Research Article

An Efficient Series Solution for Nonlinear Multiterm Fractional Differential Equations

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In this paper, we introduce an efficient series solution for a class of nonlinear multiterm fractional differential equations of Caputo type. The approach is a generalization to our recent work for single fractional differential equations. We extend the idea of the Taylor series expansion method to multiterm fractional differential equations, where we overcome the difficulty of computing iterated fractional derivatives, which are difficult to be computed in general. The terms of the series are obtained sequentially using a closed formula, where only integer derivatives have to be computed. Several examples are presented to illustrate the efficiency of the new approach and comparison with the Adomian decomposition method is performed.

1. Introduction

Fractional differential equations (FDEs) are generalization to differential equations (DEs) for noninteger orders. In recent years, FDEs caught the attention of many researchers because of their appearance in modeling several phenomenon in the physical sciences. As many FDEs do not possess exact solutions on closed forms, analytical and numerical techniques have been implemented to study these equations. Iterative methods, such as the variational iteration method (VIM) in [1], the homotopy analysis method (HAM) in [2, 3], the Adomian decomposition method (ADM) in [4–9], and the fractional differential transform method in [10], have been implemented to solve various types of FDEs. These methods produce a solution in a series form whose terms are determined sequentially. We refer the reader to [11, 12] for a comprehensive study of series solutions of fractional differential equations. Recently, we have introduced a new series solution for single fractional differential equations [13]. The new approach is a modified form of the well-known Taylor series expansion, where we overcome the difficulty of computing iterative fractional derivatives. The efficiency of the new approach has been illustrated through several examples. In this paper we extend the idea to multiterm

fractional differential equations. The presented work is a part of the Master thesis [14]. We consider the left Caputo fractional derivative $D_{0^+}^\alpha$, defined by [15, 16]

$$D_{0^+}^\alpha f(t)$$
$$= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, & \text{if } n-1 < \alpha < n, \\ f^{(n)}(t), & \text{if } \alpha = n \in \mathbb{N} \end{cases} \quad (1)$$

provided the integral exists, where Γ is the well-known Gamma function. The left Riemann-Liouville fractional integral, $I_{0^+}^\alpha$, of order $\alpha \in \mathbb{R}^+$, is defined by

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0, \ \alpha > 0. \quad (2)$$

The left Caputo derivative is related to the left Riemann-Liouville fractional integral by

$$D_{0^+}^\alpha f(t) = I_{0^+}^{n-\alpha} f^{(n)}(t), \quad n-1 < \alpha < n. \quad (3)$$

It is known that

$$((I_{0^+}^\alpha D_{0^+}^\alpha) f)(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k,$$

$$((D_{0^+}^\alpha I_{0^+}^\alpha) f)(t) = f(t). \tag{4}$$

Also, for $n - 1 < \alpha < n, n \in \mathbb{N}, t > 0$, it holds that

$$D_{0^+}^\alpha (t^\mu)$$

$$= \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, & \text{if } \mu > n-1, \\ 0, & \text{if } \mu \in \{0, 1, \ldots, n-1\}. \end{cases} \tag{5}$$

This paper is organized as follows. In Section 2, we present the series solution of nonlinear two-term fractional differential equations. We illustrate the efficiency of the presented technique through several examples. We also compare our results with the ones obtained by the Adomian decomposition method. In Section 3, we present and illustrate the efficiency of the new series solution for three-term fractional differential equations of several types. Finally, we conclude with some remarks in Section 4.

2. Two-Term Fractional Differential Equations

We start with the nonlinear two-term fractional initial value problems of the form

$$c_1 D_{0^+}^{\alpha_1} u(t) + c_2 D_{0^+}^{\alpha_2} u(t) = f(t, u(t)), \quad t > 0 \tag{6}$$

with

$$u(0) = b, \tag{7}$$

where $0 < \alpha_2 \le \alpha_1 < 1$, and c_1 and c_2 are nonzero constants. We assume that $f(t, u(t))$ is continuous and smooth with respect to $u(t)$. We also assume that α_1 and α_2 are rational numbers with $\alpha_1 = p_1/q_1$ and $\alpha_2 = p_2/q_2$, $p_1, p_2, q_1, q_2 \in \mathbb{N}$.

2.1. The Expansion Procedure. Let $q = \text{lcm}(q_1, q_2)$; we have $q = sq_1 = rq_2$ for some $s, r \in \mathbb{N}$.

In the following we expand the solution of problem (6)-(7) in an infinite series of the form

$$u(t) = \sum_{n=0}^\infty a_n t^{n/q}, \tag{8}$$

where the coefficients a_n: $n \ge 0$ have to be determined sequentially in the following manner: From the initial condition (7) we have $u(0) = b = a_0$. Since $D_{0^+}^\alpha(c) = 0$, for c being constant, we have

$$D_{0^+}^{\alpha_1} u(t) = \sum_{n=1}^\infty a_n s_n t^{n/q - p_1/q_1} = \sum_{n=1}^\infty a_n s_n t^{(n-sp_1)/q},$$

$$D_{0^+}^{\alpha_2} u(t) = \sum_{n=1}^\infty a_n r_n t^{n/q - p_2/q_2} = \sum_{n=1}^\infty a_n r_n t^{(n-rp_2)/q}, \tag{9}$$

