Research Article

An Analytical Model for Predicting the Stress Distributions within Single-Lap Adhesively Bonded Beams

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An analytical model for predicting the stress distributions within single-lap adhesively bonded beams under tension is presented in this paper. By combining the governing equations of each adherend with the joint kinematics, the overall system of governing equations can be obtained. Both the adherends and the adhesive are assumed to be under plane strain condition. With suitable boundary conditions, the stress distribution of the adhesive in the longitudinal direction is determined.

1. Introduction

The use of adhesively bonded joint, which uses metallic, composite, and ceramic materials, is of great interest to many industrial sectors including aerospace, automotive, marine, machine tools, package, and appliance industries. This widespread use of adhesive joints is due to ease of application, time and cost savings, high corrosion and fatigue resistance, crack retardance, and good damping characteristics [1–3].

Any joint occurring in practice is designed to carry a given set of loads. Most of the adherends are loaded in tension. The subsequent loads on the adhesive are then a function of the geometry of the joint. Under most operating loads and environmental conditions, the adherends behave in a linearly elastic manner. However, the adhesive may exhibit viscoelastic or nonlinear behaviour. The exact analytical solution to the problem of stress distributions in the bonded area is complex. The existing analytical studies are, therefore, based on certain simplifying assumptions with regard to the modelling of the adhesive and adherends.

In Volkersen's shear-lag analysis, it was assumed that the adhesive deforms only in shear, while the adherend deformed only in tension [4]. The consequences of the rotation of the adherends were first taken into account by Goland and Reissner [5]. The authors derived equations to evaluate the

shearing and normal stresses in the bond layer as well as those in the jointed plates, assuming that the peel and shear stresses were constants across the adhesive thickness. In Cornell's work [6], a variation and extension of Goland and Reissner's method were presented for determining the stresses in adhesive lap joints. The author assumed that the two lap joint plates act like simple beam and the more elastic adhesive layer is an infinite number of shear and tension springs. Differential equations were set up which describe the transfer of the load in one beam through the springs to the other beam. From the solution of these differential equations a fairly complete analysis of the stresses in the lap joint was obtained.

Ojalvo and Eidinoff [7] presented results of an analytical investigation on the influence of bond thickness upon the stress distribution in single adhesive lap joints. The work extended the basic approach for bonded joints, originally introduced by Goland and Reissner, through the use of a more complete shear-strain/displacement equation for the adhesive layer. The work uncovers several interesting phenomena without adding any significant complication to the analysis. Delale et al. [8] analyzed a general plane strain problem of adhesively bonded structures which consist of two different orthotropic adherends. Both the transverse shear stress effects in the adherends and the in-plane normal strain in the adhesive were taken into account. The solution was
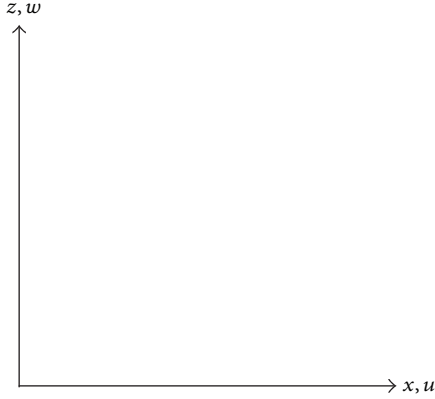
z, w

Figure 1: Coordinate system.

obtained by assuming linear stress-strain relations for the adhesive. The peak values of the shear as well as those of the normal stress in the adhesive were found to be at the edges of the overlap region. Rossettos et al. [9] established the governing equations for a step lap joint with a void using a modified shear-lag model, where the adhesive can have extensional as well as shear deformations. The model considers quadratic axial deformation across the adhesive thickness.

The objective of this work is to present an analytical model for predicting the stress distributions within a single-lap adhesively bonded beam under tension. This model is a development of the shear-lag analysis model of Volkersen [4]. The derivations are similar to those of Goland and Reissner [5] but the geometry employed is different. By combining the governing equations of each adherend with the joint kinematics, the overall system of governing equations can be obtained. Both the adherends and the adhesive are assumed to be under plane strain condition, and the adhesive stresses are assumed to be uniform across the thickness. With suitable boundary conditions, the stress distribution of the adhesive in the longitudinal direction is determined.

