduce the BVP (1.1) to an integral equation in E. To this end, we first consider the operator A defined by

$$\begin{aligned}(Au)(t) = \frac{1}{\Delta}\Bigg[&\delta y_\infty + b\int_0^\infty f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)ds \\ &+ \sum_{i=1}^{n} k_i \int_0^{\xi_i}\int_s^\infty f\big(\tau, u(\tau), u'(\tau), (Tu)(\tau), (Su)(\tau)\big)d\tau ds\Bigg] \\ &+ ty_\infty + \int_0^t \int_s^\infty f\big(\tau, u(\tau), u'(\tau), (Tu)(\tau), (Su)(\tau)\big)d\tau ds, \quad u \in Q,\end{aligned} \tag{2.11}$$

where

$$\Delta = a - \sum_{i=1}^{n} k_i > 0, \quad \delta = b + \sum_{i=1}^{n} k_i \xi_i, \quad \Delta \leq \delta. \tag{2.12}$$

Lemma 2.2. *If conditions* (H_1) *and* (H_2) *hold, then the operator A defined by* (2.11) *is a continuous operator from Q into Q.*

Proof. Firstly, we will show $A(Q) \subset Q$. Let

$$\varepsilon_0 = \frac{1}{2(1+\delta/\Delta)q^*(2+k^*+h^*)}, \qquad r = \frac{\|x_0^*\|}{N} > 0. \tag{2.13}$$

By virtue of condition (H_2), there exists a $R > r$ such that

$$\begin{gathered} \|f(t,x,y,z,w)\| \le \varepsilon_0 q(t)(\|x\| + \|y\| + \|z\| + \|w\|), \\ \forall t \in J_+, \quad x,y \in P_0, \quad z,w \in P, \ \|x\| + \|y\| + \|z\| + \|w\| > R, \\ \|f(t,x,y,z,w)\| \le m(t) + Mp(t), \\ \forall t \in J_+, \quad x,y \in P_0, \quad z,w \in P, \quad \|x\| + \|y\| + \|z\| + \|w\| \le R, \end{gathered} \tag{2.14}$$

where

$$M = \max\{g(u_0,u_1,u_2,u_3) : r \le u_i \le R \ \ (i=0,1), \ 0 \le u_i \le R \ \ (i=2,3)\}. \tag{2.15}$$

Hence,

$$\begin{aligned} \|f(t,x,y,z,w)\| \le \varepsilon_0 q(t)(\|x\| + \|y\| + \|z\| + \|w\|) + m(t) + Mp(t), \\ \forall t \in J_+, \quad x,y \in P_0, \quad z,w \in P. \end{aligned} \tag{2.16}$$

Let $u \in Q$. We have by (2.16) and Lemma 2.1

$$\begin{aligned} &\|f(t,u(t),u'(t),(Tu)(t),(Su)(t))\| \\ &\quad \le \varepsilon_0 q(t)(\|u(t)\| + \|u'(t)\| + \|(Tu)(t)\| + \|(Su)(t)\|) + m(t) + Mp(t) \\ &\quad \le \varepsilon_0 q(t)e^t(\|u\|_F + \|u'\|_C + k^*\|u\|_F + h^*\|u\|_F) + m(t) + Mp(t) \\ &\quad \le \varepsilon_0 q(t)e^t(2+k^*+h^*)\|u\|_D + m(t) + Mp(t), \quad \forall t \in J_+, \end{aligned} \tag{2.17}$$

which together with condition (H_2) implies the convergence of the infinite integral

$$\int_0^\infty \|f(t,u(t),u'(t),(Tu)(t),(Su)(t))\| dt. \tag{2.18}$$

Thus, we have

$$\begin{aligned}&\left\|\int_0^t\int_s^\infty f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)d\tau ds\right\|\\
&\quad\leq\int_0^t\int_s^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau ds\\
&\quad\leq\int_0^\infty\int_0^t\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|ds d\tau\\
&\quad\leq t\int_0^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau,\quad\forall t\in J_+,\end{aligned}\tag{2.19}$$

(2.11), (2.19), and (H_2) tell us that

$$\begin{aligned}\|(Au)(t)\|&\leq\frac{1}{\Delta}\Bigg[\delta\|y_\infty\|+b\int_0^\infty\big\|f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)\big\|ds\\
&\qquad+\sum_{i=1}^n k_i\int_0^{\xi_i}\int_s^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau\, ds\Bigg]\\
&\quad+t\|y_\infty\|+\int_0^t\int_s^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau\, ds\\
&\leq\frac{\delta}{\Delta}\|y_\infty\|+t\|y_\infty\|+\frac{b}{\Delta}\int_0^\infty\big\|f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)\big\|ds\\
&\quad+\frac{\sum_{i=1}^n k_i\xi_i}{\Delta}\int_0^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau\\
&\quad+t\int_0^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau.\end{aligned}\tag{2.20}$$