where

$$s_n = \frac{\Gamma(n/q + 1)}{\Gamma(n/q - \alpha_1 + 1)},$$

$$r_n = \frac{\Gamma(n/q + 1)}{\Gamma(n/q - \alpha_2 + 1)}. \tag{10}$$

By substituting (9) in (6) we have

$$c_1 \sum_{n=1}^\infty a_n s_n t^{(n-sp_1)/q} + c_2 \sum_{n=1}^\infty a_n r_n t^{(n-rp_2)/q}$$

$$= f\left(t, \sum_{n=0}^\infty a_n t^{n/q}\right). \tag{11}$$

Applying the well-known Taylor series method to compute the coefficients $\{a_n; n \ge 1\}$ will lead to computing iterated fractional derivatives, which are not easily computed in general. To avoid this difficulty, let $t = w^q$; we have

$$c_1 \sum_{n=1}^\infty a_n s_n w^{n-sp_1} + c_2 \sum_{n=1}^\infty a_n r_n w^{n-rp_2}$$

$$= f\left(w^q, \sum_{n=0}^\infty a_n w^n\right). \tag{12}$$

Shifting the index to zero yields

$$c_1 \sum_{n=0}^\infty a_{n+1} s_{n+1} w^{n-sp_1+1} + c_2 \sum_{n=0}^\infty a_{n+1} r_{n+1} w^{n-rp_2+1}$$

$$= f\left(w^q, \sum_{n=0}^\infty a_n w^n\right). \tag{13}$$

To avoid the singularity at $w = 0$, we multiply (13) by w^{sp_1-1}; we have

$$c_1 \sum_{n=0}^\infty a_{n+1} s_{n+1} w^n + c_2 \sum_{n=0}^\infty a_{n+1} r_{n+1} w^{n-rp_2+sp_1}$$

$$= w^{sp_1-1} f\left(w^q, \sum_{n=0}^\infty a_n w^n\right). \tag{14}$$

Now, since $\alpha_1 = p_1/q_1 = sp_1/q > rp_2/q = p_2/q_2 = \alpha_2$, thus $sp_1 - rp_2 > 0$, and (14) has no singularity at $w = 0$.

Let $k = sp_1 - rp_2 - 1 \ge 0$, and $g(w) = f(w^q, \sum_{n=0}^\infty a_n w^n)$; then (14) can be written as

$$c_1 \sum_{n=0}^k a_{n+1} s_{n+1} w^n + \sum_{n=k+1}^\infty (c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k}) w^n$$

$$= w^{k+rp_2} g(w). \tag{15}$$

We first determine the coefficients a_n for $n \le k$. By performing the nth derivative of (15) with respect to w and

substituting $w = 0$, we have

$$c_1 n! a_{n+1} s_{n+1} = \frac{d^n}{dw^n} \left(w^{k+rp_2} g(w) \right) \Big|_{w=0} \qquad (16)$$

which yields

$$a_{n+1} = \frac{1}{c_1 n! s_{n+1}} \frac{d^n}{dw^n} \left(w^{k+rp_2} f \left(w^q, \sum_{n=0}^{\infty} a_n w^n \right) \right) \Big|_{w=0}. \qquad (17)$$

Since $k + rp_2 \geq n + 1$, and f is smooth, then for $n \leq k$, we have

$$\frac{d^n}{dw^n} \left(w^{k+rp_2} f \left(w^q, \sum_{n=0}^{\infty} a_n w^n \right) \right) \Big|_{w=0} = 0, \qquad (18)$$

and hence $a_{n+1} = 0$, for $n \leq k$.

We now determine a_n, for $n \geq k+1$. By performing the nth derivative of (15) with respect to w and substituting $w = 0$, we have

$$n! \left(c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k} \right) = \frac{d^n}{dw^n} \left(w^{k+rp_2} g(w) \right) \Big|_{w=0}$$
$$= \frac{d^n}{dw^n} \left(w^{k+rp_2} f \left(w^q, \sum_{n=0}^{\infty} a_n w^n \right) \right) \Big|_{w=0}. \qquad (19)$$

Using the well-known Leibniz rule for differentiating the products, we have

$$\frac{d^n}{dw^n} \left(w^{k+rp_2} g(w) \right)$$
$$= \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{dw^j} \left(w^{k+rp_2} \right) \frac{d^{n-j}}{dw^{n-j}} \left(g(w) \right). \qquad (20)$$

Since

$$\frac{d^j}{dw^j} \left(w^{k+rp_2} \right) \Big|_{w=0} = \begin{cases} 0 & \text{if } j \neq k + rp_2, \\ j! & \text{if } j = k + rp_2, \end{cases} \qquad (21)$$

we have

$$c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k}$$
$$= \frac{1}{(n-j)!} \left(\frac{d^{n-j}}{dw^{n-j}} g(w) \right) \Big|_{w=0}, \qquad (22)$$

$$\text{where } j = k + rp_2.$$

From the last equation we determine $a_n : n \geq k + 1$ and thus the solution

$$u(t) = \sum_{n=0}^{\infty} a_n t^{n/q} = a_0 + \sum_{n=k+1}^{\infty} a_n t^{n/q} \qquad (23)$$

is obtained.

