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Research Article

Quantum Difference Langevin System with Nonlocal q-Derivative Conditions

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We introduce a new class of boundary value problems for Langevin quantum difference systems. Some new existence and uniqueness results for coupled systems are obtained by using fixed point theorems. The existence and uniqueness of solutions are established by Banach's contraction mapping principle, while the existence of solutions is derived by using Leray-Schauder's alternative. The obtained results are well illustrated with the aid of examples.

1. Introduction

Quantum calculus (q-calculus) has a rich history and the details of its basic notions, results, and methods can be found in the text [1]. Apart from the traditional treatment of quantum calculus, many interesting questions and problems, especially from theoretical point of view, either remained open or were partially answered. In recent years, the topic has attracted the attention of several researchers and a variety of new results can be found in the papers [2–12]. However, there are many aspects of boundary value problems of quantum difference equations that need attention. For instance, quantum difference Langevin systems with nonlocal q-derivative conditions are yet to be addressed.

In this paper, we investigate the sufficient conditions for existence of solutions for quantum difference Langevin system of the form

$$
\begin{gathered}
D_q\left(D_q+\lambda_1\right)x(t)=f\left(t,x(t),y(t)\right),\quad t\in J,\\
D_p\left(D_p+\lambda_2\right)y(t)=g\left(t,x(t),y(t)\right),\quad t\in J,\\
x(0)=\alpha_1 D_{z_1}x(0),\\
y(T)=D_{u_1}x(\xi),\\
y(0)=\alpha_2 D_{z_2}y(0),\\
x(T)=D_{u_2}y(\eta),
\end{gathered}
\tag{1}
$$

where $J=[0,T]$, $0<p,q,u_1,u_2,z_1,z_2<1$, are quantum numbers, $\lambda_1,\lambda_2,\alpha_1,\alpha_2\in\mathbb{R}$ are constants, $f,g\in C(J\times\mathbb{R}^2,\mathbb{R})$ are continuous functions, and $\xi,\eta\in J$ are fixed points.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [13]. For some new developments on the fractional Langevin equation in physics, see, for example, [14–22].

In this paper, we prove existence and uniqueness of solutions by using Banach's contraction principle and existence of solutions via Leray-Schauder's alternative.

The paper is organized as follows. In Section 2, we recall some preliminary results from quantum calculus needed in the sequel. Also two basic lemmas are proved. The main existence and uniqueness results are contained in Section 3. Finally, in Section 4, examples illustrating the obtained results are presented.

2. Preliminaries

Let us recall some basic concepts of q-calculus [1, 23].

Definition 1. For $0 < q < 1$, we define the q-derivative of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in J \setminus \{0\},$$
$$D_q f(0) = \lim_{t\to 0} D_q f(t). \tag{2}$$

The higher order q-derivatives are given by

$$D_q^0 f(t) = f(t),$$
$$D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{3}$$

For $x \geq 0$, we set $J_x = \{xq^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $f : J_x \to \mathbb{R}$ by

$$I_q f(x) = \int_0^x f(s)\, d_q s = \sum_{n=0}^{\infty} x(1-q) q^n f(xq^n) \tag{4}$$

provided that the series converges.

For $a, b \in J_x$, we set

$$\int_a^b f(s)\, d_q s = I_q f(b) - I_q f(a)$$
$$= (1-q) \sum_{n=0}^{\infty} q^n \left[b f(bq^n) - a f(aq^n) \right]. \tag{5}$$

Note that, for $a, b \in J_x$, we have $a = xq^{n_1}$, $b = xq^{n_2}$, for some $n_1, n_2 \in \mathbb{N}$, thus the definite integral $\int_a^b f(s) d_q s$ is just a finite sum, so no question about convergence is raised.

We note that

$$D_q I_q f(x) = f(x); \tag{6}$$

while, if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0). \tag{7}$$

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = \left(D_q g(t)\right) h(t) + g(qt) D_q h(t),$$
$$\int_0^x f(t) D_q g(t)\, d_q t = \left[f(t) g(t)\right]_0^x - \int_0^x D_q f(t) g(qt)\, d_q t. \tag{8}$$

Further, the reversing order of integration is given by

$$\int_0^t \int_0^s f(r)\, d_q r\, d_q s = \int_0^t \int_{qr}^t f(r)\, d_q s\, d_q r. \tag{9}$$

In the limit $q \to 1$ the above results correspond to their counterparts in standard calculus.