Therefore,

$$\begin{aligned}\frac{\|(Au)(t)\|}{e^t}&\leq\frac{\delta}{\Delta}\|y_\infty\|+\|y_\infty\|+\frac{b}{\Delta}\int_0^\infty\big\|f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)\big\|ds\\
&\quad+\frac{\sum_{i=1}^n k_i\xi_i}{\Delta}\int_0^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau\\
&\quad+\int_0^\infty\big\|f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)\big\|d\tau\end{aligned}$$

$$\begin{aligned}
&\leq \left(1+\frac{\delta}{\Delta}\right)\int_0^{\infty}\|f(s,u(s),u'(s),(Tu)(s),(Su)(s))\|ds+\left(\frac{\delta}{\Delta}+1\right)\|y_{\infty}\| \\
&\leq \left(1+\frac{\delta}{\Delta}\right)\left[\varepsilon_0 q^*(2+k^*+h^*)\|u\|_D+m^*+Mp^*\right]+\left(\frac{\delta}{\Delta}+1\right)\|y_{\infty}\| \\
&\leq \frac{1}{2}\|u\|_D+\left(1+\frac{\delta}{\Delta}\right)(m^*+Mp^*)+\left(\frac{\delta}{\Delta}+1\right)\|y_{\infty}\|.
\end{aligned} \tag{2.21}$$

Differentiating (2.11), we get

$$(Au)'(t)=\int_t^{\infty} f(s,u(s),u'(s),(Tu)(s),(Su)(s))ds+y_{\infty}. \tag{2.22}$$

Hence,

$$\begin{aligned}
\frac{\|(Au)'(t)\|}{e^t} &\leq \int_0^{\infty}\|f(s,u(s),u'(s),(Tu)(s),(Su)(s))ds\|+\|y_{\infty}\| \\
&\leq \varepsilon_0 q^*(2+k^*+h^*)\|u\|_D+m^*+Mp^*+\|y_{\infty}\| \\
&\leq \frac{1}{2}\|u\|_D+m^*+Mp^*+\|y_{\infty}\|, \quad \forall t\in J.
\end{aligned} \tag{2.23}$$

It follows from (2.21) and (2.23) that

$$\|(Au)(t)\|_D \leq \frac{1}{2}\|u\|_D+\left(1+\frac{\delta}{\Delta}\right)(m^*+Mp^*)+\left(\frac{\delta}{\Delta}+1\right)\|y_{\infty}\|. \tag{2.24}$$

So, $Au\in DC^1[J,E]$. On the other hand, it can be easily seen that

$$(Au)(t)\geq \frac{\delta}{\Delta}y_{\infty}\geq y_{\infty}\geq x_0^*, \quad (Au)'(t)\geq y_{\infty}\geq x_0^*, \qquad \forall t\in J. \tag{2.25}$$

So, $Au\in Q$. Thus, A maps Q into Q and (2.24) holds.

Secondly, we will show that A is continuous. Let $u_m,\overline{u}\in Q$, $\|u_m-\overline{u}\|_D\to 0$ $(m\to\infty)$. Then $r=\sup_m\|u_m\|_D<\infty$ and $\|\overline{u}\|_D\leq r$. Similar to (2.21) and (2.23), it is easy to say

$$\begin{aligned}
\|Au_m-A\overline{u}\|_D \leq \left(1+\frac{\delta}{\Delta}\right)\int_0^{\infty}&\|f(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)) \\
&-f(s,\overline{u}(s),\overline{u}'(s),(T\overline{u})(s),(S\overline{u})(s))\|ds.
\end{aligned} \tag{2.26}$$

So we get

$$\begin{aligned}
f(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)) \longrightarrow f(s,\overline{u}(s),\overline{u}'(s),(T\overline{u})(s),(S\overline{u})(s)) \\
\text{as } m\longrightarrow\infty, \quad \forall t\in J_+.
\end{aligned} \tag{2.27}$$

Then we know from (2.17) that

$$\begin{aligned}
&\left\| f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right) - f\left(s,\overline{u}(s),\overline{u}'(s),(T\overline{u})(s),(S\overline{u})(s)\right)\right\| \\
&\quad\le 2\left[\varepsilon_0 q(t)e^t(2+k^*+h^*)r + m(t) + Mp(t)\right] \\
&\quad:= \sigma(t) \in L[J_+,J], \quad m = 1,2,3,\ldots, \ \forall t \in J_+.
\end{aligned} \tag{2.28}$$