Remark 1. In (6), assuming $\alpha_1 = \alpha_2 = 1$, then $s_n = r_n = n, k = -1$ and $k + rp_2 = 0$. Thus (17) is reduced to

$$a_{n+1} = \frac{1}{c_1 (n+1)!} \frac{d^n}{dw^n} \left[f \left(w, \sum_{n=0}^{\infty} a_n w^n \right) \right] \Big|_{w=0}, \qquad (24)$$

which coincides with the coefficients obtained by the Taylor series expansion method. Same comment applies for the coefficient a_{n+1} in (22).

Remark 2. The algorithm can be generalized for the two-term fractional differential equation (6) with $0 < \alpha_2 \leq \alpha_1 < N$, for arbitrary integer N. But we have to take care of the existence of the fractional derivative $D_{0^+}^{\alpha_1}(t^{n/q})$ (see Eq. (5)), by choosing the coefficient $a_n = 0$, for $n = 1, \ldots, qN - 1$. This case is discussed in Section 3.

2.2. Numerical Results

Example 3. Consider the nonlinear two-term fractional initial value problem

$$2D_{0^+}^{1/2} u(t) + 2\Gamma \left(\frac{13}{10} \right) D_{0^+}^{1/5} u(t)$$
$$= \Gamma \left(\frac{1}{2} \right) \left(u^2 + t^{3/10} - t + 1 \right), \quad t > 0, \qquad (25)$$

with

$$u(0) = 0. \qquad (26)$$

The exact solution of the problem is $u(t) = t^{1/2}$.

Applying the current algorithm, we have $\alpha_1 = 1/2 = p_1/q_1$, $\alpha_2 = 1/5 = p_2/q_2$, $q = \text{l.c.m}(q_1, q_2) = 10$, $s = 5$, and $r = 2$. We expand the solution in an infinite series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^{n/10}. \qquad (27)$$

The initial condition in (26) yields $a_0 = 0$. We have $t = w^q = w^{10}$ and

$$g(w) = f \left(w^{10}, u(w) \right)$$
$$= \Gamma \left(\frac{1}{2} \right) \left(\left(\sum_{n=0}^{\infty} a_n w^n \right)^2 + w^3 - w^{10} + 1 \right). \qquad (28)$$

Since $g(w)$ is continuous and smooth with respect to w, we have

$$a_{n+1} = 0, \quad \text{for } n \leq k = sp_1 - rp_2 - 1 = 2. \qquad (29)$$

Thus $a_1 = a_2 = a_3 = 0$. The function $g(w)$ satisfies the assumption of the proposed algorithm, and it holds that

$$\frac{d^m}{dw^m}\left(g\left(w\right)\right)\Big|_{w=0} = \Gamma\left(\frac{1}{2}\right)\begin{cases} 1 & m = 0 \\ 3! & m = 3 \\ 6!a_3^2 & m = 6 \\ 2 \times 7!a_3a_4 & m = 7 \\ 8!a_4^2 + 2 \times 8!a_3a_5 & m = 8 \\ 2!\left(a_4a_5 + a_3a_3\right) & m = 9 \\ -10! + 10!a_5^2 + 2 \times 10!\left(a_4a_6 + a_3a_7\right) & m = 10 \\ \vdots & \vdots \end{cases} \tag{30}$$

The computation above is made using the software Mathematica version 9. For $n \geq 3$; substituting (30) in (22) yields

$$2a_{n+1}s_{n+1} + 2\Gamma\left(\frac{13}{10}\right)a_{n-2}r_{n-2}$$
$$= \frac{1}{(n-4)!}\frac{d^{n-4}}{dw^{n-4}}\left(g\left(w\right)\right)\Big|_{w=0}, \tag{31}$$

where

$$s_{n+1} = \frac{\Gamma\left((n+1)/10+1\right)}{\Gamma\left((n+1)/10+1/2\right)},$$
$$r_{n-2} = \frac{\Gamma\left((n-2)/10+1\right)}{\Gamma\left((n-2)/10+4/5\right)}. \tag{32}$$

Applying (31) together with $a_1 = a_2 = a_3 = 0$, we have

$$\begin{aligned} a_4 &= 0, \\ a_5 &= 1, \\ a_n &= 0, \\ n &\geq 6. \end{aligned} \tag{33}$$

Thus,

$$u\left(t\right) = a_5 t^{5/10} = t^{1/2}, \tag{34}$$

and the exact solution of problem (25)-(26) is obtained.

In the following we compare our results with the Adomian decomposition method (ADM). Assume that the nonlinear function $f(u) = u^2$ and the solution $u(t)$ can be expressed in the following series form:

$$u\left(t\right) = \sum_{n=0}^{\infty} u_n\left(t\right), \tag{35}$$
$$f\left(u\right) = \sum_{n=0}^{\infty} A_n,$$

where $A_n, n = 0, 1, 2, \ldots$ are the well-known Adomian polynomials that can be derived from [17]

$$A_n = \frac{1}{n!}\frac{d^n}{d\beta^n}\left[f\left(\sum_{j=0}^{\infty}\beta^j u_j\right)\right]_{\beta=0}, \quad j \geq 0. \tag{36}$$

For recent advances in the Adomian decomposition method we refer the reader to [18, 19]. For $f(u) = u^2$, we have

