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Research Article

Global External Stochastic Stabilization of Linear Systems with Input Saturation: An Alternative Approach

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This paper presents results concerning the global external stochastic stabilization for linear systems with input saturation and stochastic external disturbances under random Gaussian distributed initial conditions. The objective is to construct a class of control laws that achieve global asymptotic stability in the absence of disturbances, while guaranteeing a bounded variance of the state for all the time in the presence of disturbances. By using an alternative approach, a new class of scheduled control laws are proposed, and the global external stochastic stabilization problem can be solved only through some routine manipulations. Furthermore, the reported approach allows a larger range of the design parameter. Finally, two numerical examples are provided to validate the theoretical results.

1. Introduction

In decades, the stabilization of linear systems with input saturation has been widely studied by many researchers; see, for instance, [1–4] and the references therein, and in general, internal stabilization and external stabilization are two kinds of main research focus on this subject [5]. It is now well known [6] that global internal stabilization is possible if and only if the linear system is asymptotically null controllable with bounded controls (ANCBC); that is, the linear system is stabilizable and all its open-loop poles are located in the closed left-half plane. Generally speaking, a class of nonlinear control laws should be constructed to achieve global internal stabilization; see, for instance, [7, 8]. By designing a class of low gain control laws, the semiglobal internal stabilization framework was introduced in [9, 10] with linear control laws.

On the other hand, external stabilization requires that internal stabilization is guaranteed when considering input saturation, which means external and internal stabilization have to be achieved simultaneously. By introducing the framework of external L_p stability, it was proved in [11] that the general ANCBC systems with input saturation can achieve external L_p stability with a linear control law when the external disturbance is input-additive and the global/semiglobal internal stabilization is guaranteed with this linear law in the absence of the external disturbance. For the non-input-additive disturbance, it is known [12] that in general external L_p stability via a linear control law is almost never possible. Notable results from [13] reveal that, for double-integrator systems with input saturation and the non-input-additive disturbance, any linear control laws achieve external L_p stability for $p \in [1, 2]$, but no linear control laws can achieve external L_p stability for $p \in [2, \infty]$. Considering the input-to-state stability (ISS) framework, it is pointed out [14] that the external and internal stabilization of double-integrator systems can not be achieved via a linear control law. Moreover, it is impossible to get a good stable response if we consider such systems with the non-input-additive disturbance. Based on these observations, the external stochastic stabilization problem should be considered, where the considered ANCBC system is subject to input saturation, stochastic external disturbances, and random Gaussian distributed initial conditions; see, for instance, [5] for neutrally stable systems, [15] for a chain of integrators, and [16] for general ANCBC systems. The objective is to construct a class of nonlinear control laws that achieve global asymptotic stability in the absence of disturbances, while guaranteeing a bounded variance of the state in the presence of disturbances.

A common feature in the aforementioned works of [15, 16] lies in a conjecture, and some additional restrictions have been imposed on certain design parameter. In this paper, we revisit the simultaneous external and internal stabilization of general ANCBC systems with input saturation and stochastic external disturbances. The main challenge of this study is how to solve the global external stochastic stabilization problem only through some routine manipulations. Moreover, it is worth pointing out that the main contributions are stated as follows.

(1) An alternative approach is first introduced, and the global external stochastic stabilization can be solved through some routine manipulations, which means that the conjecture/lemma in [15, 16] is no longer needed.

(2) Compared with existing results [15, 16], there is no restriction imposed on the design parameter, which allows a larger range.

Notations. The notations A^T and $\mathrm{tr}(A)$ denote the transpose and trace of matrix A, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $P > 0$ (≥ 0) means that P is a real symmetric positive (semipositive) definite matrix. $I_n \in \mathbb{R}^{n\times n}$ and 0 represent, respectively, the identity matrix and zero matrix.

2. Problem Formulation

We consider the following stochastic differential equation:

$$dx(t) = Ax(t)\,dt + B\delta(u(t))\,dt + E\,dw(t), \tag{1}$$

where the state x, the control input u, and the disturbance w are vector-valued signals of dimensions n, m, and l, respectively. Here w is a Wiener process (a Brownian motion) with mean 0 and rate Q. The initial condition $x_i(0) \in \mathbb{R}^n$ of (1) is a Gaussian random vector which is independent of w, and $\delta(\cdot)$ is a standard saturation function given as

$$\delta(u) = \begin{cases} u, & \text{if } |u| \leq 1, \\ \mathrm{sign}(u), & \text{if } |u| > 1. \end{cases} \tag{2}$$

We firstly assume that the pair (A, B) is stabilizable and all the eigenvalues of A are in the closed left-half plane. As is well known, all eigenvalues of A that have negative real parts will not affect the stabilizability property of the system. Without loss of generality, we will give the following assumption.