It follows from (2.27) and (2.28) and the dominated convergence theorem that

$$\begin{aligned}
\lim_{m\to\infty}\int_0^\infty &\left\| f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)\right. \\
&\left. - f\left(s,\overline{u}(s),\overline{u}'(s),(T\overline{u})(s),(S\overline{u})(s)\right)\right\| = 0.
\end{aligned} \tag{2.29}$$

It follows from (2.26) and (2.29) that $\|Au_m - A\overline{u}\|_D \to 0$ as $m \to \infty$, and the continuity of A is proved. □

Lemma 2.3. *Let conditions* (H_1) *and* (H_2) *be satisfied. Then* $u \in Q \cap C^2[J_+,E]$ *is a solution of the BVP* (1.1) *if and only if* $u \in Q$ *is a solution of the following integral equation:*

$$\begin{aligned}
u(t) = \frac{1}{\Delta}\Bigg[&\delta y_\infty + b\int_0^\infty f\left(s,u(s),u'(s),(Tu)(s),(Su)(s)\right)ds \\
&+ \sum_{i=1}^n k_i \int_0^{\xi_i}\int_s^\infty f\left(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\right)d\tau\, ds\Bigg] \\
&+ ty_\infty + \int_0^t\int_s^\infty f\left(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\right)d\tau\, ds.
\end{aligned} \tag{2.30}$$

Proof. Integrating the differential equation in (1.1) from t to ∞, one has

$$u'(t) = y_\infty + \int_t^\infty f\left(s,u(s),u'(s),(Tu)(s),(Su)(s)\right)ds. \tag{2.31}$$

Then, integrating (2.31) from 0 to t, we have

$$u(t) = u(0) + ty_\infty + \int_0^t\int_s^\infty f\left(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\right)d\tau\, ds. \tag{2.32}$$

By Lemma 2.2, we know that $\int_0^t\int_s^\infty f(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau))d\tau\, ds$ is convergent. Since $au(0) - bu'(0) - \sum_{i=1}^n k_i u(\xi_i) = \theta$, we can compute those coefficients $u(0)$, $u'(0)$, $u(\xi_i)$ and then obtain (2.30).

Conversely, if u is a solution of integral equation (2.30), then direct differentiation gives the proof. □

Lemma 2.4 (Mönch Fixed Point Theorem [10]). *Let Q be a closed convex set of E and $u \in Q$. Assume that the continuous operator $A : Q \to Q$ has the following property: $V \subset \overline{co}(\{u\} \bigcup F(V)) \Rightarrow V$ is relatively compact. Then F has a fixed point in Q.*

Lemma 2.5. *Let H be a bounded set in $DC^1[J,E]$. Suppose that $H'(t)$ is equicontinuous on any finite subinterval $[0,c]$ $(c>0)$ of J and $e^{-t}\|u^{(i)}\| \to 0$ as $t \to \infty$ uniformly for $u \in H$ $(i = 0,1)$. Then*

$$\alpha_D(H) = \max\left\{\sup_{t\in J}\left[e^{-t}\alpha(H(t))\right], \sup_{t\in J}\left[e^{-t}\alpha\left(H^{(1)}(t)\right)\right]\right\}, \tag{2.33}$$

where $H^{(i)}(t) = \{(x_m)^{(i)}(t) : x_m \in H,\ m = 1,2,3,\ldots\}$ $(i = 0,1)$, α and α_D denote the Kuratowski measure of noncompactness of bounded sets in E and $DC^1[J,E]$, respectively.

Proof. The proof is similar to [11, Lemma 7], we omit it. □

Lemma 2.6. *If condition (H_4) is satisfied, then $x, y \in Q$, $x^{(i)} \le y^{(i)}$, $t \in J$ $(i = 0,1)$ imply that $(Ax)^{(i)} \le (Ay)^{(i)}$, $t \in J$ $(i = 0,1)$.*

Proof. It is easy to see that this lemma follows from (2.11) and (2.22) and condition (H_4). □

3. Main Results

In the following, we will give the main results of this paper.

Theorem 3.1. *Assume that (H_1)–(H_3) hold, then the BVP (1.1) has a positive solution $\overline{u} \in DC^1[J,E] \bigcap C^2[J_+,E]$ satisfying $(\overline{u})^{(i)}(t) \ge x_0^*$ for $t \in J$ $(i = 0,1)$.*

Proof. By Lemma 2.2, the operator A defined by (2.11) is a continuous operator from Q into Q. And, by Lemma 2.3, we only need to show that A has a fixed point $\overline{u}$ in Q.