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \end{aligned} \tag{37}$$

$$\vdots$$

Applying the Riemann-Liouville fractional integral operator $I_{0^+}^{1/2}$ to (25) and substituting

$$I_{0^+}^{1/2}D_{0^+}^{1/5}u\left(t\right) = I_{0^+}^{3/10}u\left(t\right), \tag{38}$$

we have

$$u\left(t\right) = \frac{\Gamma\left(1/2\right)}{2}I_{0^+}^{1/2}\left(1 + t^{3/10} - t\right) - \Gamma\left(\frac{13}{10}\right)I_{0^+}^{3/10}u\left(t\right)$$
$$+ \frac{\Gamma\left(1/2\right)}{2}I_{0^+}^{1/2}\left(u^2\left(t\right)\right)$$
$$= t^{1/2} + 0.853958t^{4/5} - 0.666667t^{3/2}$$
$$- \Gamma\left(\frac{13}{10}\right)I_{0^+}^{3/10}\left(\sum_{n=0}^{\infty}u_n\left(t\right)\right)$$
$$+ \frac{\Gamma\left(1/2\right)}{2}I_{0^+}^{1/2}\left(\sum_{n=0}^{\infty}A_n\right) = \sum_{n=0}^{\infty}u_n\left(t\right). \tag{39}$$

Set

$$u_0 = t^{1/2} + 0.853958t^{4/5} - 0.666667t^{3/2} \qquad (40)$$

and balancing (39) yields

$$u_{n+1} = -\Gamma\left(\frac{13}{10}\right) I_{0^+}^{3/10}(u_n) + \frac{\Gamma(1/2)}{2} I_{0^+}^{1/2}(A_n),$$

$$n = 0, 1, 2, \ldots \qquad (41)$$

Evaluating the first 5 term of the power series solution, we obtain the approximate solution $u^{(4)}(t) = \sum_{n=0}^{4} u_n(t)$, where

$$u^{(4)}(t) = t^{0.5} + 0.257998t^2 - 2.798359t^{2.7}$$

$$- 1.454522t^3 + 8.855605t^{3.4} + 9.799202t^{3.7}$$

$$+ 2.317936t^4 - 11.599906t^{4.1}$$

$$- 22.429449t^{4.4} - 11.273123t^{4.7}$$

$$+ 6.685081t^{4.8} - 1.388180t^5$$

$$+ 23.109057t^{5.1} + 20.278401t^{5.4}$$

$$- 1.404129t^{5.5} + 5.305238t^{5.7}$$

$$- 11.070905t^{5.8} + 0.278605t^6 \qquad (42)$$

$$- 17.334122t^{6.1} - 7.918245t^{6.4}$$

$$+ 2.012699t^{6.5} - 0.885674t^{6.7}$$

$$+ 7.156236t^{6.8} + 5.827855t^{7.1}$$

$$+ 1.140180t^{7.4} - 1.153180t^{7.5}$$

$$- 2.127835t^{7.8} - 0.742611t^{8.1}$$

$$+ 0.309488t^{8.5} + 0.244589t^{8.8}$$

$$- 0.032578t^{9.5},$$

which is not the exact solution. Solving the problem by the ADM with $u_0 = t^{0.5}$ yields $U^{(4)}(t) = \sum_{n=0}^{4} U_n(t)$, where

$$U^{(4)}(t) = t^{0.5} - 0.853958t^{0.8} + 0.682106t^{1.1}$$

$$- 0.515733t^{1.4} + 0.666667t^{1.5}$$

$$+ 0.0372208t^{1.7} - 1.527774t^{1.8}$$

$$+ 2.25291t^{2.1} - 2.68363t^{2.4} + 0.711111t^{2.5} \qquad (43)$$

$$- 2.576649t^{2.8} + 5.473545t^{3.1}$$

$$+ 0.8533333t^{3.5} - 4.19714t^{3.8}$$

$$+ 1.078789t^{4.5},$$

which is again not the exact solution. Figure 1 depicts the exact solution $u(t)$ obtained by the current algorithm and
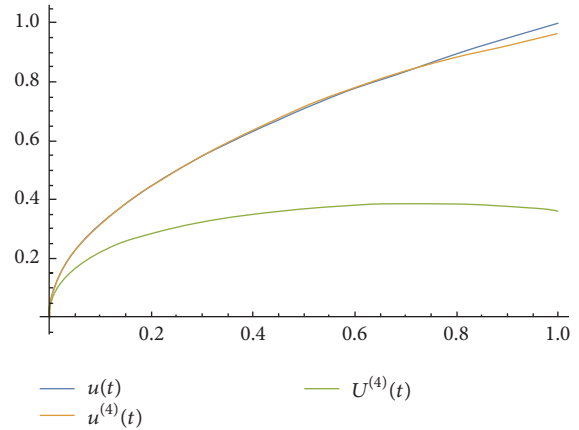

FIGURE 1: Comparison between the exact and approximate solutions obtained by the ADM for problem ((25)-(26)).

the approximate solutions $u^{(4)}(t)$ and $U^{(4)}(t)$ obtained by the Adomian decomposition method.

