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*Research Article*

# Stability for a New Class of GNOVI with $(\gamma_G, \lambda)$-Weak-GRD Mappings in Positive Hilbert Spaces

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By using ordered fixed point theory, we set up a new class of GNOVI structures (general nonlinear ordered variational inclusions) with $(\gamma_G, \lambda)$-weak-GRD mappings, discuss an existence theorem of solution, consider a perturbed Ishikawa iterative algorithm and the convergence of iterative sequences generated by the algorithm, and show the stability of algorithm for GNOVI structures in positive Hilbert spaces. The results in the instrument are obtained.

## 1. Introduction

Stability for variational inequality or general nonlinear ordered variational inclusions problems are of course powerful tools to deal with the problems occurring in control, nonlinear programming, economics, engineering sciences and optimization, and so forth. In recent years, there are some achievements in terms of systems of inequalities [1], weak vector variational inequality [2], differential mixed variational inequalities [3], and so forth. Moreover, Jin [4] studied the stability for strong nonlinear quasi-variational inclusion involving H-accretive operators in 2006. After that the authors investigated some the stability problems of perturbed Ishikawa iterative algorithms for nonlinear variational inclusion problems involving $(A, \eta)$-accretive mappings [5, 6].

On the other hand, in 1972, Amann [7] had the number of solutions of nonlinear equations in ordered Banach spaces. Focusing on the work done related to the fixed points of nonlinear increasing operators in ordered Banach spaces, it is worth mentioning that work done by Du [8] is quite interesting and applicable in pure and applied sciences. From 2008, the authors have some results with regard to the approximation algorithm, the approximation solution for a variety of generalized nonlinear ordered variational inequalities, ordered equations and inclusions, and sensitivity analysis for a class of parametric variational inclusions in ordered Banach spaces (see [7–22]). For related work, we refer the reader to [1–36] and the references therein.

Taking into account the importance of above-mentioned research works, in this paper, a new class of generalized nonlinear ordered variational inclusion structures, GNOVI structures, are introduced in positive Hilbert spaces. By using the resolvent operator for $(\gamma_G, \lambda)$-weak-GRD set-valued mappings and fixed point theory, an existence theorem of solution for the GNOVI frameworks is established, a perturbed Ishikawa iterative algorithm is suggested, and the stability and the convergence of iterative sequences generated by the algorithm are discussed in positive Hilbert spaces. In this field, the results in the instrument are obtained.

## 2. Preliminaries and a New Class of GNOVI Structures

Let us recall the following results and concepts for research stability for a new class of GNOVI with $(\gamma_G, \lambda)$-weak-GRD mappings in positive Hilbert spaces.

Let $\mathfrak{R}$ be real set, let $\mathbf{H}$ be Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, a norm $\|\cdot\|$, and a zero element $\theta$, let nonempty closed convex subsets $\mathbf{C} \subseteq \mathbf{H}$ be a cone, let $N$ be a normal constant of $\mathbf{C}$, and let relation $\leq$ defined by a normal cone $\mathbf{C}$

be a partial ordered relation in $\mathbf{H}$; then $\mathbf{H}$ formats an ordered Hilbert space for the ordered relation $\leq$ and $x$ and $y$ are said to be compared to each other (denoted by $x \propto y$) for $x, y \in \mathbf{H}$ if $x \leq y$ or $y \leq x$ holds in $\mathbf{H}$. If supper$\{u, v\}$ express the least upper bound of a binary set $\{u, v\}$ and $\inf\{u, v\}$ express the greatest lower bound of a binary set $\{u, v\}$ on the partial ordered relation $\leq$ for any $u, v \in \mathbf{H}_P$, supper$\{u, v\}$ and $\inf\{u, v\}$ exist, and some binary operators can be defined as follows:

(i) $x \wedge y = \inf\{u, v\}$.

(ii) $u \vee v = \mathrm{supper}\{u, v\}$.

(iii) $u \oplus v = (u - v) \vee (v - u)$.

$\wedge$, $\vee$, and $\oplus$ are called AND, OR, and XOR operations, respectively; then $(\mathbf{H}, \vee, \wedge, \leq)$ is an ordered lattice [35].

*Definition 1.* An ordered Hilbert space $\mathbf{H}$ with an inner product $\langle\cdot,\cdot\rangle$ is said to be a positive Hilbert space (denoted by $\mathbf{H}_P$) with a partially ordered relation $\leq$, if

$$x \geq \theta, \quad y \geq \theta \quad \text{for any } x, y \in \mathbf{H}; \tag{1}$$

then $\langle x, y\rangle \geq 0$ holds, or $\mathbf{H}$ with an inner product $\langle\cdot,\cdot\rangle$ is said to be a nonpositive Hilbert space (denoted by $\mathbf{H}_N$) with a partially ordered relation $\leq$.

As an example, let $\mathbf{C}_1 = \{(u, v) \mid 0 \leq u, v,\ u, v \in \mathfrak{R}\}$ be closed convex subsets and let $\leq$ defined by a normal cone $\mathbf{C}_1$ be a partial ordered relation in $\mathfrak{R}^2$ (denoted by $\mathbf{H}_{\mathrm{NC}_1}$); it is clear that $\mathfrak{R}^2$ is a positive Hilbert space with the partially ordered relation $\leq$. However, when letting $\mathbf{C}_2 = \{(u, v) \mid 0 \leq u,\ |v| \leq 2u,\ u \in \mathfrak{R}\}$, then $\mathbf{C}_2$ is closed convex subsets. Obviously, $\mathfrak{R}^2$ is a nonpositive Hilbert space with $\leq$ because $\langle(u, u), (u, -1.5u)\rangle = -0.5u^2 < 0$ for $\theta \leq (u, u), (u, -1.5u) \in \mathbf{C}_2$ (denoted by $\mathbf{H}_{\mathrm{NC}_2}$).