Assumption 1. The pair (A, B) is controllable and all the eigenvalues of A are on the imaginary axis.

In what follows, the global external stochastic stabilization problem can be defined as follows.

Definition 2. Consider system (1); the global external stochastic stabilization problem is to find a control law $u = f(x)$ such that, for all possible values for the rate Q of the stochastic process w, the following properties hold.

(i) In the absence of the disturbance w, the equilibrium point $x = 0$ of system (1) with $u = f(x)$ is globally asymptotically stable.

(ii) In the presence of the disturbance w, the variance of the state of the controlled system (1) with $u = f(x)$ is bounded over $t \geq 0$.

To solve the global external stochastic stabilization problem, a class of possibly nonlinear control laws are proposed in [15] as

$$u = F_{\varepsilon(x)}x, \tag{3}$$

where the parameter $\varepsilon(x)$ is scheduled according to

$$\varepsilon(x) = \max\left\{\varepsilon(x) \in (0, 1] \mid \varepsilon(x)\,x^T P_{\varepsilon(x)} x \leq c\right\} \tag{4}$$

provided c is sufficiently small, and the state control gain is given as

$$F_{\varepsilon(x)} = -B^T P_{\varepsilon(x)}, \tag{5}$$

where $P_{\varepsilon(x)}$ is the positive definite solution to the following parameter dependent algebraic Riccati equation (ARE):

$$A^T P_{\varepsilon(x)} + P_{\varepsilon(x)}A - P_{\varepsilon(x)}BB^T P_{\varepsilon(x)} = -\varepsilon(x)\,P_{\varepsilon(x)}, \tag{6}$$

where $\varepsilon(x) > 0$.

To avoid the trivial results, the case that $\varepsilon(x) = 0$ is not considered in this paper. Meanwhile, it has been shown in [15], using a class of possibly nonlinear control laws (3), that the global external stochastic stabilization problem is solved provided the following conjecture holds.

Conjecture 3. *There exists a scalar α such that*

$$\varepsilon(x)\,\frac{dP_{\varepsilon(x)}}{d\varepsilon} \leq \alpha P_{\varepsilon(x)}, \tag{7}$$

for all $\varepsilon(x) \in (0, 1]$.

Although it is proved in [16] that the above conjecture always holds, it is noted that the parameter $\varepsilon(x)$ is restricted to the interval $(0, 1]$. As can be seen in the following section, the above conjecture/lemma is no longer needed, a new class of scheduled control laws will be proposed, and we can solve the global external stochastic stabilization only through some routine manipulations. Furthermore, there is no restriction imposed on the positive design parameter $\varepsilon(x)$, which allows a larger range.

3. Main Results

To find a solution to the problem of global external stochastic stabilization, the following parameter dependent ARE should be revisited:

$$A^T P(\varepsilon) + P(\varepsilon)A - P(\varepsilon)BB^T P(\varepsilon) = -\varepsilon P(\varepsilon), \tag{8}$$

where $\varepsilon > 0$ is a scalar. This parameter dependent ARE was first introduced in [17] for a linear system, and the properties can be summarized as follows.

Lemma 4 (see [17]). *Let Assumption 1 hold. Then, for an arbitrary $\varepsilon > 0$, the ARE (8) has a unique positive solution $P(\varepsilon) = W^{-1}(\varepsilon)$, where $W(\varepsilon)$ satisfies the parametric Lyapunov equation:*

$$W\left(A+\frac{\varepsilon}{2}I_n\right)^T+\left(A+\frac{\varepsilon}{2}I_n\right)W = BB^T,$$
$$\frac{dP(\varepsilon)}{d\varepsilon} > 0, \quad \forall\varepsilon > 0, \tag{9}$$
$$\lim_{\varepsilon\to 0} P(\varepsilon) = 0.$$

Furthermore, $P(\varepsilon)$ is a polynomial function in ε for a scalar input. In general, $P(\varepsilon)$ is a rational function in ε.