Choose

$$R > 2\left[\left(1+\frac{\delta}{\Delta}\right)(m^* + Mp^*) + \left(\frac{\delta}{\Delta}+1\right)\|y_\infty\|\right], \tag{3.1}$$

and let $Q_1 = \{u \in Q : \|u\|_D \le R\}$. Obviously, Q_1 is a bounded closed convex set in the space $DC^1[J,E]$. It is easy to see that Q_1 is not empty since $(\delta/\Delta + 1)y_\infty \in Q_1$. It follows from (2.24) and (3.1) that $u \in Q_1$ implies $Au \in Q_1$, that is, A maps Q_1 into Q_1.

Now, we are in position to show that $A(Q_1)$ is relatively compact. Let $V = \{u_m : m = 1,2,\ldots\} \subset Q_1$ satisfying $V \subset \overline{co}(\{u\} \bigcup \{(AV)\})$ for some $u \in Q_1$. Then $\|u_m\|_D \le R$. So we have by (2.11) and (2.22) that

$$\begin{aligned}(Au_m)(t) &= \frac{1}{\Delta}\left[\delta y_\infty + b\int_0^\infty f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)ds\right.\\ &\qquad\left. + \sum_{i=1}^{n} k_i\int_0^{\xi_i}\int_s^\infty f\left(\tau,u_m(\tau),u_m'(\tau),(Tu_m)(\tau),(Su_m)(\tau)\right)d\tau\, ds\right]\\ &\quad + ty_\infty + \int_0^t\int_s^\infty f\left(\tau,u_m(\tau),u_m'(\tau),(Tu_m)(\tau),(Su_m)(\tau)\right)d\tau\, ds\\ &= \frac{1}{\Delta}\left[\delta y_\infty + b\int_0^\infty f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)ds\right.\\ &\qquad\left. + \sum_{i=1}^{n} k_i\int_0^{\xi_i}\int_s^\infty f\left(\tau,u_m(\tau),u_m'(\tau),(Tu_m)(\tau),(Su_m)(\tau)\right)d\tau\, ds\right]\\ &\quad + ty_\infty + \int_0^t sf\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)ds\\ &\quad + \int_t^\infty tf\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)ds,\\ (Au_m)'(t) &= y_\infty + \int_t^\infty f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)ds.\end{aligned} \tag{3.2}$$

So for any $t_1, t_2 \in [0,c]$ $(c>0)$, $t_1 < t_2$, we have that

$$\begin{aligned}\left\|(Au_m)'(t_2)-(Au_m)'(t_1)\right\| &\le \int_{t_1}^{t_2}\left\|f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)\right\|ds\\ &\le \varepsilon_0(2+k^*+h^*)R\int_{t_1}^{t_2}q(s)e^s ds + \int_{t_1}^{t_2}\left[m(s)+Mp(s)\right]ds.\end{aligned} \tag{3.3}$$

(3.3) implies that $\{(Au_m)'(t)\}$ $(m = 1,2,3,\ldots)$ is equicontinuous on any finite subinterval of J.

On the other hand, by (2.17), (2.20), (3.2) we can obtain

$$\begin{aligned}\|(Au_m)(t)\| &\le \frac{1}{2}\|u_m\|_D + \left(t+\frac{\delta}{\Delta}\right)(m^*+Mp^*) + \left(\frac{\delta}{\Delta}+t\right)\|y_\infty\|, \quad \forall t\in J_+,\\ \left\|(Au_m)'(t)\right\| &\le \frac{1}{2}\|u_m\|_D + m^* + Mp^* + \|y_\infty\|, \quad \forall t \in J_+,\end{aligned} \tag{3.4}$$

which implies that $e^{-t}\|(Au_m)^{(i)}(t)\| \to 0$ as $t \to \infty$ $(i = 0,1)$ uniformly for $m = 1,2,3,\ldots$. Hence, by Lemma 2.5 we have

$$\alpha_D(AV) = \max\left\{\sup_{t\in J}\left[e^{-t}\alpha((AV)(t))\right], \sup_{t\in J}\left[e^{-t}\alpha\left((AV)^{(1)}(t)\right)\right]\right\}, \tag{3.5}$$

where $AV = \{Au_m : u_m \in V,\ m = 1,2,3,\ldots\}$, and $(AV)^{(1)} = \{(Au_m)^{(1)}(t) : u_m \in V,\ m = 1,2,3,\ldots\}$.