The following results and structural relationships are achievements gained by some folks in ordered Banach spaces (see [8, 9, 16–28, 30–35]), and they are as same as right in positive Hilbert space $\mathbf{H}_P$.

**Theorem 2** (see [9, 35]). *Let $\mathfrak{R}$ be real set, let $\mathbf{H}$ be an ordered Hilbert space, and let $(\mathbf{H}, \vee, \wedge, \leq)$ be an ordered lattice; then the following relations hold:*

(1) *If $u$ and $v$ can be compared, then $\theta \leq u \oplus v$.*

(2) $\theta \leq u \oplus \theta$.

(3) *If $\lambda \geq 0$, then $\lambda(u \vee v) = \lambda u \vee \lambda v$.*

(4) *Let $\lambda$ be real; then $(\lambda u) \oplus (\lambda v) = |\lambda|(u \oplus v)$.*

(5) *$(u+w)\wedge(v+w)$ exists and $(u+w)\wedge(v+w) = (u\wedge v)+w$.*

(6) *$(u+w)\vee(v+w)$ exists and $(u+w)\vee(v+w) = (u\vee y)+w$.*

(7) *If $u, v$, and $w$ can be compared to each other, then $(u \oplus v) \leq u \oplus w + w \oplus v$.*

(8) *Let $(u+v)\vee(w+z)$ exist, and if $x \propto u, v$ and $y \propto u, v$, then $(u+v)\oplus(w+z) \leq (u\oplus w+v\oplus z)\wedge(u\oplus z+v\oplus w)$.*

(9) *If $u, v, w$, and $z$ can be compared to each other, then $(u\wedge v)\oplus(w\wedge z) \leq ((u\oplus w)\vee(v\oplus z))\wedge((u\oplus z)\vee(v\oplus w))$.*

(10) *$(\alpha u) \oplus (\beta u) = |\alpha - \beta| u$ for $\alpha, \beta \in \mathfrak{R}$.*

*For arbitrariness, $u, v, w$, and $z \in \mathbf{H}$.*

**Theorem 3** (see [8, 9, 16]). *Let $\mathbf{H}$ be an ordered Hilbert space and let $\leq$ be a partial ordered relation in $\mathbf{H}$; then the following conclusions hold:*

(i) *If $x \propto y$, then (1) supper$\{x, y\}$ and $\inf\{x, y\}$ exist, (2) $x - y \propto y - x$, and (3) $\theta \leq (x - y) \vee (y - x)$.*

(ii) *If, for any natural number $n$, $x \propto y_n$ and $y_n \to y^*$ $(n \to \infty)$, then $x \propto y^*$.*

**Theorem 4** (see [22]). *If $\mathbf{H}_P$ is a positive Hilbert space and $\leq$ is a partial ordered relation in $\mathbf{H}_P$, then the inequalities,*

(1) *if $\theta \leq z$, $u \leq v$, then $\langle u, z\rangle \leq \langle v, z\rangle$,*

(2) *if $\theta \leq z$, then $\langle u \wedge v, z\rangle \leq \langle u, z\rangle \wedge \langle v, z\rangle$, $\langle u, z\rangle \vee \langle v, z\rangle \leq \langle u \vee v, z\rangle$,*

(3) *if $\theta \leq z$, then $\langle u, z\rangle+\langle v, z\rangle-\langle u, z\rangle\wedge\langle v, z\rangle \leq \langle u\vee v, z\rangle$,*

(4) *if $\theta \leq z$, then $\langle u, z\rangle \vee \langle v, z\rangle + \langle u \wedge v, z\rangle \leq \langle u + v, z\rangle$,*

(5) *if $\theta \leq z$, then $\langle u, z\rangle \oplus \langle v, z\rangle \leq \langle u \oplus v, z\rangle$,*

*hold for $u, v, z, w \in \mathbf{H}_P$.*

It is worth noting that (1)–(5) metric inequalities in Theorem 4 are failure in nonpositive Hilbert space $\mathbf{H}_N$, for example, $\mathbf{H}_{\mathrm{NC}_2}$.

*Definition 5.* Let $\mathbf{H}_P$ be a real positive Hilbert space, and let $Q : \mathbf{H}_P\times\mathbf{H}_P \to \mathbf{H}_P$ be a mapping. The mapping $Q : \mathbf{H}\times\mathbf{H} \to \mathbf{H}$ is said to be ordered Lipschitz continuous mapping with constants $(\mu, \nu)$; if $u \propto v$ and $w \propto z$, then $Q(w, u) \propto Q(z, v)$ and there exist constants $\mu, \nu > 0$ such that

$$Q(w, u) \oplus Q(z, v) \leq \mu (w \oplus z) + \nu (u \oplus v). \tag{2}$$

*Definition 6.* Let $\mathbf{H}_P$ be a real positive Hilbert space, let $M : \mathbf{H}_P \to \mathrm{CB}(\mathbf{H}_P)$ be a set-valued mapping, and let $G : \mathbf{H}_P \to \mathbf{H}_P$ be a strong comparison and $\beta$-ordered compressed mapping.

(1) $M$ is said to be a weak comparison mapping with respect to $G$; if, for any $x, y \in X$, $x \propto y$, then there exist $v_x \in M(G(x))$ and $v_y \in M(G(y))$ such that $x \propto v_x$, $y \propto v_y$, and $v_x \propto v_y$, where $v_x$ and $v_y$ are said to be weak-comparison elements, respectively.

(2) $M$ with respect to $G$ is said to be a $\lambda$-weak ordered different comparison mapping with respect to $G$; if there exists a constant $\lambda > 0$ such that, for any $x, y \in \mathbf{H}_P$, there exist $v_x \in M(G(x))$, $v_y \in M(G(y))$, $\lambda(v_x - v_y) \propto x - y$ holds, where $v_x$ and $v_y$ are said to be $\lambda$-elements, respectively.

(3) $M$ is said to be an ordered rectangular mapping, if, for each $x, y \in \mathbf{H}_P$, and any $v_x \in M(x)$ and any $v_y \in M(y)$ such that $\langle v_x \odot v_y, -(x \oplus y)\rangle = 0$ holds.