Remark 5. According to [18], one property of (8) can be found as

$$\operatorname{tr}\left(B^T P(\varepsilon) B\right) = n\varepsilon. \tag{10}$$

In what follows, we propose a new class of scheduled control laws to solve the global external stochastic stabilization problem, which can be carried out in three steps.

Step 1. Let

$$\varepsilon(x) = \max\left\{\varepsilon(x)\in(0,\gamma) \mid \operatorname{tr}\left(B^T P_{\varepsilon(x)}B\right)x^T P_{\varepsilon(x)}x \le 1\right\}, \tag{11}$$

where $\gamma > 0$ is a scalar.

Step 2. Solve the following ARE:

$$A^T P_{\varepsilon(x)} + P_{\varepsilon(x)}A - P_{\varepsilon(x)}BB^T P_{\varepsilon(x)} = -\varepsilon(x) P_{\varepsilon(x)}, \tag{12}$$

where the existence of $P_{\varepsilon(x)}$ is guaranteed by Lemma 4.

Step 3. Construct a control law $u = f(x)$ as

$$u = F_{\varepsilon(x)}x, \tag{13}$$

where $F_{\varepsilon(x)} = -B^T P_{\varepsilon(x)}$ is the state control gain.

Remark 6. It is observed from (11) that if needed, the scalar $\gamma > 0$ can be given by any positive values, which means there is no restriction imposed on the positive parameter $\varepsilon(x)$. Compared with the scheduled parameter $\varepsilon(x)$ in (4), the positive parameter $\varepsilon(x)$ in (11) allows a larger range. Furthermore, it is known from [17] that $e^{-\varepsilon(x)/2}$ represents the convergence rate of the closed-loop system. In fact, a larger parameter $\varepsilon(x)$ may indicate a slower convergence speed.

The following lemmas play crucial roles.

Lemma 7. *For any positive scalar γ and all $\varepsilon(x)\in(0,\gamma)$, one has*

$$\frac{dP_{\varepsilon(x)}}{d\varepsilon} = -\frac{1}{\varepsilon(x)}P_{\varepsilon(x)}, \tag{14}$$

where $P_{\varepsilon(x)}\in\mathbb{R}^{n\times n}$ is defined in (12).

Proof. Define

$$V(x) = x^T P_{\varepsilon(x)}x, \tag{15}$$

where x is the state of system (1). Note from (11) that $\operatorname{tr}(B^T P_{\varepsilon(x)}B)x^T P_{\varepsilon(x)}x \le 1$, and we have

$$d\left[\operatorname{tr}\left(B^T P_{\varepsilon(x)}B\right)x^T P_{\varepsilon(x)}x\right] = 0. \tag{16}$$

According to the Itô stochastic differential rule, we obtain

$$\begin{aligned}\operatorname{tr}\left(B^T P_{\varepsilon(x)}B\right)\left[x^T\frac{dP_{\varepsilon(x)}}{d\varepsilon}x\,d\varepsilon + 2x^T P_{\varepsilon(x)}dx\right] &= -x^T P_{\varepsilon(x)}x\frac{d\left[\operatorname{tr}\left(B^T P_{\varepsilon(x)}B\right)\right]}{d\varepsilon}d\varepsilon \\ &\quad - \operatorname{tr}\left(B^T P_{\varepsilon(x)}B\right)\operatorname{tr}\left(E^T P_{\varepsilon(x)}E\right)dt.\end{aligned} \tag{17}$$

Note that

$$\begin{aligned}dV(x) &= x^T\frac{dP_{\varepsilon(x)}}{d\varepsilon}x\,d\varepsilon + 2x^T P_{\varepsilon(x)}dx \\ &\quad + \operatorname{tr}\left(E^T P_{\varepsilon(x)}E\right)dt,\end{aligned} \tag{18}$$

and substituting (17), we find that

$$dV(x) = -\operatorname{tr}^{-1}\left(B^T P_{\varepsilon(x)}B\right)x^T P_{\varepsilon(x)}x\frac{d\left[\operatorname{tr}\left(B^T P_{\varepsilon(x)}B\right)\right]}{d\varepsilon}d\varepsilon. \tag{19}$$

According to (10), we can continue (19) as

$$\frac{dV(x)}{d\varepsilon} = -\frac{1}{\varepsilon(x)}x^T P_{\varepsilon(x)}x. \tag{20}$$

It follows from (15) that

$$x^T\frac{dP_{\varepsilon(x)}}{d\varepsilon}x = -\frac{1}{\varepsilon(x)}x^T P_{\varepsilon(x)}x. \tag{21}$$

This implies

$$\frac{dP_{\varepsilon(x)}}{d\varepsilon} = -\frac{1}{\varepsilon(x)}P_{\varepsilon(x)}, \tag{22}$$

which completes the proof. □

Remark 8. Compared with (7) in Conjecture 3, the above relationship between $dP_{\varepsilon(x)}/d\varepsilon$ and $P_{\varepsilon(x)}$ in Lemma 7 is analytically discovered through some routine manipulations. Meanwhile, if needed, the scalar γ can be given by any positive values, which indicates a larger range.