Example 4. Consider the nonlinear two-term fractional initial value problem

$$D_{0^+}^{4/5}u(t) + D_{0^+}^{1/2}u(t) = \frac{u(t)\sin(t)}{u^2(t) + 1}, \quad t > 0, \qquad (44)$$

with

$$u(0) = 1. \qquad (45)$$

This example has been discussed in [20], where the problem is transformed to a fractional integral equation, and then the Adams-Bashforth-Moulton method is used with step size $h = 1/50$ to approximate the solution.

Applying the current algorithm, we have $p_1 = 4, q_1 = 5, p_2 = 1, q_2 = 2, q = \text{l.c.m}(5,2) = 10, s = 2, r = 5, k = 2$, and

$$a_{n+1}s_{n+1} + a_{n-2}r_{n-2}$$

$$= \frac{1}{(n-7)!} \left(\frac{d^{n-7}}{dw^{n-7}} f\left(w^{10}, \sum_{n=0}^{\infty} a_n w^n\right) \right)\Bigg|_{w=0}, \qquad (46)$$

where

$$f\left(w^{10}, \sum_{n=0}^{\infty} a_n w^n\right) = \frac{\sin\left(w^{10}\right) \sum_{n=0}^{\infty} a_n w^n}{\left(\sum_{n=0}^{\infty} a_n w^n\right)^2 + 1}. \qquad (47)$$

TABLE 1: The error of Example 4 for $N = 35$ and different values of t.

t_i	$E_{35}(t_i)$
0.1	1.16844×10^{-10}
0.2	8.36081×10^{-9}
0.3	9.93967×10^{-8}
0.4	5.28261×10^{-7}
0.5	1.93074×10^{-6}
0.6	5.47664×10^{-6}
0.7	0.0000130833
0.8	0.0000276624
0.9	0.0000534896
1.0	0.0000967893

TABLE 2: The error of Example 4 at $t = 1$ and for different values of N.

N	$E_N(1)$
5	0.0681689
10	0.0681689
15	0.0681689
20	0.067155
25	0.0248829
30	0.00882027
35	0.0000967893

The initial condition $u(0) = 1$ yields $a_0 = 1$. We apply (46) to compute the first 13 nonzero terms of a_n, and we have

$$u(t) = 1 + 0.298242t^{1.8} - 0.227519t^{2.1}$$
$$+ 0.167717t^{2.4} - 0.119885t^{2.7} + 0.0833333t^3$$
$$- 0.0564631t^{3.3} + 0.0373656t^{3.6}$$
$$- 0.0280303t^{3.8} - 0.0241927t^{3.9} \tag{48}$$
$$+ 0.0179008t^{4.1} + 0.0153477t^{4.2}$$
$$- 0.0112111t^{4.4}.$$

Since the exact solution of problem (44)-(45) is not available in a closed form, we define the error $E_N(t)$ by

$$E_N(t_i) = \int_0^{t_i} \left(\mathscr{P}u_N(t)\right)^2 dt, \quad t \in [0, t_i], \tag{49}$$

where

$$\mathscr{P}u(t) = D_{0^+}^{4/5}u(t) + D_{0^+}^{1/2}u(t) - \frac{u(t)\sin(t)}{u^2(t) + 1}, \tag{50}$$
$$u_N(t) = \sum_{k=0}^{N} a_k t^{k/10}.$$

Tables 1 and 2 present the error E_N for different values of N. One can see that the error decreases with N and more accuracy can be achieved by considering more terms. Also, the error increases with t, as any other series solutions.

3. Three-Term Fractional Differential Equations

We consider the nonlinear three-term fractional initial value problem of the form

$$c_1 D_{0^+}^{\alpha_1}u(t) + c_2 D_{0^+}^{\alpha_2}u(t) + c_3 D_{0^+}^{\alpha_3}u(t) = f(t, y(t)),$$
$$t > 0, \tag{51}$$

with

$$u^{(i)}(0) = b_i, \quad i = 0, 1, \ldots, N_1 - 1, \tag{52}$$

where

$$0 < \alpha_3 \le \alpha_2 \le \alpha_1 < N_1,$$
$$N_1 - 1 < \alpha_1 < N_1,$$
$$N_2 - 1 < \alpha_2 \le N_2 \le N_1, \tag{53}$$
$$0 < N_3 - 1 < \alpha_3 \le N_3 \le N_2,$$

and c_1, c_2, and c_3 are nonzero constants. We assume that $f(t, y(t))$ is continuous and smooth with respect to $u(t)$. We also assume that α_1, α_2, and α_3 are rational numbers with $\alpha_1 = p_1/q_1, \alpha_2 = p_2/q_2$, and $\alpha_3 = p_3/q_3$, where $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{N}$. Let $q = \text{lcm}(q_1, q_2, q_3)$; we have $q = sq_1 = rq_2 = vq_3$ for s, r and $v \in \mathbb{N}$.

Applying analogous steps for the case of the two-term fractional equations, we seek a solution of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^{n/q} = \sum_{n=0}^{(N_1-1)q} a_n t^{n/q} + \sum_{n=(N_1-1)q+1}^{\infty} a_n t^{n/q}$$
$$= \sum_{n=0}^{N_1-1} a_{nq} t^n + \sum_{n=(N_1-1)q+1}^{\infty} a_n t^{n/q}. \tag{54}$$

From the initial condition (52) we have $u^{(i)}(0) = b_i = i!a_{iq}$, and thus, $a_{iq} = b_i/i!, i = 0, 1, \ldots, N_1 - 1$.