(4) $M$ is said to be a $\gamma_G$-ordered rectangular mapping with respect to $G$; if there exists a constant $\gamma_G \geq 0$, for any $x, y \in \mathbf{H}_P$, there exist $v_x \in M(G(x))$ and $v_y \in M(G(y))$ such that

$$\left\langle v_x \odot v_y, -(G(x) \oplus (y))\right\rangle \geq \gamma_G \left\|G(x) \oplus G(y)\right\|^2 \quad (3)$$

holds, where $v_x$ and $v_y$ are said to be $\gamma_G$-elements, respectively.

(5) A weak comparison mapping $M$ with respect to $B$ is said to be a $(\gamma_G, \lambda)$-weak-GRD mapping with respect to $B$, if $M$ is a $\gamma_G$-ordered rectangular and $\lambda$-weak ordered different comparison mapping with respect to $B$ and $(G + \lambda M)(\mathbf{H}_P) = \mathbf{H}_P$ for $\lambda > 0$, and there exist $v_x \in M(G(x))$ and $v_y \in M(G(y))$ such that $v_x$ and $v_y$ are $(\gamma_G, \lambda)$-elements, respectively.

*Remark 7* (see [9]). Let $\mathbf{H}_P$ be a real positive Hilbert space, let $G : \mathbf{H}_P \to \mathbf{H}_P$ be a single-valued mapping, and let $M : \mathbf{H}_P \to \mathrm{CB}(\mathbf{H}_P)$ be a set-valued mapping; then one has the following:

(i) If $G = I$ (identical mapping), then a $\gamma_I$-ordered rectangular mapping must be ordered rectangularly in [15].

(ii) An ordered RME mapping must be $\lambda$-weak-GRD in [15].

(iii) A $\lambda$-ordered monotone mapping must be $\lambda$-weak ordered different comparison [22].

**Theorem 8** (see [22]). *Let $\mathbf{H}_P$ be a real positive Hilbert space with normal constant $N$, and let $G$ be a strong comparison and $\beta$-ordered compressed mapping. Let $M : \mathbf{H}_P \to CB(\mathbf{H}_P)$ be an $\alpha_I$-weak ordered rectangular set-valued mapping and $I$ is an identical mapping. Let mapping $R_G^{M,\lambda} = (G + \lambda M)^{-1} : \mathbf{H}_P \to 2^{\mathbf{H}_P}$ be an inverse mapping of $(G + \lambda M)$.*

*If $\alpha_I\lambda > \beta > 0$, $\lambda(\alpha_I \wedge \gamma_G) > \beta > 0$, and $M : \mathbf{H}_P \to CB(\mathbf{H}_P)$ is a $(\gamma_G, \lambda)$-weak-GRD set-valued mapping with respect to $R_G^{M,\lambda}$, then the resolvent operator $R_G^{M,\lambda}$ of $M$ is a single-valued comparison, and*

$$\left\|R_G^{M,\lambda}(u) \oplus R_G^{M,\lambda}(v)\right\| \leq \frac{1}{\gamma_G\lambda - \beta}\|u \oplus v\|, \quad (4)$$

*for $z_u \in M(R_G^{M,\lambda}(u))$ and $z_v \in M(R_G^{M,\lambda}(v))$, which are $\alpha_I$, $\gamma_G$, and $\lambda$-elements, respectively.*

Let $\mathfrak{R}$ be real set, and let $\mathbf{H}_P$ be a real positive Hilbert space with normal constant $N$, a norm $\|\cdot\|$, an inner product $\langle\cdot,\cdot\rangle$, and zero $\theta$. Let $M : \mathbf{H}_P \to \mathrm{CB}(\mathbf{H}_P)$ and $\rho M(\mathbf{H}_P) = \{\rho v \mid v \in M(\mathbf{H}_P)\}$ be two set-valued mappings, and let $g : \mathbf{H}_P \to \mathbf{H}_P$ and $F : \mathbf{H}_P \times \mathbf{H}_P \to \mathbf{H}_P$ be two single-valued nonlinear ordered compression mappings. We consider the following structures.

For $\rho > 0$ and any $\xi \in \mathfrak{R}$, find $x \in \mathbf{H}_P$ such that

$$\theta \in \rho M(x) - \xi F(x, g(x)), \quad (5)$$

which is called a new class of general nonlinear ordered variational inclusion structures (GNOVI structures) in positive Hilbert spaces.

*Remark 9.* (i) If $\rho = 1$, $\xi = 0$, and $M(x) = A(g(x))$, then problem (5) becomes the ordered variational inequality $A(g(x)) \geq \theta$, which was studied by Li [9].

(ii) If $\rho = 1$ and $\xi = 0$, then problem (5) becomes the ordered variational inequality $\theta \in M(x)$, which was studied by Li [10].

(iii) If $\rho = \omega$, $\xi = -1$, and $F(x, g(x)) = f(x) - w(w \in X)$, then problem (5) becomes the ordered variational inequality $w \in f(x) + \omega M(x)$, which was studied by Li et al. [20].

## 3. Existence Theorem of the Solution for GNOVI Structures

In this section, by using Definition 1 and Theorems 2–4 and 8, we study a new class of general nonlinear ordered variational inclusion structures in positive Hilbert spaces.