2. Formulation of the Problem

Figure 1 shows the coordinate system used in this study; x and z are coordinates, while u and w represent displacements. A single-lap-jointed beam is shown in Figure 2. The joint length is $2c$. The two adherends are considered to be of equal thickness t and equal length $(l + 2c)$. Two points a and b are the points at the centre of two free edge of the single-lap-jointed beam. The adhesive thickness is η. The joint width is assumed to be large compared with the adherend thickness. If the system is now loaded by tensile forces T per unit of adherend width at the points a and b, the line of action of the forces will be aob.

For studying the system deformation, two coordinate systems (x_1, z_1) and (x_2, z_2) are introduced in Figure 2. The first system is used to analyze the behavior of the left-hand adherend under load. The coordinate x_1 has its origin at the point a, extends along the longitudinal axis of the

adherend, and is positive to the right. w_1 represents transverse deformation of the adherend from the unloaded condition and is positive upwards. A similar definition is taken for (x_2, z_2) with reference to the joint.

If M_1 is equal to the bending moment in the adherend at station x_1 and M_2 is equal to the moment in the joint at station x_2, each per unit of width, then

$$M_1 = T\left[\tan\alpha_n x_1 - w_1\right] \quad (0 \le x_1 \le l),$$

$$M_2 = T\left[\tan\alpha_n (l + x_2) - w_2 - \frac{t + \eta}{2}\right] \quad (0 \le x_2 \le c), \tag{1}$$

where

$$\tan\alpha_n = \frac{(t + \eta)/2}{l + c}. \tag{2}$$

As $\eta \ll t$ and $t \ll l$, (1) then reduce to the simplified forms by using Taylor series:

$$M_1 = T\left[\alpha_n x_1 - w_1\right] \quad (0 \le x_1 \le l),$$

$$M_2 = T\left[\alpha_n (l + x_2) - w_2 - \frac{t}{2}\right] \quad (0 \le x_2 \le c). \tag{3}$$

As for deformations w_1 and w_2, we have

$$\frac{d^2 w_1}{dx_1^2} = -\frac{M_1}{D_1},$$

$$\frac{d^2 w_2}{dx_2^2} = -\frac{M_2}{D_2}, \tag{4}$$

where D_1 and D_2 are the flexural rigidities of the adherend and joint, respectively.

From (3) and (4), we have

$$\frac{d^2 w_1}{dx_1^2} = -\frac{T}{D_1}\left[\alpha_n x_1 - w_1\right] \quad (0 \le x_1 \le l), \tag{5}$$

$$\frac{d^2 w_2}{dx_2^2} = -\frac{T}{D_2}\left[\alpha_n (l + x_2) - w_2 - \frac{t}{2}\right] \quad (0 \le x_2 \le c). \tag{6}$$

From the four boundary conditions

$$\text{at } x_1 = 0, \qquad w_1 = 0,$$

$$\text{at } x_1 = l, \qquad x_2 = 0, \qquad w_1 = w_2,$$

$$\text{at } x_1 = l, \qquad x_2 = 0, \qquad \frac{dw_1}{dx_1} = \frac{dw_2}{dx_2}, \tag{7}$$

$$\text{at } x_2 = c, \qquad w_2 = 0.$$

The two preceding equations (5) and (6) have solutions of the form

$$w_1 = A_1 \cosh b_1 x_1 + B_1 \sinh b_1 x_1 + \alpha_n x_1 \quad (0 \le x_1 \le l),$$

$$w_2 = A_2 \cosh b_2 x_2 + B_2 \sinh b_2 x_2 + \alpha_n \left(l + x_2 - \frac{t}{2}\right)$$
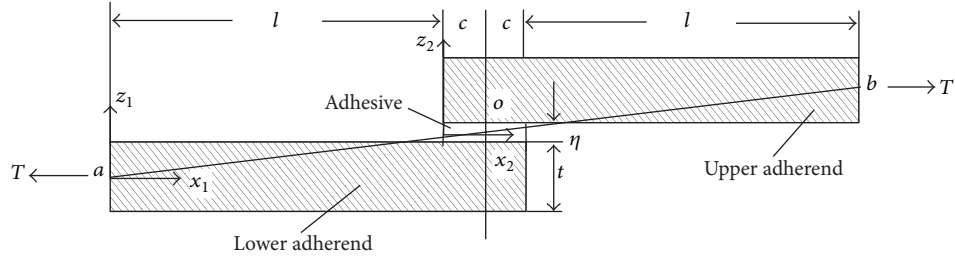
$$(0 \le x_2 \le c), \tag{8}$$

FIGURE 2: A single-lap-jointed beam.