Lemma 2. *Let $f : J \to \mathbb{R}$ be a continuous function and $0 < p$, $q < 1$. Then we have the following:*

(i)

$$D_p \left[\int_0^t f(s)\, d_q s \right] = \frac{1}{(1-p)t} \int_{pt}^t f(s)\, d_q s, \quad t \neq 0, \tag{10}$$

$$\lim_{t\to 0} D_p \left[\int_0^t f(s)\, d_q s \right] = f(0);$$

(ii)

$$D_p \left[\int_0^t \int_0^r f(s)\, d_q s\, d_q r \right] = \int_0^{pt} f(s)\, d_q s + \int_{pt}^t \frac{(t-qs)}{(1-p)t} f(s)\, d_q s, \quad t \neq 0, \tag{11}$$

$$\lim_{t\to 0} D_p \left[\int_0^t \int_0^r f(s)\, d_q s\, d_q r \right] = 0.$$

Proof. To prove (i), using the definition of p-derivative, we have

$$D_p \left[\int_0^t f(s)\, d_q s \right] = \frac{1}{(1-p)t} \left[\int_0^t f(s)\, d_q s - \int_0^{pt} f(s)\, d_q s \right] = \frac{1}{(1-p)t} \int_{pt}^t f(s)\, d_q s, \quad t \neq 0. \tag{12}$$

For $t \to 0$, we obtain

$$\lim_{t\to 0} D_p \left[\int_0^t f(s)\, d_q s \right] = \lim_{t\to 0} D_p \left[t(1-q) \sum_{n=0}^{\infty} q^n f(tq^n) \right] = \lim_{t\to 0} \frac{(1-q)}{(1-p)} \left[\sum_{n=0}^{\infty} q^n f(tq^n) - p \sum_{n=0}^{\infty} q^n f(ptq^n) \right] = f(0). \tag{13}$$

Next, we will show that (ii) holds. From the reversing order of integration, the double q-integral can be reduced to a single integral as

$$\int_0^t \int_0^r f(s)\, d_q s\, d_q r = \int_0^t (t - qs) f(s)\, d_q s. \tag{14}$$

Taking the p-derivative to the both sides of the above equation, it follows that

$$\begin{aligned} D_p\left[\int_0^t\int_0^r f(s)\,d_qs\,d_qr\right] &= D_p\left[\int_0^t (t-qs)\,f(s)\,d_qs\right] \\ &= \frac{1}{(1-p)t}\left[\int_0^t (t-qs)\,f(s)\,d_qs\right. \\ &\quad\left.+\int_0^{pt}(qs-pt)\,f(s)\,d_qs\right] \\ &= \frac{1}{(1-p)t}\left[\int_0^t (t-qs)\,f(s)\,d_qs\right. \\ &\quad\left.-\int_0^{pt}(t-qs)\,f(s)\,d_qs+\int_0^{pt}(t-pt)\,f(s)\,d_qs\right] \\ &= \int_0^{pt} f(s)\,d_qs+\int_{pt}^t\frac{(t-qs)}{(1-p)t}f(s)\,d_qs. \end{aligned} \tag{15}$$

Since

$$\begin{aligned} \int_{pt}^t\frac{(t-qs)}{(1-p)t}f(s)\,d_qs &= \frac{1}{(1-p)}\int_{pt}^t f(s)\,d_qs \\ &\quad-\frac{q}{(1-p)t}\int_{pt}^t sf(s)\,d_qs=\frac{(1-q)}{(1-p)} \\ &\quad\cdot\sum_{n=0}^{\infty}q^n\left[tf(tq^n)-ptf(ptq^n)\right]-\frac{q(1-q)}{(1-p)t} \\ &\quad\cdot\sum_{n=0}^{\infty}q^n\left[t^2q^nf(tq^n)-(pt)^2q^nf(ptq^n)\right], \end{aligned} \tag{16}$$

it is easy to see that

$$\lim_{t\to 0}D_p\left[\int_0^t\int_0^r f(s)\,d_qs\,d_qr\right]=0. \tag{17}$$

This completes the proof. □

Let

$$A=(1+\alpha_1\lambda_1)(1+\alpha_2\lambda_2)\left[1-\Omega_1\Omega_2\right], \tag{18}$$

with

$$\begin{aligned} \Omega_1 &= T+\frac{\alpha_1}{1+\alpha_1\lambda_1}, \\ \Omega_2 &= T+\frac{\alpha_2}{1+\alpha_2\lambda_2}. \end{aligned} \tag{19}$$