Lemma 9. *Let Assumption 1 hold. Then one has*

$$\left\|F_{\varepsilon(x)}x\right\| \le 1. \tag{23}$$

Proof. In view of (11), it is clear that

$$\begin{aligned} \left\|F_{\varepsilon(x)}x\right\|^2 &= \left\|-B^T P_{\varepsilon(x)}x\right\|^2 = x^T P_{\varepsilon(x)} B B^T P_{\varepsilon(x)} x \\ &\leq \operatorname{tr}\left(B^T P_{\varepsilon(x)} B\right) x^T P_{\varepsilon(x)} x, \end{aligned} \tag{24}$$

which is less than or equal to 1. □

Lemma 10 (see [19]). *Let $V_n \geq 0$ be random variables and let $\{\mathscr{F}_n\}$ be a filtration to which $\{V_n\}$ is adapted and suppose that there exist constants $a > 0$, J and $M < \infty$, and $p > 2$ such that $\mathbb{E}V_0^s$ is bounded for $s \in [1, p]$ and for all n*

$$\begin{aligned} &\mathbb{E}\left(V_{n+1} - V_n \mid \mathscr{F}_n\right) < -a, \\ &\qquad\qquad \text{on the event that } \{V_n > J\}, \end{aligned} \tag{25}$$

$$\mathbb{E}\left(\left|V_{n+1} - V_n\right|^p \mid \mathscr{F}_n\right) \leq M. \tag{26}$$

Then for any $r \in (0, p-1)$, there is $c = c(a, J, M, p, r)$ such that $\mathbb{E}V_n^r < c$ for all n.

The following is the main result on the global external stochastic stabilization problem.

Theorem 11. *Let Assumption 1 hold. Consider the system described by stochastic differential equation (1); the control law (13) can solve the global external stochastic stabilization problem as defined in Definition 2.*

Proof. By using (1) and (13), we obtain

$$dx = Ax\,dt + B\delta\left(F_{\varepsilon(x)}x\right)dt + E\,dw. \tag{27}$$

Recalling from Lemma 9 that $\|F_{\varepsilon(x)}x\| \leq 1$, we can continue (27) as follows:

$$dx = \left(A - BB^T P_{\varepsilon(x)}\right)x\,dt + E\,dw. \tag{28}$$

Consider a Lyapunov function as

$$V(x) = x^T P_{\varepsilon(x)} x. \tag{29}$$

According to Lemma 4, it is easily verified that $V(x)$ is a global Lyapunov function for system (28) in the absence of the disturbance w, which establishes global asymptotic stability.

It remains to show that the variance of the state of system (28) is bounded in the presence of the disturbance w. According to Lemma 10, we will use $V_t = x^T P_{\varepsilon(x)} x$ for $t = n$ and $\{\mathscr{F}_t\}$ is the natural filtration generated by the stochastic process x.

Firstly, we will prove (26) for arbitrary integers p. According to the Itô stochastic differential rule, we obtain

$$\begin{aligned} dV(x) &= \frac{\partial V(x)}{\partial \varepsilon}dt + 2x^T P_{\varepsilon(x)}dx + \operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)dt \\ &= x^T \frac{dP_{\varepsilon(x)}}{d\varepsilon}x\,dt + 2x^T P_{\varepsilon(x)}dx \\ &\quad + \operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)dt. \end{aligned} \tag{30}$$

Note from Lemma 7 that

$$\frac{dP_{\varepsilon(x)}}{d\varepsilon} = -\frac{1}{\varepsilon(x)}P_{\varepsilon(x)}. \tag{31}$$