**Lemma 10.** *Let $\mathbf{H}_P$ be a real positive Hilbert space with normal constant $N$, let $G$ be a strong comparison and $\beta$-ordered compressed mapping, and let $M : \mathbf{H}_P \to CB(\mathbf{H}_P)$ be a $(\gamma_G, \lambda)$-weak ordered GRD set-valued mapping with respect to $R_G^{M,\lambda}$. Let $g : \mathbf{H}_P \to \mathbf{H}_P$ and $F : \mathbf{H}_P \times \mathbf{H}_P \to \mathbf{H}_P$ be two single-valued nonlinear mappings. Then inclusion problem (5) has a solution $x^*$ if and only if $x^* = R_G^{M,\lambda}(G(x^*) + \lambda(\xi/\rho)F(x^*, g(x^*)))$ in $\mathbf{H}_P$.*

*Proof.* For $\rho > 0$, take notice of the fact that $\theta \in \rho M(x) - \xi F(x, g(x))$ if and only if $G(x) + \lambda(\xi/\rho)F(x, g(x)) \in G(x) + \lambda M(x)$; this directly follows from the definition of $R_G^{\rho M,\lambda}$ and problem (5). □

**Theorem 11.** *Let $\mathfrak{R}$ be real set, and let $\mathbf{H}_P$ be a real positive Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and a normal constant $N$. Let $G$ be a strong comparison and $\beta$-ordered compressed mapping, let $M : \mathbf{H}_P \to CB(\mathbf{H}_P)$ be an $\alpha_I$-ordered rectangular and $(\gamma_G, \lambda)$-weak-GRD set-valued mapping with respect to $R_G^{M,\lambda}$, and let $v_x \in M(R_G^{M,\lambda}(x))$ and $v_y \in M(R_G^{M,\lambda}(y))$ be $\alpha_I$, $\lambda$, and $\gamma_G$-elements, respectively. Let $F : \mathbf{H}_P \times \mathbf{H}_P \to \mathbf{H}_P$ be an ordered Lipschitz continuous mapping with constants $(\mu, \nu)$, and let $g : \mathbf{H}_P \to \mathbf{H}_P$ be single-valued nonlinear $\varphi$-ordered compression mapping. If $M, G, g, \lambda(\xi/\rho)F(x, g(x))$ and $G(x) + \lambda(\xi/\rho)F(x, g(x))$ are compared to each other and $\beta$ and $\lambda$ satisfy*

$$0 < \beta < \lambda\left(\frac{\gamma_G}{2} \wedge \alpha_I\right) \wedge 1, \quad (6)$$

$$\rho\beta(N + 1) + \lambda(\mu + \nu\varphi|\xi|)N < \rho\gamma_G\lambda, \quad (7)$$

*then there exists a solution $x^*$ of GNOVI structures (5), which is a fixed point of $R_G^{M,\lambda}(G(x) + \lambda(\xi/\rho)F(x, g(x)))$.*

*Proof.* Let $\mathbf{H}_P$ be a positive Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and a normal constant $N$, let $G$ be a strong comparison and $\beta$-ordered compression mapping, and let $M(x) = \{v \mid v \in M(x)\} : \mathbf{H}_P \to \mathrm{CB}(\mathbf{H}_P)$ $(\rho > 0)$ be a $(\gamma_G, \lambda)$-weak-GRD set-valued mapping with respect to $R_G^{M,\lambda}$. □

Since $\alpha_I, \beta, \gamma_G, \lambda > 0$ and by condition (6), we have

$$\lambda(\alpha_I \wedge \gamma_G) \geq \lambda\left(\frac{\gamma_G}{2} \wedge \alpha\right) = \lambda\frac{\gamma_G}{2} \wedge \lambda\alpha_I > \beta > 0,$$
$$1 > \frac{\beta}{\gamma_G\lambda - \beta} > 0. \tag{8}$$

By Theorems 4 and 8(4) in [9] and the conditions, if $x_1 \propto x_2$, then $R_G^{M,\lambda}(G + \lambda(\xi/\rho)F(\cdot, g(\cdot)))(x_1) \propto R_G^{M,\lambda}(G(\cdot) + \lambda(\xi/\rho)F(\cdot, g(\cdot)))(x_2)$ for $x_1, x_2 \in \mathbf{H}_P$, and

$$\begin{aligned}
&\left\| R_G^{\rho M,\lambda}\left(G + \lambda\frac{\xi}{\rho}F(\cdot, g(\cdot))\right)(x_1) \right.\\
&\quad \left. \oplus R_G^{\rho M,\lambda}\left(G + \lambda\frac{\xi}{\rho}F(\cdot, g(\cdot))\right)(x_2)\right\| \\
&\leq \frac{1}{\gamma_G\lambda - \beta}\left\|\left(G + \lambda\frac{\xi}{\rho}F(\cdot, g(\cdot))\right)(x_1)\right.\\
&\quad \left. \oplus \left(G + \lambda\frac{\xi}{\rho}F(\cdot, g(\cdot))\right)(x_2)\right\| \\
&\leq \frac{N}{\gamma_G\lambda - \beta}\left(\|G(x_1) \oplus G(x_2)\|\right.\\
&\quad \left. + \lambda\frac{|\xi|}{\rho}\|F(x_1, g(x_1)) \oplus F(x_2, g(x_2))\|\right) \\
&\leq \frac{N}{\gamma_G\lambda - \beta}\left(\beta\|x_1 \oplus x_2\|\right.\\
&\quad \left. + \lambda\frac{|\xi|}{\rho}(\mu\|x_1 \oplus x_2\| + \nu\|g(x_1) \oplus g(x_2)\|)\right) \\
&\leq \frac{N}{\gamma_G\lambda - \beta}\left(\beta\|x_1 \oplus x_2\|\right.\\
&\quad \left. + \lambda\frac{|\xi|}{\rho}(\mu + \nu\varphi)\|x_1 \oplus x_2\|\right) \leq N \\
&\quad \cdot \frac{\rho\beta + \lambda(\mu + \nu\varphi|\xi|)}{\rho(\gamma_G\lambda - \beta)}\|x_1 \oplus x_2\|.
\end{aligned} \tag{9}$$

It follows that $R_G^{M,\lambda}(G + (\lambda/\rho)w)$ has a fixed point $x^*$, which is a solution $x^*$ for GNOVI (5), from Lemma 10 and $N((\rho\beta + \lambda(\mu + \nu\varphi|\xi|))/\rho(\gamma_G\lambda - \beta)) < 1$ for (7).