Using the fact that $D_{0^+}^{\alpha}(t^n) = 0$, for $\alpha > n \in \mathbb{N}$, we have

$$D_{0^+}^{\alpha_1}u(t) = \sum_{n=(N_1-1)q+1}^{\infty} a_n s_n t^{(n-sp_1)/q},$$

$$D_{0^+}^{\alpha_2}u(t) = \sum_{n=N_2}^{N_1-1} a_{nq}\rho_{nq}t^{n-\alpha_2} + \sum_{n=(N_1-1)q+1}^{\infty} a_n r_n t^{(n-rp_2)/q}, \tag{55}$$

$$D_{0^+}^{\alpha_3}u(t) = \sum_{n=N_3}^{N_1-1} a_{nq}v_{nq}t^{n-\alpha_3} + \sum_{n=(N_1-1)q+1}^{\infty} a_n v_n t^{(n-vp_3)/q},$$

where $s_n = \Gamma(n/q+1)/\Gamma(n/q-\alpha_1+1), r_n = \Gamma(n/q+1)/\Gamma(n/q-\alpha_2+1), \rho_n = \Gamma(n+1)/\Gamma(n+1-\alpha_2), v_n = \Gamma(n/q+1)/\Gamma(n/q-\alpha_3+1)$, and $v_n = \Gamma(n+1)/\Gamma(n+1-\alpha_3)$.

Substituting (55) in (51), we have

$$c_1 \sum_{n=(N_1-1)q+1}^{\infty} a_n s_n t^{(n-sp_1)/q} + c_2 \sum_{n=N_2}^{N_1-1} a_{nq}\rho_{nq} t^{n-\alpha_2}$$

$$+ c_2 \sum_{n=(N_1-1)q+1}^{\infty} a_n r_n t^{(n-rp_2)/q} + c_3 \sum_{n=N_3}^{N_1-1} a_{nq}\nu_{nq} t^{n-\alpha_3} \quad (56)$$

$$+ c_3 \sum_{n=(N_1-1)q+1}^{\infty} a_n \nu_n t^{(n-\nu p_3)/q} = f\left(t, \sum_{n=0}^{\infty} a_n t^{n/q}\right).$$

Since the coefficients $a_{nq}, n = 0, \ldots, N_1 - 1$, are known, we write the last equation in the form

$$c_1 \sum_{n=(N_1-1)q+1}^{\infty} a_n s_n t^{(n-sp_1)/q} + c_2 \sum_{n=(N_1-1)q+1}^{\infty} a_n r_n t^{(n-rp_2)/q}$$

$$+ c_3 \sum_{n=(N_1-1)q+1}^{\infty} a_n \nu_n t^{(n-\nu p_3)/q}$$

$$= f\left(t, \sum_{n=0}^{\infty} a_n t^{n/q}\right) - c_2 \sum_{n=N_2}^{N_1-1} a_{nq}\rho_{nq} t^{n-\alpha_2} \quad (57)$$

$$- c_3 \sum_{n=N_3}^{N_1-1} a_{nq}\nu_{nq} t^{n-\alpha_3}.$$

Let $w = t^{1/q}$; then (57) reduces to

$$\sum_{n=(N_1-1)q+1}^{\infty} c_1 a_n s_n w^{n-sp_1} + \sum_{n=(N_1-1)q+1}^{\infty} c_2 a_n r_n w^{n-rp_2}$$

$$+ \sum_{n=(N_1-1)q+1}^{\infty} c_3 a_n \nu_n w^{n-\nu p_3}$$

$$= f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right) - \sum_{n=N_2}^{N_1-1} c_2 a_{nq}\rho_{nq} w^{q(n-\alpha_2)} \quad (58)$$

$$- \sum_{n=N_3}^{N_1-1} c_3 a_{nq}\nu_{nq} w^{q(n-\alpha_3)}.$$

Substituting $z = (N_1 - 1)q + 1$ and

$$g(w) = f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right) - \sum_{n=N_2}^{N_1-1} c_2 a_{nq}\rho_{nq} w^{q(n-\alpha_2)}$$

$$- \sum_{n=N_3}^{N_1-1} c_3 a_{nq}\nu_{nq} w^{q(n-\alpha_3)} \quad (59)$$

and shifting the indices to zero, we have

$$\sum_{n=0}^{\infty} c_1 a_{n+z} s_{n+z} w^{n-sp_1+z} + \sum_{n=0}^{\infty} c_2 a_{n+z} r_{n+z} w^{n-rp_2+z}$$

$$+ \sum_{n=0}^{\infty} c_3 a_{n+z} \nu_{n+z} w^{n-\nu p_3+z} = g(w). \quad (60)$$

To avoid singularity we multiply both sides in (60) by w^{sp_1-z}; we have

$$\sum_{n=0}^{\infty} c_1 a_{n+z} s_{n+z} w^{n} + \sum_{n=0}^{\infty} c_2 a_{n+z} r_{n+z} w^{n-rp_2+sp_1}$$

$$+ \sum_{n=0}^{\infty} c_3 a_{n+z} \nu_{n+z} w^{n-\nu p_3+sp_1} = w^{sp_1-z} g(w). \quad (61)$$

Equation (61) can be written as

$$\sum_{n=0}^{k_1-1} c_1 a_{n+z} s_{n+z} w^{n} + \sum_{n=k_1}^{k_2-1} \left(c_1 a_{n+z} s_{n+z} + c_2 a_{n+z-k_1} r_{n+z-k_1}\right)$$

$$\cdot w^n + \sum_{n=k_2}^{\infty} \left(c_1 a_{n+z} s_{n+z} + c_2 a_{n+z-k_1} r_{n+z-k_1}\right. \quad (62)$$

$$\left. + c_3 a_{n+z-k_2} \nu_{n+z-k_2}\right) w^n = w^{sp_1-z} g(w),$$

where $k_1 = sp_1 - rp_2$ and $k_2 = sp_1 - \nu p_3$.