It follows that

$$\begin{aligned} dV &= -\frac{1}{\varepsilon(x)}x^T P_{\varepsilon(x)} x + 2x^T P_{\varepsilon(x)} dx \\ &\quad + \operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)dt \\ &= -\left[\frac{1}{\varepsilon(x)} + \varepsilon(x)\right]x^T P_{\varepsilon(x)} x\,dt + \operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)dt \\ &\quad + 2x^T P_{\varepsilon(x)} E\,dw. \end{aligned} \tag{32}$$

Define

$$f_1(x) = -\left[\frac{1}{\varepsilon(x)} + \varepsilon(x)\right]x^T P_{\varepsilon(x)} x + \operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right) \tag{33}$$

which satisfies

$$\begin{aligned} \left\|f_1(x)\right\| &\leq \left\|\left[\frac{1}{\varepsilon(x)} + \varepsilon(x)\right]x^T P_{\varepsilon(x)} x\right\| \\ &\quad + \left\|\operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)\right\| \\ &\leq \left\|\left[\frac{1}{\varepsilon(x)} + \varepsilon(x)\right]\frac{1}{n\varepsilon(x)}\right\| + \left\|\operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)\right\| \\ &\leq M_1, \end{aligned} \tag{34}$$

for some suitable constant M_1, where we have used $\operatorname{tr}(B^T P_{\varepsilon(x)} B)x^T P_{\varepsilon(x)} x \leq 1$. Moreover, define

$$f_2(x) = 2x^T P_{\varepsilon(x)} E \tag{35}$$

which satisfies

$$\begin{aligned} \left\|f_2(x)\right\| &\leq 2\left|x^T P_{\varepsilon(x)} x\right|^{1/2}\left\|P_{\varepsilon(x)}^{1/2}\right\|\|E\| \\ &= 2\left|\operatorname{tr}\left(B^T P_{\varepsilon(x)} B\right)x^T P_{\varepsilon(x)} x\right|^{1/2}\left|\operatorname{tr}\left(B^T P_{\varepsilon(x)} B\right)\right|^{-1/2} \\ &\quad \cdot \left\|P_{\varepsilon(x)}^{1/2}\right\|\|E\| \leq 2\left|n\varepsilon(x)\right|^{-1/2}\left\|P_{\varepsilon(x)}^{1/2}\right\|\|E\| \leq M_2, \end{aligned} \tag{36}$$

for some suitable constant M_2. Thus we have

$$\left|V_t - V_n\right| \leq \left|\int_n^t f_1(x)\,dt + \int_n^t f_2(x)\,d\omega\right|. \tag{37}$$

It follows from [20] that all higher order moments of $\int_n^t f_2(x)d\omega$ are bounded, and given that $f_1(x)$ is bounded, we trivially find that $\mathbb{E}(|V_t - V_n|^p \mid \mathscr{F}_n)$ is bounded for any p and for any $t \in [n, n+1]$ which implies (26).

Next, we will prove (25). Since $\operatorname{tr}(B^T P_{\varepsilon(x)} B)x^T P_{\varepsilon(x)} x \leq 1$, there exists $\kappa > 0$ such that $x^T P_{\varepsilon(x)} x \geq \kappa$. Consider J_1 such that $V > J_1$ implies that

$$\operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right) \leq \frac{\kappa}{2}\left(\frac{1}{\varepsilon(x)} + \varepsilon(x)\right). \tag{38}$$

According to (32), we have

$$\begin{aligned}&\mathbb{E}\left(\left|V_{n+1}-V_n\right|^p \mid \mathscr{F}_n\right)\\&\leq \int_n^{n+1}\left[-\frac{\kappa}{\varepsilon(x)}-\kappa\varepsilon(x)+\operatorname{tr}\left(E^T P_{\varepsilon(x)} E\right)\right] dt.\end{aligned} \tag{39}$$

Assuming $V_n > J$ with $J > J_1$, and using Doob's martingale inequality [21], we have

$$\mathbb{P}\left[\sup_{t\in[n,n+1]}\left|V_t-V_n\right| > J-J_1\right] \leq \frac{\mathbb{E}\left[\left|V_t-V_n\right|^2\right]}{\left(J-J_1\right)^2}. \tag{40}$$

That is, for any ρ, we can choose J such that

$$\mathbb{P}\left[\inf_{t\in[n,n+1]} V_t < J_1 \mid V_n > J\right] < \rho. \tag{41}$$