## 4. Stability of Algorithm for GNOVI Structures

*Definition 12.* Let $T : \mathbf{H}_P \to \mathbf{H}_P$ be a self-mapping, $u_0 \in \mathbf{H}_P$, and let $u_{n+1} = h(T, u_n)$ define an iteration procedure which yields a sequence of points $\{u_n\}_{u=0}^{\infty}$ in $\mathbf{H}_P$. Suppose that $\{u \in \mathbf{H}_P : Tu = u\} \neq \emptyset$ and $\{u_n\}_{n=0}^{\infty}$ converge to a fixed point $u^*$ of $T$. Let $\{w_n\} \subset \mathbf{H}_P$ and let $\varepsilon_n = \|w_{n+1} - h(T, w_n)\|$. If $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $u_n \to u^*$, then the iteration procedure defined by $u_{n+1} = h(T, u_n)$ is said to be $T$-stable or stable with respect to $T$.

**Lemma 13** (see [36]). *Let $\{\xi_n\}_{n=0}^{\infty}$ be a nonnegative real sequence and let $\{\zeta_n\}_{n=0}^{\infty}$ be a real sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty}\zeta_n = \infty$. If there exists a positive integer $n_1$ such that*

$$\xi_{n+1} \leq (1 - \zeta_n)\xi_n + \zeta_n\eta_n, \quad \forall n \geq n_1, \tag{10}$$

*where $\eta_n \geq 0$ for all $n \geq 0$ and $\eta_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty}\xi_n = 0$.*

Based on Theorem 11, we can develop a new Ishikawa iterative sequence for solving problem (5) as follows.

*Algorithm 14.* Let $\mathfrak{R}$ be real set, and let $\mathbf{H}_P$ be a real positive Hilbert space with normal constant $N$. Let $\{\omega_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ be two sequences such that $\omega_n, \sigma_n \in [0, 1]$ and $\sum_{n=0}^{\infty}\omega_n = \infty$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences in $\mathbf{H}_P$ introduced to take into account possible inexact computation, where $a_n \oplus \theta = a_n$ and $b_n \oplus \theta = b_n$ $(n = 0, 1, 2, \cdots)$. For any given $x_0 \in \mathbf{H}_P$, the perturbed Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned}
x_{n+1} &= (1 - \omega_n)x_n \\
&\quad + \omega_n\left[R_G^{M,\lambda}\left(G(y_n) + \lambda\frac{\xi}{\rho}F(y_n, g(y_n))\right)\right] \\
&\quad + \omega_n a_n, \\
y_n &= (1 - \sigma_n)x_n \\
&\quad + \sigma_n\left[R_G^{M,\lambda}\left(G(x_n) + \lambda\frac{\xi}{\rho}F(x_n, g(x_n))\right)\right] \\
&\quad + \sigma_n b_n.
\end{aligned} \tag{11}$$

Let $\{z_n\}_{n=0}^{\infty}$ be any sequence in $X$ and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$\begin{aligned}
\varepsilon_n &= \left\| z_{n+1} \right.\\
&\quad \left. - \left[R_G^{M,\lambda}\left(G(t_n) + \lambda\frac{\xi}{\rho}F(t_n, g(t_n))\right) + \omega_n a_n\right]\right\|, \\
t_n &= (1 - \sigma_n)z_n \\
&\quad + \sigma_n\left(R_G^{M,\lambda}\left(G(z_n) + \lambda\frac{\xi}{\rho}F(z_n, g(z_n))\right)\right) + \omega_n b_n,
\end{aligned} \tag{12}$$

where $\rho, \lambda > 0$, $\xi \in \mathfrak{R}$ and $n = 0, 1, 2, \ldots$.

*Remark 15.* For a suitable choice of the mappings $A$, $g$, $f$, $B$, $\sigma_n$, $\omega_n$, $\rho$, and $\xi$ and space $\mathbf{H}_P$, then Algorithm 14 can be degenerated to known the algorithms in [9].

**Theorem 16.** *Let $\mathfrak{R}, \mathbf{H}_P, M, g, F$ be the same as in Theorem 11, and let $\{\omega_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ be two sequences such that $\omega_n, \sigma_n \in [0, 1]$ and $\sum_{n=0}^{\infty}\omega_n = \infty$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences in $\mathbf{H}_P$ introduced to take into account possible* *inexact computation, where* $a_n \oplus \theta = a_n$ *and* $b_n \oplus \theta = b_n$ $(n = 0, 1, 2, \cdots)$. *If condition*

$$\rho\beta + \lambda(\mu + \nu\varphi|\xi|) < \rho(\lambda\gamma_G - \beta)\min\left\{\frac{1}{N}, \frac{1}{2}\right\} \tag{13}$$

*holds, then one has the following:*

(i) *If* $\lim_{n\to\infty}\|a_n \vee -a_n\| = \lim_{n\to\infty}\|b_n \vee -b_n\| = 0$, *then sequence* $\{x_n\}$ *generated by (11) converges strongly to* $x^* \in \mathbf{H}_P$, *and* $x^*$ *is a unique solution of problem (5).*

(ii) *Moreover, if* $0 < \varphi \le \omega_n$, *then* $\lim_{n\to\infty} z_n = x^*$ *if and only if* $\lim_{n\to\infty}\varepsilon_n = 0$, *where* $\varepsilon_n$ *is defined by (12); that is, sequence* $\{x_n\}$ *generated by (11) is* $T$*-stable.*

*Proof.* Let $\mathfrak{R}, \mathbf{H}_P, M, g, F$ be the same as in Theorem 11. If (13) holds then (7) is true.

In the first place, we show that (i) is right.