where [5]

$$A_1 = \frac{-(t/2)\sinh(b_2 c)}{\sinh(b_1 l)\cosh(b_2 c) + (b_2/b_1)\sinh(b_1 l)\cosh(b_2 c)},$$

$$A_2 = \frac{(t/2)(b_2/b_1)\cosh(b_1 l)\sinh(b_2 c)}{\sinh(b_1 l)\cosh(b_2 c) + (b_2/b_1)\sinh(b_1 l)\cosh(b_2 c)},$$

$$B_1 = \frac{-(t/2)\cosh(b_2 c)}{\sinh(b_1 l)\cosh(b_2 c) + (b_2/b_1)\sinh(b_1 l)\cosh(b_2 c)},$$

$$B_2 = \frac{(t/2)(b_2/b_1)\cosh(b_1 l)\cosh(b_2 c)}{\sinh(b_1 l)\cosh(b_2 c) + (b_2/b_1)\sinh(b_1 l)\cosh(b_2 c)},$$

$$b_1^2 = \frac{T}{D_1},$$

$$b_2^2 = \frac{T}{D_2}.$$

$$(9)$$

3. Stress Distributions in the Adhesive Layer

Figure 3 shows elements of the upper and lower adherends. M, V, and T are the bending moments, vertical shear, and axial tension in the adherends. The subscripts u and l designate quantities pertaining to the upper and lower adherend, respectively. σ_0 and τ_0 are the transverse normal stress and the shear stress, respectively.

The conditions of moment equilibrium for the elements of the adherends are

$$\frac{dM_u}{dx} - V_u + \tau_0 \frac{t}{2} = 0,$$

$$\frac{dM_l}{dx} - V_l + \tau_0 \frac{t}{2} = 0.$$

$$(10)$$

The conditions of horizontal force equilibrium are

$$\frac{dT_u}{dx} - \tau_0 = 0,$$

$$\frac{dT_l}{dx} + \tau_0 = 0.$$

$$(11)$$

The conditions of vertical force equilibrium for the elements of the adherends are

$$\frac{dV_u}{dx} - \sigma_0 = 0,$$

$$\frac{dV_l}{dx} + \sigma_0 = 0.$$

$$(12)$$

The transverse deflections of the upper and lower adherends are denoted by v_u and v_l, respectively, both measured positively upward. Then the thin plate theory gives

$$\frac{d^2 v_u}{dx^2} = -\frac{M_u}{D_1},$$

$$\frac{d^2 v_l}{dx^2} = -\frac{M_l}{D_1},$$

$$(13)$$

$$\text{with } D_1 = \frac{Et^3}{12(1-v^2)},$$

where E and v are Young's modulus and Poisson's ratio of adherends, respectively. D_1 refers to the flexural rigidity of the adherends.

The longitudinal displacements of the adherend at the adherend boundaries adjacent to the adhesive are denoted by u_u and u_l. Then from the stress-strain relations

$$\frac{du_u}{dx} = \frac{1}{E}\left(\frac{T_u}{t} - 6\frac{M_u}{t^2}\right),$$

$$\frac{du_l}{dx} = \frac{1}{E}\left(\frac{T_l}{t} + 6\frac{M_l}{t^2}\right).$$

$$(14)$$

Let E_{ad} and G_{ad} refer to Young's and shear moduli, respectively, of the adhesive material; then

$$\frac{\tau_0}{G_{ad}} = \frac{u_u - u_l}{\eta},$$

$$\frac{\sigma_0}{E_{ad}} = \frac{v_u - v_l}{\eta}.$$

$$(15)$$

Combining equations from (10) to (14) and differentiating equation (15) we have

$$\frac{d^3\tau_0}{dx^3} - \frac{8G_{ad}}{Et\eta}\frac{d\tau_0}{dx} = 0,$$

$$(16)$$

$$\frac{d^4\sigma_0}{dx^4} + \frac{24(1-v^2)E_{ad}}{Et^3\eta}\sigma_0 = 0.$$
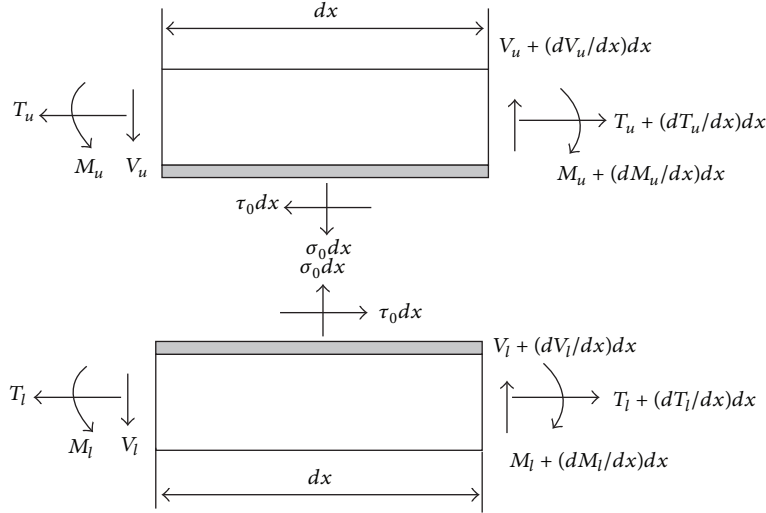
$$(17)$$

FIGURE 3: Free body diagram of joint with adhesive layer.