However, for $V_n > J$, we have

$$\begin{aligned}&\mathbb{E}\left(V_{n+1}-V_n \mid \mathscr{F}_n\right)\\&\leq -\mathbb{P}\left[\inf_{t\in[n,n+1]} V_t > J_1 \mid V_n > J\right] T_1\\&\quad+\mathbb{P}\left[\inf_{t\in[n,n+1]} V_t < J_1 \mid V_n > J\right] T_2,\end{aligned} \tag{42}$$

where $-T_1 = (\kappa/2)(1/\varepsilon(x)+\varepsilon(x))$ is an upper bound for $-\kappa(1/\varepsilon(x)+\varepsilon(x))+\operatorname{tr}(E^T P_{\varepsilon(x)}E)$ and $T_2 = \operatorname{tr}(E^T P_{\varepsilon(x)}E)$. Since we have given $V > J_1$, we obtain

$$\mathbb{E}\left(V_{n+1}-V_n \mid \mathscr{F}_n\right) \leq -T_1(1-\rho)+T_2\rho, \tag{43}$$

which is less than $-a$ for $a = T_1/2$ provided ρ is small enough. This implies that (25) is satisfied. Therefore, we conclude from Lemma 10 that $\mathbb{E}V_t^r$ is bounded for all positive integers r.

In what follows, we will prove that the variance of x is bounded under the condition that $\mathbb{E}V_t^r$ is bounded for all positive integers r. Since $P_{\varepsilon(x)}^{-1}$ is a rational function in $\varepsilon(x)$, there exist $\eta > 0$ and an integer q such that

$$P_{\varepsilon(x)}^{-1} \leq \eta\varepsilon(x)^{-q}, \tag{44}$$

which implies

$$P_{\varepsilon(x)} \geq \eta^{-1}\varepsilon(x)^q. \tag{45}$$

This yields

$$x^T P_{\varepsilon(x)} x \geq \eta^{-1}\varepsilon(x)^q x^T x. \tag{46}$$

It follows from $\operatorname{tr}(B^T P_{\varepsilon(x)} B) x^T P_{\varepsilon(x)} x \leq 1$ that

$$x^T x \leq \eta\varepsilon(x)^{-q} x^T P_{\varepsilon(x)} x \leq \frac{\eta}{n\varepsilon(x)^{(q+1)}}, \tag{47}$$

which implies the expectation of the term on the right is bounded. Therefore, we have $\mathbb{E}x^T x < \infty$, which completes the proof. □

Remark 12. From Theorem 11, we can know that the global external stochastic stabilization problem is solved for the case of Assumption 1. Moreover, it is easy to prove that the results can be extended to the case that the pair (A, B) is stabilizable and all the eigenvalues of A are in the closed left-half plane.

4. Simulation Examples

In this section, we provide two numerical examples to illustrate the effectiveness of the proposed control laws.

Example 1. Consider a one-link manipulator with an elastic joint actuated by a DC motor [22, 23] with stochastic disturbances. Then, the dynamics are given as

$$dx(t) = Ax(t)\,dt + B\delta(u(t))\,dt + E\,d\omega(t), \tag{48}$$

where

$$\begin{aligned}A &= \begin{bmatrix}0 & 1 & 0 & 0\\-48.6 & -1.26 & 48.6 & 0\\0 & 0 & 0 & 10\\1.95 & 0 & -1.95 & 0\end{bmatrix},\\B &= \begin{bmatrix}0\\21.6\\0\\0\end{bmatrix},\\E &= \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.\end{aligned} \tag{49}$$

Since the eigenvalues of A are $\{-0.3625, -0.4487 + 8.2203i, -0.4487 - 8.2203i, 0\}$, it is easily verified that the pair (A, B) is stabilizable and all the eigenvalues of A are in the closed left-half plane. We choose $\gamma = 1.5$, which means $\varepsilon(x) \in (0, 1.5)$. Following the design procedure from (11) to (13) in this paper, the simulation results are given in Figure 1, where Figure 1(a) shows the state trajectories and Figure 1(b) shows the evolution of scheduled parameter $\varepsilon(x(t))$. It is seen that the results yield a bounded variance of the state, and the state is bounded by 0.1. Moreover, it seems that the state responses converge to constant values. However, from the data sheet of the simulation example, the state response of zero eigenvalue converges to zero with very small stochastic values (see the green line), and state responses of other eigenvalues converge to constant values. This may arise from the reason that three eigenvalues are on the left-half plane and only one zero eigenvalue is on the imaginary axis.