It follows from (2.18) that the infinite integral $\int_0^\infty \|f(t,u_m(t),u_m'(t),(Tu_m)(t),(Su_m)(t))\|dt$ is convergence uniformly for $m = 1,2,3,\ldots$. So, for any $\varepsilon > 0$, we can choose a large $T > 0$ such that

$$\int_T^\infty \|f(t,u_m(t),u_m'(t),(Tu_m)(t),(Su_m)(t))\|dt < \varepsilon \tag{3.6}$$

holds for any m. Then, by [6, Theoerm 1.2.3], (3.6), and (H_3), we obtain

$$\begin{aligned}
e^{-t}\alpha((AV)(t)) &\le 2\frac{t}{e^t}\int_t^T \alpha\big(f(s,V(s),V'(s),(TV)(s),(SV)(s))\big)ds + 2\varepsilon \\
&\quad + 2\frac{s}{e^t}\int_0^t \alpha\big(f(s,V(s),V'(s),(TV)(s),(SV)(s))\big)ds \\
&\le 4\int_0^\infty \alpha\big(f(s,V(s),V'(s),(TV)(s),(SV)(s))\big)ds + 2\varepsilon \\
&\le 4\alpha_D(V)\int_0^\infty [L_0(s) + L_1(s) + L_2(s)k^* + L_3(s)h^*]e^s ds + 2\varepsilon, \\
e^{-t}\alpha\Big((AV)^{(1)}(t)\Big) &\le 2\int_0^\infty \alpha\big(f(s,V(s),V'(s),(TV)(s),(SV)(s))\big)ds + 2\varepsilon \\
&\le 2\alpha_D(V)\int_0^\infty [L_0(s) + L_1(s) + L_2(s)k^* + L_3(s)h^*]e^s ds + 2\varepsilon.
\end{aligned} \tag{3.7}$$

By (3.7) and noting the fact that $\varepsilon > 0$ is arbitrary, we see that

$$a_D(AV) \le 4L^* a_D(V) \le a_D(V). \tag{3.8}$$

On the other hand, $\alpha_D(V) \le \alpha_D\{\overline{co}(\{u\}\bigcup(AV))\} = \alpha_D(AV)$. Then $\alpha_D(V) = 0$, that is, V is relatively compact in $DC^1[J,E]$.

Hence, the Mönch fixed point theorem implies that A has a fixed point $\overline{u}$ in Q_1 and this theorem is proved. □

Theorem 3.2. *Let cone P be normal and let conditions (H_1)–(H_4) be satisfied. Then the BVP* (1.1) *has a positive solution $y \in Q\bigcap[J_+,E]$ which is minimal in the sense that $u^{(i)}(t) \ge y^{(i)}(t),\ t \in J\ (i = 0,1)$ for any positive solution $u \in Q\bigcap[J_+,E]$. Moreover, $\|y\|_D \le 2\gamma + \|x_0\|_D$, where*

$$\gamma = \left(1 + \frac{\delta}{\Delta}\right)(m^* + Mp^*) + \left(\frac{\delta}{\Delta} + 1\right)\|y_\infty\| \tag{3.9}$$

and there exists a monotone iterative sequence $\{u_m\}$ *such that* $u_m^{(i)} \to y^{(i)}$ *as* $m \to \infty$ $(i = 0, 1)$ *uniformly on* J *and* $u_m''(t) \to y''(t)$ *as* $m \to \infty$ *for any* $t \in J_+$*, where*

$$\begin{aligned} u_0(t) = {} & \frac{1}{\Delta}\left[\delta y_\infty + b\int_0^\infty f(s, x_0^*, x_0^*, \theta, \theta)ds + \sum_{i=1}^n k_i \int_0^{\xi_i}\int_s^\infty f(\tau, x_0^*, x_0^*, \theta, \theta)d\tau\, ds\right] \\ & + ty_\infty + \int_0^t\int_s^\infty f(\tau, x_0^*, x_0^*, \theta, \theta)d\tau\, ds, \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_m(t) = {} & \frac{1}{\Delta}\Bigg[\delta y_\infty + b\int_0^\infty f(s, u_{m-1}(s), u_{m-1}'(s), (Tu_{m-1})(s), (Su_{m-1})(s))ds \\ & \quad + \sum_{i=1}^n k_i \int_0^{\xi_i}\int_s^\infty f(\tau, u_{m-1}(\tau), u_{m-1}'(\tau), (Tu_{m-1})(\tau), (Su_{m-1})(\tau))d\tau\, ds\Bigg] \\ & + ty_\infty + \int_0^t\int_s^\infty f(\tau, u_{m-1}(\tau), u_{m-1}'(\tau), (Tu_{m-1})(\tau), (Su_{m-1})(\tau))d\tau\, ds. \end{aligned} \tag{3.11}$$

Proof. From (3.10), we can find that $u_0 \in C[J, E]$ and

$$u_0'(t) = \int_t^\infty f(s, x_0^*, x_0^*, \theta, \theta)ds + y_\infty. \tag{3.12}$$