Let $x^*$ be a unique solution of problem (5); then we have

$$\begin{aligned} x^* &= (1-\omega_n)x^* \\ &\quad + \omega_n\left[R_G^{M,\lambda}\left(G(x^*) + \lambda\frac{\xi}{\rho}F(x^*, g(x^*))\right)\right] \\ &= (1-\sigma_n)x^* \\ &\quad + \sigma_n\left[R_G^{M,\lambda}\left(G(x^*) + \lambda\frac{\xi}{\rho}F(x^*, g(x^*))\right)\right]. \end{aligned} \tag{14}$$

From (11), (14), and (9) and Theorems 2, 4, and 11, it follows that

$$\begin{aligned} \theta &\le x_{n+1} \oplus x^* \le (1-\omega_n)(x_n \oplus x^*) + \omega_n(a_n \oplus \theta) \\ &\quad + \omega_n\left(\left(R_G^{M,\lambda}\left(G(y_n) + \lambda\frac{\xi}{\rho}F(y_n, g(y_n))\right)\right)\right. \\ &\quad \left.\oplus\left(R_G^{M,\lambda}\left(G(x^*) + \lambda\frac{\xi}{\rho}F(x^*, g(x^*))\right)\right)\right) \le (1 \\ &\quad - \omega_n)(x_n \oplus x^*) + h\omega_n(y_n \oplus x^*) + \omega_n(a_n \oplus \theta), \end{aligned} \tag{15}$$

where

$$h = \frac{\rho\beta + \lambda(\mu + \nu\varphi|\xi|)}{\rho(\gamma_G\lambda - \beta)}. \tag{16}$$

Similarly, we can prove that

$$\begin{aligned} \theta &\le y_n \oplus x^* \le (1-\sigma_n)(x_n \oplus x^*) + \sigma_n(b_n \oplus \theta) \\ &\quad + \sigma_n\left(\left(R_G^{M,\lambda}\left(G(y_n) + \lambda\frac{\xi}{\rho}F(y_n, g(y_n))\right)\right)\right. \\ &\quad \left.\oplus\left(R_G^{M,\lambda}\left(G(x^*) + \lambda\frac{\xi}{\rho}F(x^*, g(x^*))\right)\right)\right) \le (1 \\ &\quad - \sigma_n)(x_n \oplus x^*) + \sigma_n h(x_n \oplus x^*) + \sigma_n(b_n \oplus \theta). \end{aligned} \tag{17}$$

It follows that

$$\begin{aligned} \theta &\le x_{n+1} \oplus x^* \le (1-\omega_n)(x_n \oplus x^*) \\ &\quad + h\omega_n((1-\sigma_n)(x_n \oplus x^*) + \sigma_n h(x_n \oplus x^*) + \sigma_n b_n) \\ &\quad + \omega_n(a_n \oplus \theta) \le (1-\omega_n)(x_n \oplus x^*) \\ &\quad + h((1-\sigma_n)(x_n \oplus x^*) + \sigma_n h(x_n \oplus x^*) \\ &\quad + \sigma_n(b_n \oplus \theta)) + \omega_n(a_n \oplus \theta) \le (1 - \omega_n(1-2h)) \\ &\quad \cdot (x_n \oplus x^*) + \omega_n(h\sigma_n(b_n \oplus \theta) + (a_n \oplus \theta)), \end{aligned} \tag{18}$$

from (15), (16), and (17) and Theorems 2 and 4.

By assumption (13), we have $0 < 1 - 2h < 1$ and deduce

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1 - \omega_n(1-2h))N\|x_n - x^*\| \\ &\le +\omega_n(1-2h)N\left(\frac{h\|b_n \vee -b_n\| + \|a_n \vee -a_n\|}{1-2h}\right), \end{aligned} \tag{19}$$

for (18) and Theorem 11, and $a_n \oplus \theta = a_n \vee -a_n$ and $b_n \oplus \theta = b_n \vee -b_n$.

Let

$$\begin{aligned} \eta_n &= \|x_n - x^*\|, \\ \zeta_n &= \omega_n(1-2h)N, \\ \chi_n &= \frac{h\|b_n \vee -b_n\| + \|a_n \vee -a_n\|}{1-2h}; \end{aligned} \tag{20}$$

then (20) can be written as

$$\eta_{n+1} \le (1-\zeta_n)\xi_n + \zeta_n\chi_n. \tag{21}$$

It follows from Lemma 13 and $\lim_{n\to\infty}\|a_n \vee -a_n\| = \lim_{n\to\infty}\|b_n \vee -b_n\| = 0$ that $\xi_n \to 0$ $(n \to \infty)$, and so $\{x_n\}$ converges strongly to unique solution $x^*$ of problem (5).

There is one more point; we prove (ii).

Let $Q(x) = R_G^{M,\lambda}(G(x) + \lambda(\xi/\rho)F(x, g(x)))$ for $x \in X_P$. By (12) and Theorems 2 and 4 and (14), we obtain

$$\begin{aligned} \theta &\le z_{n+1} \oplus x^* \\ &\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n Q(t_n) + \omega_n a_n)) \\ &\qquad + ((1-\omega_n)z_n + \omega_n Q(t_n) + \omega_n a_n) \oplus x^* \\ &\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n Q(t_n) + \omega_n a_n)) \\ &\qquad + ((1-\omega_n)z_n + \omega_n Q(t_n) + \omega_n a_n) \\ &\qquad \oplus ((1-\omega_n)x^* + \omega_n Q(x^*)) \\ &\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n Q(t_n) + \omega_n a_n)) \\ &\qquad + (1-\omega_n)(z_n \oplus x^*) + \omega_n Q(t_n) \oplus Q(x^*) \\ &\qquad + \omega_n(a_n \oplus \theta) \\ &\le (z_{n+1} \oplus ((1-\omega_n)z_n + \omega_n Q(t_n) + \omega_n(a_n \oplus \theta))) \\ &\qquad + (1 - \omega_n(1-2h))(z_n \oplus x^*) \\ &\qquad + \omega_n(h\sigma_n(b_n \oplus \theta) + (a_n \oplus \theta)). \end{aligned} \tag{22}$$