Example 2. In this example, to make a comparison, we consider a double integrator with input saturation and stochastic external disturbances (borrowed from [16]):

$$\begin{aligned}\dot{x}_1 &= x_2,\\\dot{x}_2 &= \delta(u) + w,\end{aligned} \tag{50}$$

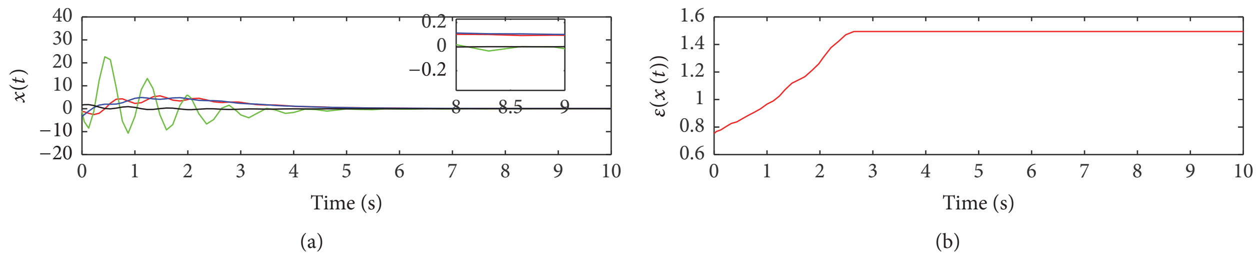

Figure 1: The state trajectories of one-link manipulator and the evolution of the scheduled parameter $\varepsilon(x(t))$.

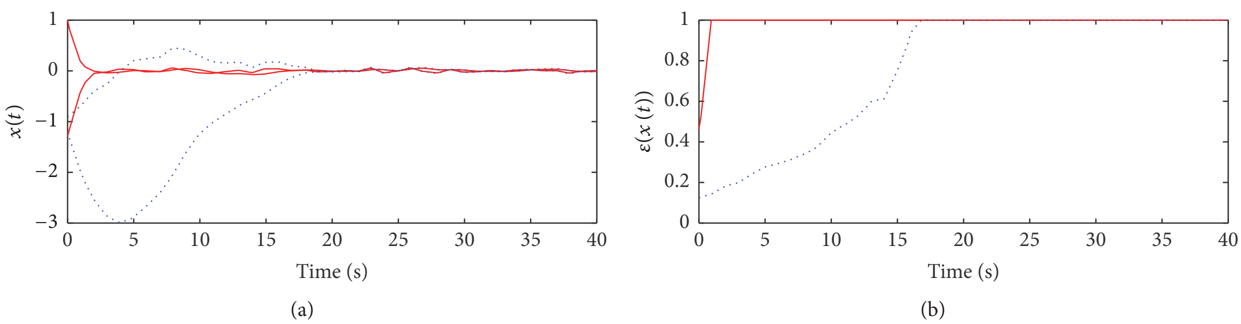

Figure 2: The state trajectories of double integrator and the evolution of the scheduled parameter $\varepsilon(x(t))$.

which indicates that the system matrices are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{51}$$

$$E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is well known that, for the above system, the pair (A, B) is stabilizable and all the eigenvalues of A are in the closed left-half plane. Solving the ARE (8), we obtain

$$P(\varepsilon) = \begin{bmatrix} \varepsilon^3 & \varepsilon^2 \\ \varepsilon^2 & 2\varepsilon \end{bmatrix}. \tag{52}$$

We choose $\gamma = 1$, which means $\varepsilon(x) \in (0, 1)$. According to the design procedure from (11) to (13), the simulation results are given in Figure 2 with red and solid line. It is shown that the variance of the state is bounded, and the convergence time is less than 10 seconds. Furthermore, we conduct additional simulations to compare the performance between the proposed control laws in this paper and the class of control laws of (2) in [16]. Based on the same conditions, the simulation results are shown in Figure 2 with blue and dotted line. It is observed from Figure 2 that the proposed control laws in this paper need less time to achieve global external stochastic stabilization. This may arise from the reason that the positive scalar c in (4) is sufficiently small, which may indicate a slower convergence speed.

5. Conclusions

This paper presents results on the global external stochastic stabilization of linear systems with input saturation. A new scheduled nonlinear controller design procedure is presented for the general ANCBC systems with input saturation and stochastic external disturbances. By making use of some routine manipulations, the global external stochastic stabilization has been achieved without the conjecture required in existing results. Moreover, the reported approach allows a larger range of certain design parameter.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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