By (3.10) and (3.12), we have $u_0^{(i)} \geq y_\infty \geq x_0^*$ $(i = 0, 1)$ and

$$\begin{aligned} \|u_0(t)\| \leq {} & \frac{1}{\Delta}\left[\delta\|y_\infty\| + b\int_0^\infty \|f(s, x_0^*, x_0^*, \theta, \theta)\|ds + \sum_{i=1}^n k_i \int_0^{\xi_i}\int_s^\infty \|f(\tau, x_0^*, x_0^*, \theta, \theta)\|d\tau\, ds\right] \\ & + t\|y_\infty\| + \int_0^t\int_s^\infty \|f(\tau, x_0^*, x_0^*, \theta, \theta)\|d\tau\, ds \\ \leq {} & \frac{\delta}{\Delta}\|y_\infty\| + t\|y_\infty\| + \frac{b}{\Delta}\int_0^\infty \|f(s, x_0^*, x_0^*, \theta, \theta)\|ds \\ & + \frac{\sum_{i=1}^n k_i\xi_i}{\Delta}\int_0^\infty \|f(\tau, x_0^*, x_0^*, \theta, \theta)\|d\tau + t\int_0^\infty \|f(\tau, x_0^*, x_0^*, \theta, \theta)\|d\tau, \\ \|u_0'(t)\| \leq {} & \int_t^\infty \|f(s, x_0^*, x_0^*, \theta, \theta)\|ds + \|y_\infty\|. \end{aligned} \tag{3.13}$$

(2.18) implies that $e^{-t}\|u_0^{(i)}\| \to 0$ as $t \to \infty$ $(i = 0, 1)$. Hence, $\|u_0\|_F, \|u_0\|_C < \infty$ $(i = 0, 1)$, which implies $u_0 \in DC^1[J, E]$. From (2.11) and (3.11) we get

$$u_m(t) = (Au_{m-1})(t), \quad \forall t \in J,\ m = 1, 2, 3, \ldots. \tag{3.14}$$

By Lemma 2.2, we get $u_m \in Q$ and

$$\|u_m(t)\|_D = \|(Au_{m-1})(t)\|_D \leq \frac{1}{2}\|u_{m-1}\|_D + \gamma, \quad m = 1,2,3,\ldots, \tag{3.15}$$

By Lemma 2.6 and (3.15), we have

$$u_0^* \leq u_0^{(i)}(t) \leq u_1^{(i)}(t) \leq \cdots \leq u_m^{(i)}(t) \leq \cdots, \quad \forall t \in J \ (i = 0,1). \tag{3.16}$$

It follows from (3.15), by induction, that

$$\|u_m(t)\|_D \leq \gamma + \frac{1}{2}\gamma + \cdots \left(\frac{1}{2}\right)^{m-1}\gamma + \left(\frac{1}{2}\right)^{m}\|u_0\|_D \leq 2\gamma + \|u_0\|_D, \quad m = 1,2,3,\ldots. \tag{3.17}$$

Let $K = \{u \in Q : \|u\| \leq 2\gamma + \|u_0\|_D\}$. So, K is a bounded closed convex set in the space $DC^1[J,E]$ and operator A maps K into K. Clearly, K is not empty since $u_0 \in K$.

Let $W = \{u_m : m = 0,1,2,\ldots\}$ and $AW = \{Au_m : m = 0,1,2,\ldots\}$. Obviously, $W \subset K$ and $W = \{u_0\}\bigcup(AW)$. Similarly, as the proof in Theorem 3.1, we can obtain $\alpha_D(W) = 0$, that is, W is relatively compact in $DC^1[J,E]$. So there exist a $y \in DC^1[J,E]$ and a subsequence $\{u_{m_j} : j = 1,2,3,\ldots\} \subset W$ such that $\{u_{m_j}^{(i)} : j = 1,2,3,\ldots\}$ converges to $y^{(i)}(t)$ uniformly on J $(i = 0,1)$. Since that P is normal and $\{u_m^{(i)}(t) : m = 1,2,3,\ldots\}$ is nondecreasing on account of (3.16), it is easy to see that the entire sequence $\{u_m^{(i)}(t) : m = 1,2,3,\ldots\}$ converges to $y^{(i)}(t)$ uniformly on J $(i = 0,1)$. By $u_m \in K$ and K is a closed convex set in space $DC^1[J,E]$, we have $y \in K$. Let $t \in J$ be arbitrarily fixed. It is clear that

$$H(t,s)\|u_m(s) - y(s)\| \longrightarrow 0, \quad \text{as } m \longrightarrow \infty, \ \forall s \in J \tag{3.18}$$

and, by (3.17), we have

$$H(t,s)\|u_m(s) - y(s)\| \leq 2(2\gamma + \|u_0\|_D)H(t,s)e^s = \rho(s) \in L[J,J], \quad \forall s \in J \ (m = 1,2,\ldots). \tag{3.19}$$