Lemma 3. *Let $A\neq 0$ and the functions $\phi,\psi\in C(J,\mathbb{R})$. Then $x,y\in C(J,\mathbb{R})$ are solutions of the problem*

$$\begin{aligned} D_q\left(D_q+\lambda_1\right)x(t) &= \phi(t),\quad t\in J, \\ D_p\left(D_p+\lambda_2\right)y(t) &= \psi(t),\quad t\in J, \\ x(0) &= \alpha_1D_{z_1}x(0), \\ y(T) &= D_{u_1}x(\xi), \\ y(0) &= \alpha_2D_{z_2}y(0), \\ x(T) &= D_{u_2}y(\eta), \end{aligned} \tag{20}$$

if and only if

$$\begin{aligned} x(t) &= -\lambda_1\int_0^t x(s)\,d_qs+\int_0^t(t-qs)\,\phi(s)\,d_qs \\ &\quad+\frac{[t(1+\alpha_1\lambda_1)+\alpha_1](1+\alpha_2\lambda_2)}{A}\left[\frac{\lambda_1}{(1-u_1)\xi}\right. \\ &\quad\cdot\int_{u_1\xi}^{\xi}x(s)\,d_qs-\int_0^{u_1\xi}\phi(s)\,d_qs \\ &\quad-\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}\phi(s)\,d_qs-\lambda_2\int_0^T y(s)\,d_ps \\ &\quad+\int_0^T(T-ps)\,\psi(s)\,d_ps+\Omega_2\left(-\lambda_1\int_0^T x(s)\,d_qs\right. \\ &\quad+\int_0^T(T-qs)\,\phi(s)\,d_qs+\frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta}y(s)\,d_ps \\ &\quad\left.\left.-\int_0^{u_2\eta}\psi(s)\,d_ps-\int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}\psi(s)\,d_ps\right)\right], \\ y(t) &= -\lambda_2\int_0^t y(s)\,d_ps+\int_0^t(t-ps)\,\psi(s)\,d_ps \\ &\quad+\frac{[t(1+\alpha_2\lambda_2)+\alpha_2](1+\alpha_1\lambda_1)}{A}\left[\Omega_1\left(\frac{\lambda_1}{(1-u_1)\xi}\right.\right. \\ &\quad\cdot\int_{u_1\xi}^{\xi}x(s)\,d_qs-\int_0^{u_1\xi}\phi(s)\,d_qs \\ &\quad-\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}\phi(s)\,d_qs-\lambda_2\int_0^T y(s)\,d_ps \\ &\quad\left.+\int_0^T(T-ps)\,\psi(s)\,d_ps\right)-\lambda_1\int_0^T x(s)\,d_qs \\ &\quad+\int_0^T(T-qs)\,\phi(s)\,d_qs+\frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta}y(s)\,d_ps \\ &\quad\left.-\int_0^{u_2\eta}\psi(s)\,d_ps-\int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}\psi(s)\,d_ps\right]. \end{aligned} \tag{21}$$

Proof. Simplifying the first two equations of problem (20) and applying the double quantum integral, we obtain

$$\begin{aligned} x(t) &= -\lambda_1\int_0^t x(s)\,d_qs+\int_0^t(t-qs)\,\phi(s)\,d_qs+C_1t \\ &\quad+C_2, \\ y(t) &= -\lambda_2\int_0^t y(s)\,d_ps+\int_0^t(t-ps)\,\psi(s)\,d_ps+C_3t \\ &\quad+C_4, \end{aligned} \tag{22}$$

where C_i, $i = 1,2,3,4 \in \mathbb{R}$. From the conditions $x(0) = \alpha_1 D_{z_1} x(0)$, $y(0) = \alpha_2 D_{z_2} y(0)$, Lemma 2, and $\alpha_1\lambda_1, \alpha_2\lambda_2 \neq -1$, we have

$$C_2 = \frac{\alpha_1 C_1}{1+\alpha_1\lambda_1}, \qquad C_4 = \frac{\alpha_2 C_3}{1+\alpha_2\lambda_2}. \tag{23}$$