The boundary conditions can be written as

$$
\begin{aligned}
&\text{at } x = c, \qquad M_u = T_u = V_u = 0, \\
&M_l = M_0, \qquad V_l = V_0, \qquad T_l = T, \\
&\text{at } x = -c, \qquad M_l = T_l = V_l = 0, \\
&M_u = -M_0, \qquad V_u = V_0, \qquad T_u = T.
\end{aligned}
\tag{18}
$$

From (8) and (9) we have

$$
M_0 = (M_1)_{x_1=l} = -D_1 \left(\frac{d^2 w_1}{dx_1^2} \right)_{x_1=l} = k\frac{Tt}{2},
$$

$$
V_0 = \left(\frac{dM_1}{dx_1} \right)_{x_1=l} = kT\left[3\left(1-v^2\right) \frac{T}{Et} \right]^{1/2},
$$

$$
k = \frac{\sinh(b_1 l) \cosh(b_2 c)}{\sinh(b_1 l)\cosh(b_2 c) + (b_1/b_2)\cosh(b_1 l)\sinh(b_2 c)}.
\tag{19}
$$

On the basis of the differential equations (16) and (17) and the boundary conditions, the distributions of σ_0 and τ_0 in the adhesive are obtained in the form

$$
\begin{aligned}
\sigma_0 = \frac{pt^2}{\Delta c^2} \Bigg[&\left(R_2 \lambda^2 \frac{k}{2} + \lambda k' \cosh\lambda\cos\lambda \right) \cosh\lambda\frac{x}{c}\cos\lambda\frac{x}{c} \\
&+ \left(R_1 \lambda^2 \frac{k}{2} + \lambda k' \sinh\lambda\sin\lambda \right) \sinh\lambda\frac{x}{c}\sin\lambda\frac{x}{c} \Bigg]
\end{aligned}
$$

$$
\tau_0 = -\frac{pt}{8c}\left[\frac{\beta c}{t}(1+3k)\frac{\cosh(\beta x/t)}{\sinh(\beta c/t)} + 3(1-k) \right],
\tag{20}
$$

where

$$
\lambda = \gamma\frac{c}{t}, \qquad \gamma^4 = 6\frac{E_{ad}}{E}\frac{t}{\eta}, \qquad \beta^2 = 8\frac{G_{ad}}{E}\frac{t}{\eta},
$$

$$
R_1 = \cosh\lambda\sin\lambda + \sinh\lambda\cos\lambda,
$$

$$
R_2 = \sinh\lambda\cos\lambda - \cosh\lambda\sin\lambda,
$$

$$
\Delta = \frac{1}{2}(\sinh 2\lambda + \sin 2\lambda), \qquad k = \frac{2M_0}{pt^2}, \qquad k' = \frac{V_0 c}{pt^2},
$$

$$
b_1 = 2\left[\frac{3p\left(1-v^2\right)}{Et^2} \right]^{1/2}, \qquad b_2 = \frac{b_1}{2\sqrt{2}}.
\tag{21}
$$

If the materials and the thickness of the adherends are different, then (17) may become a 7th order differential equation. It is obvious that the analytical model is very complex and only the stress distributions of the adhesive in the longitudinal direction are determined. In addition, the adhesive stresses actually are not uniform across the thickness. In other words, the simplifications have restricted the results. Usually, when we consider different boundary conditions by a closed-form analysis, the limitation is how tractable a realistic mathematical model is within an algebraic solution. To overcome this limitation, the finite element analysis (FEA) technique can be employed.

4. Summary

An analytical model was presented for predicting the stress distributions within a single-lap adhesively bonded beam under tension. By combining the governing equations of each adherend with the joint kinematics, the overall system of governing equations can be obtained. Both the adherends and the adhesive are assumed to be under plane strain condition, and the adhesive stresses are assumed to be uniform across the thickness. With suitable boundary conditions, the stress

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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