Applying analogs steps for deriving (22), we have

$$c_1 a_{n+z} s_{n+z} + c_2 a_{n+z-k_1} r_{n+z-k_1} + c_3 a_{n+z-k_2} \nu_{n+z-k_2}$$

$$= \frac{1}{(n-j)!} \left[\frac{d^{n-j}}{dw^{n-j}} g(w)\right]_{w=0} \quad \text{for } n \geq k_2, \quad (63)$$

where $j = sp_1 - z$. From the last recursion we determine $a_{n+z}, n \geq k_2$.

Remark 5. Our derivation is based on the facts that the nonlinear function is smooth and the fractional differential equations is of constant coefficients. In case if one of these conditions does not hold, a modified treatment will be considered as we will see in Example 7.

3.1. Numerical Results

Example 6. Consider the Bagely-Torvik initial value problem

$$D_{0^+}^{5/2} u(t) + D_{0^+}^{2} u(t) - 2\sqrt{\pi} D_{0^+}^{1/2} u(t) + 4u(t) = g(t),$$
$$t \in [0, 1], \quad (64)$$

$$u(0) = u'(0) = u''(0) = 0, \quad (65)$$

where

$$g(t) = 4t^9 - \frac{131072}{12155} t^{17/2} + 72t^7 + \frac{49152}{143\sqrt{\pi}} t^{13/2}. \quad (66)$$

This example has been discussed in [21] using a Chebyshev spectral method, where the solution has been approximated by the shifted Chebyshev polynomials with different degrees. Then the exact solution $u(t) = t^9$ was obtained by considering the shifted Chebyshev polynomial of degree 9. We mention here that there are simple typos in presenting

the example in (64) and in (66) and we correct them here. Applying the current algorithm we have

$$p_1 = 5,$$
$$p_2 = 4,$$
$$p_3 = 1,$$
$$q_1 = q_2 = q_3 = 2,$$
$$q = \mathrm{lcm}(q_1, q_2, q_3) = 2,$$
$$s = 1,$$
$$r = 1,$$
$$v = 1,$$
$$k_1 = 1,$$
$$k_2 = 4.$$

$$(67)$$

We expand the solution in infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{n/2}$. The initial condition in (65) yields $a_0 = a_2 = a_4 = 0$. Let $t = w^2$; applying (59) we have

$$g(w) = 4w^{18} - \frac{131072}{12155} w^{17} + 72w^{14} + \frac{49152}{143\sqrt{\pi}} w^{13} - 4 \left(\sum_{n=0}^{\infty} a_n w^n \right),$$

$$(68)$$

and it holds that

$$\frac{\mathrm{d}^m}{\mathrm{d}w^m} g(w) \Big|_{w=0}$$

$$= \begin{cases} -4a_1 & \text{if } m = 1 \\ -4 \times 2! a_2 & \text{if } m = 2 \\ \vdots & \vdots \\ 13! \left(\dfrac{49152}{143\sqrt{\pi}} - 4a_{13} \right) & \text{if } m = 13 \\ 14! (72 - 4a_{14}) & \text{if } m = 14 \\ -4 \times 15! a_{15} & \text{if } m = 15 \\ -4 \times 16! a_{16} & \text{if } m = 16 \\ 17! \left(-\dfrac{131072}{12155} - 4a_{17} \right) & \text{if } m = 17 \\ 18! (4 - 4a_{18}) & \text{if } m = 18 \\ -4! a_{19} & \text{if } m = 19 \\ \vdots & \vdots \end{cases}$$

$$(69)$$

Using (63) and (69), we have

$$a_{n+5} s_{n+5} + a_{n+4} r_{n+4} - 2\sqrt{\pi} a_{n+1} v_{n+1}$$
$$= \frac{1}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}w^n} g(w) \right) \Big|_{w=0}, \quad n \ge 4,$$

$$(70)$$

where

$$s_{n+5} = \frac{\Gamma((n+5)/2+1)}{\Gamma((n+5)/2-3/2)},$$

$$r_{n+4} = \frac{(n+4)(n+2)}{4},$$

$$(71)$$

$$v_{n+1} = \frac{\Gamma((n+1)/2+1)}{\Gamma((n+1)/2+1/2)}.$$

Since $g(w)$ is smooth, then $a_{n+5} = 0$, for $n < 4$. We now apply the last recursion together with $a_0 = \cdots = a_8 = 0$, to compute a_{n+5}, for $n \ge 4$. For $n = 4$, we have

$$a_9 s_9 + a_8 r_8 - 2\sqrt{\pi} a_5 v_5 = g^{(4)}(0) = 0,$$

$$(72)$$

and thus $a_9 = 0$.