As in the proof of inequality (19) and Theorem 11, we have

$$\begin{aligned}
&\|z_{n+1}-x^*\| \\
&\quad\le N\|z_{n+1}-[(1-\omega_n)z_n+\omega_n Q(t_n)+\omega_n a_n]\| \\
&\qquad+N(1-\omega_n(1-2h))\|z_n-x^*\| \\
&\qquad+N\omega_n(h\sigma_n\|b_n\vee -b_n\|+\|a_n\vee -a_n\|) \\
&\quad\le N\varepsilon_n+N(1-\omega_n(1-2h))\|z_n-x^*\| \\
&\qquad+N\omega_n(h\|b_n\vee -b_n\|+\|a_n\vee -a_n\|).
\end{aligned} \tag{23}$$

Since $0<\varphi\le\omega_n$, by (23), we have

$$\begin{aligned}
&\|z_{n+1}-x^*\|\le[1-\omega_n(1-2h)]N\|z_n-x^*\| \\
&\quad+(1-2h) \\
&\quad\cdot\omega_n N\left(\frac{h\|b_n\vee -b_n\|+\|a_n\vee -a_n\|}{1-2h}+\frac{\varepsilon_n}{\varphi(1-2h)}\right).
\end{aligned} \tag{24}$$

Suppose that $\lim_{n\to\infty}\varepsilon_n=0$; we have $\lim_{n\to\infty}z_n=x^*$ for $\sum_{n=0}^{\infty}\omega_n=\infty$, Theorem 3, and $\lim_{n\to\infty}\|a_n\vee -a_n\|=\lim_{n\to\infty}\|b_n\vee -b_n\|=0$.

Conversely, if $\lim_{n\to\infty}z_n=x^*$, then, by (14) and $\lim_{n\to\infty}\|a_n\vee -a_n\|=\lim_{n\to\infty}\|b_n\vee -b_n\|=0$, we get

$$\begin{aligned}
&\theta\le z_{n+1}\oplus[(1-\omega_n)z_n+\omega_n Q(t_n)+\omega_n a_n]\le(z_{n+1} \\
&\quad\oplus x^*)+[(1-\omega_n)z_n+\omega_n Q(t_n)+\omega_n a_n]\oplus x^* \\
&\quad\le(z_{n+1}\oplus x^*)+[(1-\omega_n)(z_n\oplus x^*) \\
&\quad+\omega_n(Q(t_n)\oplus Q(x^*))+\omega_n(a_n\oplus\theta)]\le(z_{n+1} \\
&\quad\oplus x^*)+(1-\omega_n(1-2h))(z_n\oplus x^*)\le+\omega_n(h\sigma_n(b_n\oplus\theta) \\
&\quad+(a_n\oplus\theta)).
\end{aligned} \tag{25}$$

From (12) and Theorem 11, it follows that

$$\begin{aligned}
&\|\varepsilon_n\|=\|z_{n+1}-[(1-\omega_n)z_n+\omega_n Q(t_n)+\omega_n a_n]\| \\
&\quad\le N\|z_{n+1}\oplus[(1-\omega_n)z_n+\omega_n Q(t_n)+\omega_n(a_n\oplus\theta)]\| \\
&\quad\le N\|z_{n+1}\oplus x^*\|+N(1-\omega_n(1-2h))\|z_n-x^*\| \\
&\quad\le+N\omega_n(h\|b_n\vee -b_n\|+\|a_n\vee -a_n\|);
\end{aligned} \tag{26}$$

we can have

$$\lim_{n\to\infty}\|\varepsilon_n\|=0. \tag{27}$$

Sequence $\{x_n\}$ generated by (11) is $T$-stable. This completes the proof. □

*Remark 17.* For a suitable choice of the mappings $M,g,F,\rho,\xi$, we can obtain known results [9, 22] as special cases of Theorem 11.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

The main idea of this paper was proposed by Hong Gang Li. Hong Gang Li, Yongqin Yang, Mao Ming Jin, and Qinghua Zhang prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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## References

[1] S. M. Robinson, "Stability theory for systems of inequalities—part II: differentiable nonlinear systems," *SIAM Journal on Numerical Analysis*, vol. 13, no. 4, pp. 497–513, 1974.

[2] S. J. Li and C. R. Chen, "Stability of weak vector variational inequality," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 4, pp. 1528–1535, 2009.

[3] X. Wang, W. Li, X.-S. Li, and N.-J. Huang, "Stability for differential mixed variational inequalities," *Optimization Letters*, vol. 8, no. 6, pp. 1873–1887, 2014.

[4] M.-M. Jin, "Perturbed algorithm and stability for strongly nonlinear quasi-variational inclusion involving $H$-accretive operators," *Mathematical Inequalities & Applications*, vol. 9, no. 4, pp. 771–779, 2006.

[5] H. G. Li, "Perturbed ishikawa iterative algorithm and stability for nonlinear mixed quasi-variational inclusions involving $(A,\eta)$-accretive mappings," *Advances in Variational and Hemivariational Inequalities*, vol. 11, no. 1, Article ID 41C50, 2008.

[6] H.-g. Li, "Stability of perturbed algorithm for nonlinear random multi-valued mixed variational inclusions involving nonlinear random $(A,\eta)$-monotone mappings," *Advances in Nonlinear Variational Inequalities*, vol. 12, no. 1, pp. 37–50, 2009.

[7] H. Amann, "On the number of solutions of nonlinear equations in ordered Banach space," *Journal of Functional Analysis*, vol. 11, no. 3, pp. 346–384, 1972.

[8] Y. H. Du, "Fixed points of increasing operators in ordered Banach spaces and applications," *Applicable Analysis*, vol. 38, no. 1-2, pp. 1–20, 1990.

[9] H.-G. Li, "Approximation solution for generalized nonlinear ordered variational inequality and ordered equation in ordered Banach space," *Nonlinear Analysis Forum*, vol. 13, no. 2, pp. 205–214, 2008.