It follows from (3.18) and (3.19) and the dominated convergence theorem that

$$\|(Su_m)(s) - (Sy)(s)\| \leq \int_0^\infty H(t,s)\|u_m(s) - y(s)\|ds \longrightarrow 0 \quad (m \longrightarrow \infty), \tag{3.20}$$

which implies that $(Su_m)(t) \to (Sy)(t)$, as $m \to \infty$ for any $t \in J$. Hence,

$$\begin{gathered} f(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)) \longrightarrow f(s,y(s),y'(s),(Ty)(s),(Sy)(s)) \\ \text{as } m \longrightarrow \infty, \quad \forall s \in J_+. \end{gathered} \tag{3.21}$$

By virtue of (3.17) and Lemma 2.1, we have

$$\begin{aligned}&\left\|f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right)-f\left(s,y(s),y'(s),(Ty)(s),(Sy)(s)\right)\right\|\\&\quad\le 2\left[\varepsilon_0 q(s)e^s\left(2+k^*+h^*\right)\|u_m\|_D+m(s)+Mp(s)\right]\\&\quad\le 2\left[\varepsilon_0 q(s)e^s\left(2+k^*+h^*\right)\left(2\gamma+\|u_0\|_D\right)+m(s)+Mp(s)\right].\end{aligned}\tag{3.22}$$

Now, noting (3.21) and (3.22) and taking limit as $m\to\infty$ in (3.11), we obtain

$$\begin{aligned}y(t)=\frac{1}{\Delta}\Bigg[&\delta y_\infty+b\int_0^\infty f\left(s,y(s),y'(s),(Ty)(s),(Sy)(s)\right)ds\\&+\sum_{i=1}^n k_i\int_0^{\xi_i}\int_s^\infty f\left(\tau,y(\tau),y'(\tau),(Ty)(\tau),(Sy)(\tau)\right)d\tau\, ds\Bigg]\\&+ty_\infty+\int_0^t\int_s^\infty f\left(\tau,y(\tau),y'(\tau),(Ty)(\tau),(Sy)(\tau)\right)d\tau\, ds,\end{aligned}\tag{3.23}$$

which follows from Lemma 2.3 that $y\in K\bigcap C^2[J_+,E]$ and $y(t)$ is positive solution of the BVP (1.1). Differentiating (3.11) twice, we get

$$u_m''=-f\left(s,u_m(s),u_m'(s),(Tu_m)(s),(Su_m)(s)\right),\quad \forall s\in J_+,\ m=1,2,3,\ldots.\tag{3.24}$$

Hence, by (3.21), we obtain

$$\lim_{m\to\infty}u_m''=-f\left(s,y(s),y'(s),(Ty)(s),(Sy)(s)\right),\quad \forall s\in J_+.\tag{3.25}$$

Let $u(t)$ be any other positive solution of the BVP (1.1). By Lemma 2.3, we have $u(t)\in Q$ and $u(t)=(Au)(t)$, for $t\in J$. It is clear that $u^{(i)}(t)\ge x_0^*>0$ for any $t\in J$ $(i=0,1)$. So, by Lemma 2.6, we can have $u^{(i)}(t)\ge u_0^{(i)}$ for any $t\in J$ $(i=0,1)$. Assume that $u^{(i)}\ge u_{m-1}^{(i)}(t)$ for any $t\in J$, $m\ge 1$ $(i=0,1)$. Then, it follows from Lemma 2.6 that $(Au)^{(i)}\ge (Au_{m-1})^{(i)}(t)$ for any $t\in J$, $(i=0,1)$, that is, $u^{(i)}(t)\ge u_m^{(i)}(t)$ for any $t\in J$, $(i=0,1)$. Hence, by induction, we get

$$u^{(i)}(t)\ge u_m^{(i)}(t),\quad \forall t\in J,\ (i=0,1;\ m=0,1,2,3,\ldots).\tag{3.26}$$

Now, taking limits as $m\to\infty$ in (3.26), we get $u^{(i)}(t)\ge y^{(i)}(t)$ for $t\in J$ $(i=0,1)$, and this completes the proof. □

Theorem 3.3. *Let cone P be fully regular and let conditions $(H_1),(H_2),(H_4)$ be satisfied. Then the conclusion of Theorem 3.2 holds also.*

Proof. The proof is almost the same as that of Theorem 3.2. The only difference is that, instead of using condition (H_3), the conclusion $\alpha_D(W)=0$ was implied directly by (3.16) and (3.17) and full regularity of P and Lemma 2.5. □