Applying analogous arguments yields $a_{10} = \cdots = a_{17} = 0$. For $n = 13$, we have

$$a_{18} s_{18} + a_{17} r_{17} - 2\sqrt{\pi} a_{14} v_{14} = \frac{49152}{143\sqrt{\pi}} - 4a_{13},$$

$$(73)$$

which, together with $a_{17} = a_{14} = a_{13} = 0$, implies that $a_{18} = 1$. For $n = 14$, we have

$$a_{19} s_{19} + a_{18} r_{18} - 2\sqrt{\pi} a_{15} v_{15} = 72 - 4a_{14},$$

$$(74)$$

but $a_{18} = 1$ and $a_{15} = a_{14} = 0$; thus $a_{19} = 0$. Applying the same steps yields $a_{20} = a_{21} = 0$. For $n = 17$, we have

$$a_{22} s_{22} + a_{21} r_{21} - 2\sqrt{\pi} a_{18} v_{18} = -\frac{131072}{12155} - 4a_{17},$$

$$(75)$$

and thus $a_{22} = 0$.

Proceeding in the same manner, we have $a_{n+1} = 0$ for $n \ge 18$. Thus,

$$u(t) = a_{18} t^{18/2} = t^9,$$

$$(76)$$

and the exact solution of problem (64)-(65) is obtained.

In the following example we show that the current algorithm can be applied to more general multiterm fractional differential equations which are not necessary of the form in (51).

Example 7. Consider the nonlinear three-term fractional initial value problem

$$t D_{0^+}^{4/3} u(t) + D_{0^+}^{1/3} u(t) + t^{19/6} D_{0^+}^{1/6} u(t)$$
$$= \frac{33}{4\Gamma(8/3)} t^{8/3} + \frac{1296}{935\Gamma(5/6)} u^2(t),$$

$$(77)$$

with

$$u(0) = u'(0) = 0.$$

$$(78)$$

The exact solution for this problem is $u(t) = t^3$.

Applying the current algorithm we have $q = 6$. We then expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{n/6}$. From the initial condition (78), we have $a_0 = a_6 = 0$. To guarantee the existence of the fractional derivative $D_{0^+}^{4/3} u(t)$ we choose $a_1 = \cdots = a_5 = 0$. Let $w = t^{1/6}$; then (77) reduces to

$$a_1 r_1 + \sum_{n=0}^{18} \left(a_{n+2} s_{n+2} + a_{n+2} r_{n+2} \right) w^n$$

$$+ \sum_{n=19}^{\infty} \left(a_{n+2} s_{n+2} + a_{n+2} r_{n+2} + a_{n-18} v_{n-18} \right) w^n \qquad (79)$$

$$= \frac{33 w^{16}}{4 \Gamma(8/3)} + \frac{1296}{935 \Gamma(5/6)} \left(\sum_{n=0}^{\infty} a_n w^n \right)^2,$$

where $s_n = \Gamma(n/6 + 1)/\Gamma(n/6 - 1/3)$, $r_n = \Gamma(n/6 + 1)/\Gamma(n/6 + 2/3)$, and $v_n = \Gamma(n/6 + 1)/\Gamma(n/6 + 5/6)$.

Performing the nth derivatives one can see that $a_n = 0$ for $n < 16$. For $n = 16$, we have

$$a_{18} s_{18} + a_{18} r_{18} = \frac{33}{4 \Gamma(8/3)}, \qquad (80)$$

which yields $a_{18} = 1$.

Also, $a_{19} = \cdots = a_{35} = 0$.

For $n = 36$, we have

$$a_{38} s_{38} + a_{38} r_{38} + a_{18} v_{18} = \frac{1296}{935 \Gamma(5/6)}, \qquad (81)$$

which yields $a_{38} = 0$.

Following analogous steps, we have $a_n = 0$ for $n > 36$. Thus

$$u(t) = a_{18} t^{18/6} = t^3, \qquad (82)$$

which is the exact solution.

4. Conclusion

For fractional differential equations of order $n - 1 < \alpha < n$, it is common to obtain a series solution in the form $\sum_{n=0}^{\infty} a_n t^{\alpha(n)}$. The question is how to obtain the coefficients $a_n, n = 0, 1, \ldots$. Naturally, if the problem is of fractional order, the differentiation is also of fractional order. In this paper, we presented a new algorithm for obtaining a series solution for nonlinear multiterm fractional differential equations of Caputo type, where we overcome the use of fractional differentiation. We employed a transformation that allows us to use ordinary differentiation rather than fractional differentiation to recursively compute the coefficient of the series expansion. Then the terms of the series, a_n, are computed sequentially using a closed form formula. We applied the new algorithm to several types of multiterm fractional differential equations, where accurate solutions as well as exact solutions in closed forms have been obtained. For one example it is noted that the current algorithm is more efficient than the ADM as it is more easier to apply and it produces the exact solution while

the ADM does not. We have developed the new algorithm for two- and three-term fractional equations, while the idea can be extended to multiterm fractional equations of arbitrary order; obtaining a general formula in this case is not an easy task. The current algorithm can be modified to deal with the fractional multiterm time-diffusion equations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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