[10] H.-G. Li, "Nonlinear inclusion problem involving $(\alpha,\lambda)$-NODM set-valued mappings in ordered Hilbert space," *Applied Mathematics Letters*, vol. 25, no. 10, pp. 1384–1388, 2012.

[11] D. J. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, no. 5, pp. 623–632, 1987.

[12] D. J. Guo, "Fixed points of mixed monotone operators with applications," *Applicable Analysis*, vol. 31, no. 3, pp. 215–224, 1988.

[13] S.-S. Chang and N. J. Huang, "Generalized complementarity problems for fuzzy mappings," *Fuzzy Sets and Systems*, vol. 55, no. 2, pp. 227–234, 1993.

[14] X. P. Ding and C. L. Luo, "Perturbed proximal point algorithms for general quasi-variational-like inclusions," *Journal of Computational and Applied Mathematics*, vol. 113, no. 1-2, pp. 153–165, 2000.

[15] H. Li, "Sensitivity analysis for general nonlinear ordered parametric variational inequalities with restricted-accretive mapping in ordered Banach spaces," *Nonlinear Functional Analysis and Applications*, vol. 17, no. 1, pp. 109–118, 2012.

[16] H.-G. Li, "Approximation solution for a new class of general nonlinear ordered variational inequalities and ordered equations in ordered Banach space," *Nonlinear Analysis Forum*, vol. 14, pp. 89–97, 2009.

[17] H.-G. Li, "Nonlinear inclusion problem for ordered RME set-valued mappings in ordered hilbert space," *Nonlinear Functional Analysis and Applications*, vol. 16, no. 1, pp. 1–8, 2011.

[18] H. G. Li, D. Qiu, and Y. Zou, "Characterizations of weak-*ANODD* set-valued mappings with applications to an approximate solution of GNMOQV inclusions involving ⊕ operator in ordered Banach spaces," *Fixed Point Theory and Applications*, vol. 2013, article 241, 2013.

[19] H.-G. Li, D. Qiu, and M. Jin, "GNM ordered variational inequality system with ordered Lipschitz continuous mappings in an ordered Banach space," *Journal of Inequalities and Applications*, vol. 2013, article 514, 2013.

[20] H. G. Li, L. P. Li, and M. M. Jin, "A class of nonlinear mixed ordered inclusion problems for ordered $(\alpha_A,\lambda)$-ANODM set-valued mappings with strong comparison mapping A," *Fixed Point Theory and Applications*, vol. 2014, article 79, 2014.

[21] H. G. Li, L. P. Li, J. M. Zheng, and M. M. Jin, "Sensitivity analysis for generalized set-valued parametric ordered variational inclusion with $(\alpha, \lambda)$-*NODSM* mappings in ordered Banach spaces," *Fixed Point Theory and Applications*, vol. 2014, article 122, 2014.

[22] H. G. Li, X. B. Pan, Z. Y. Deng, and C. Y. Wang, "Solving *GNOVI* frameworks involving $(\gamma_G, \lambda)$-weak-GRD set-valued mappings in positive Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2014, article 146, 2014.

[23] N. J. Huang and Y. P. Fang, "Generalized m-accretive mappings in Banach spaces," *Journal of Sichuan University*, vol. 38, no. 4, pp. 591–592, 2001.

[24] Y.-P. Fang, N.-J. Huang, and H. B. Thompson, "A new system of variational inclusions with $(H, \eta)$-monotone operators in Hilbert spaces," *Computers & Mathematics with Applications*, vol. 49, no. 2-3, pp. 365–374, 2005.

[25] H.-Y. Lan, Y. J. Cho, and R. U. Verma, "Nonlinear relaxed cocoercive variational inclusions involving $(A, \eta)$-accretive mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1529–1538, 2006.

[26] D. Qiu and L. Shu, "Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings," *Information Sciences*, vol. 178, no. 18, pp. 3595–3604, 2008.

[27] R. U. Verma, "A hybrid proximal point algorithm based on the (A,$\eta$)-maximal monotonicity framework," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 142–147, 2008.

[28] H. G. Li, L. P. Li, and M. M. Jin, "A class of nonlinear mixed ordered inclusion problems for ordered $(\alpha_A, \lambda)$-ANODM set-valued mappings with strong comparison mapping A," *Fixed Piont Theory And Applications*, vol. 2014, p. 79, 2014.

[29] A. Hassouni and A. Moudafi, "A perturbed algorithm for variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 185, no. 3, pp. 706–712, 1994.

[30] R. P. Agarwal, Y. J. Cho, and N. J. Huang, "Sensitivity analysis for strongly nonlinear quasi-variational inclusions," *Applied Mathematics Letters*, vol. 13, no. 6, pp. 19–24, 2000.

[31] R. U. Verma, "Sensitivity analysis for relaxed cocoercive nonlinear quasivariational inclusions," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2006, Article ID 52041, 9 pages, 2006.

[32] H.-G. Li, "Iterative algorithm for a new class of generalized nonlinear fuzzy set-variational inclusions involving $(H,\eta)$-monotone mappings," *Advances in Nonlinear Variational Inequalities*, vol. 10, no. 1, pp. 89–100, 2007.

[33] H.-G. Li, A. Xu, and M. Jin, "A hybrid proximal point three-step algorithm for nonlinear set-valued quasi-variational inclusions system involving $(A, \eta)$-accretive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 635382, 24 pages, 2010.

[34] H. G. Li, A. J. Xu, and M. M. Jin, "An ishikawa-hybrid proximal point algorithm for nonlinear set-valued inclusions problem based on $(A,\eta)$-accretive framework," *Fixed Point Theory and Applications*, vol. 2010, Article ID 501293, 12 pages, 2010.

[35] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, 1974.

[36] X. L. Weng, "Fixed point iteration for local strictly pseudo-contractive mapping," *Proceedings of the American Mathematical Society*, vol. 113, no. 3, pp. 727–731, 1991.