4. An Example

Example 4.1. Consider the following infinite system of scalar second-order multipoint singular integrodifferential equation:

$$
\begin{aligned}
-u_n''(t) ={}& \frac{e^{-2t}}{n^2\sqrt{t}}\left(1+u_n+u_{2n}'+\frac{1}{2n^2u_n(t)}+\frac{1}{8n^3u_{2n}'(t)}\right)^{1/2} \\
&+\frac{e^{-3t}}{\sqrt{nt}}\left(\int_0^t \frac{e^{-s(t+1)}u_n(s)ds}{(1+ts+s)^3}\right)^{1/3} \\
&+\frac{e^{-4t}}{n\sqrt{t}(1+t)}\left(\int_0^{\infty} e^{-2s}\cos^2(t-s)u_{2n}(s)ds\right)^{1/5}, \quad \forall t\in J_+, \\
80u_n(0)-3u_n'(0)-&\sum_{i=1}^{7} iu_n\left(\frac{i+3}{4}\right)=0, \qquad \lim_{t\to+\infty}u_n'(t)=\frac{1}{n^2}.
\end{aligned}
\tag{4.1}
$$

Conclusion. Infinite system (4.2) has a minimal positive solution $u_n(t)$ satisfying $u_n(t), u_n'(t) \geq 1/n^2$ for $0 \leq t \leq +\infty$ $(n = 1,2,3,\ldots)$, and this minimal solution can be obtained by taking limits from some iterative sequences.

Proof. Let $E = l^1 = \{u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty\}$ with the norm $\|u\| = \sum_{n=1}^{\infty} |u_n|$. Choose $P = \{u = (u_n) \in l^1 : u_n \geq 0,\ n = 1,2,\ldots\}$. It is easy to see that E is weakly sequence complete, and P is a normal cone in E. Thus, P is fully regular.

Now we consider the infinite system (4.2), which can be regarded as the BVP (1.1) with $a = 80$, $b = 3$, $k_i = i$, $\xi_i = (i+3)/4$ $(i = 1,2,\ldots,7)$, $y_\infty = (1, 1/4, 1/9, \ldots)$. So we have

$$
\Delta = 80-\sum_{i=1}^{7} i = 52, \quad \delta = 3+\sum_{i=1}^{7} i\frac{i+3}{4} = 59, \quad \Delta \leq \delta. \tag{4.2}
$$

In this situation,

$$
\begin{gathered}
u = (u_1, u_2, \ldots, u_n, \ldots), \qquad v = (v_1, v_2, \ldots, v_n, \ldots), \\
w = (w_1, w_2, \ldots, w_n, \ldots), \qquad z = (z_1, z_2, \ldots, z_n, \ldots), \\
K(t,s) = \frac{e^{-s(t+1)}}{(1+ts+s)^3}, \qquad H(t,s) = e^{-2s}\cos^2(t-s), \\
f = (f_1, f_2, \ldots, f_n, \ldots),
\end{gathered}
\tag{4.3}
$$

in which

$$f_n(t,u,v,w,z) = \frac{e^{-2t}}{n^2\sqrt{t}}\left(1 + u_n + v_{2n} + \frac{1}{2n^2 u_n(t)} + \frac{1}{8n^3 v_{2n}(t)}\right)^{1/2} + \frac{e^{-3t}}{\sqrt{nt}} w_n^{1/3} + \frac{e^{-4t}}{n\sqrt{t}(1+t)} z_{2n}^{1/5}. \tag{4.4}$$

Let $x_0^* = y_\infty = (1, 1/4, 1/9, \ldots)$. Then $P_{0\lambda} = \{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \geq \lambda/n^2,\ n = 1, 2, \ldots\}$ for any $\lambda > 0$ and the condition (H_1) holds for $k^* < 1/2$ and $h^* < 1$. It is clear that $f \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P \times P, P]$ for any $\lambda > 0$. By (4.4) we get

$$\|f(t,u,v,w,z)\| \leq \frac{e^{-2t}}{\sqrt{t}}\left\{\left(\frac{11}{4} + \|u\| + \|v\|\right)^{1/2} + \|w\|^{1/3} + \|z\|^{1/5}\right\}. \tag{4.5}$$

So, the condition (H_2) is satisfied for $m(t) = 0$, $p(t) = q(t) = e^{-2t}/\sqrt{t}$ and

$$g(u_0, u_1, u_2, u_3) = \left(\frac{11}{4} + u_0 + u_1\right)^{1/2} + u_2^{1/3} + u_3^{1/5}. \tag{4.6}$$

It is easy to see that (H_4) holds. Thus, our conclusion follows from Theorem 3.3 immediately. □

Remark 4.2. It is easy to see that the Example 4.1 cannot be solved by [3–11, 13].

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