Using the coupled nonlocal boundary conditions and Lemma 2, we get the following system:

$$\begin{aligned}
&-\lambda_2 \int_0^T y(s)\,d_ps + \int_0^T (T-ps)\,\psi(s)\,d_ps + \left(\frac{T(1+\alpha_2\lambda_2)+\alpha_2}{1+\alpha_2\lambda_2}\right) C_3 \\
&= -\frac{\lambda_1}{(1-u_1)\xi}\int_{u_1\xi}^{\xi} x(s)\,d_qs + \int_0^{u_1\xi}\phi(s)\,d_qs + \int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}\phi(s)\,d_qs + C_1, \\
&-\lambda_1 \int_0^T x(s)\,d_qs + \int_0^T (T-qs)\,\phi(s)\,d_qs + \left(\frac{T(1+\alpha_1\lambda_1)+\alpha_1}{1+\alpha_1\lambda_1}\right) C_1 \\
&= -\frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta} y(s)\,d_ps + \int_0^{u_2\eta}\psi(s)\,d_ps + \int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}\psi(s)\,d_ps + C_3.
\end{aligned} \tag{24}$$

Solving system (24) for constants C_1 and C_3, we have

$$\begin{aligned}
C_1 &= \frac{1}{A}\Bigg[\frac{\lambda_1}{(1-u_1)\xi}\int_{u_1\xi}^{\xi} x(s)\,d_qs - \int_0^{u_1\xi}\phi(s)\,d_qs - \int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}\phi(s)\,d_qs - \lambda_2\int_0^T y(s)\,d_ps \\
&\quad + \int_0^T (T-ps)\,\psi(s)\,d_ps + \Omega_2\Bigg(-\lambda_1\int_0^T x(s)\,d_qs + \int_0^T (T-qs)\,\phi(s)\,d_qs + \frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta} y(s)\,d_ps \\
&\quad - \int_0^{u_2\eta}\psi(s)\,d_ps - \int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}\psi(s)\,d_ps\Bigg)\Bigg], \\
C_3 &= \frac{1}{A}\Bigg[\Omega_1\Bigg(\frac{\lambda_1}{(1-u_1)\xi}\int_{u_1\xi}^{\xi} x(s)\,d_qs - \int_0^{u_1\xi}\phi(s)\,d_qs - \int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}\phi(s)\,d_qs \\
&\quad - \lambda_2\int_0^T y(s)\,d_ps + \int_0^T (T-ps)\,\psi(s)\,d_ps\Bigg) - \lambda_1\int_0^T x(s)\,d_qs + \int_0^T (T-qs)\,\phi(s)\,d_qs \\
&\quad + \frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta} y(s)\,d_ps - \int_0^{u_2\eta}\psi(s)\,d_ps - \int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}\psi(s)\,d_ps\Bigg].
\end{aligned} \tag{25}$$

Substituting all values of constants C_i, $i = 1,2,3,4$, in (22), we obtain the solutions of system (20) as in (21). The converse follows by direct computation. This completes the proof. □

3. Main Results

In this section, we are going to prove the existence and uniqueness of solutions for the Langevin quantum difference system (1) with coupled boundary q-derivative conditions by using fixed point theorems. Let $\mathcal{X} = C([0,T],\mathbb{R})$ be the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ with the norm defined by $\|x\| = \sup_{t\in J}|x(t)|$. Obviously $(\mathcal{X}, \|\cdot\|)$ is a Banach space. In addition the product space $(\mathcal{X}\times\mathcal{X}, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 3, we define an operator $G : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ by

$$G(x,y)(t) = \begin{pmatrix} G_1(x,y)(t) \\ G_2(x,y)(t) \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned}
G_1(x,y)(t) &= -\lambda_1\int_0^t x(s)\,d_qs + \int_0^t (t-qs) \cdot f(s,x(s),y(s))\,d_qs \\
&\quad + \frac{[t(1+\alpha_1\lambda_1)+\alpha_1](1+\alpha_2\lambda_2)}{A}\Bigg[\frac{\lambda_1}{(1-u_1)\xi} \cdot \int_{u_1\xi}^{\xi} x(s)\,d_qs - \int_0^{u_1\xi} f(s,x(s),y(s))\,d_qs \\
&\quad - \int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi} f(s,x(s),y(s))\,d_qs - \lambda_2\int_0^T y(s)\,d_ps + \int_0^T (T-ps) \\
&\quad \cdot g(s,x(s),y(s))\,d_ps + \Omega_2\Bigg(-\lambda_1\int_0^T x(s)\,d_qs + \int_0^T (T-qs)\, f(s,x(s),y(s))\,d_qs + \frac{\lambda_2}{(1-u_2)\eta} \\
&\quad \cdot \int_{u_2\eta}^{\eta} y(s)\,d_ps - \int_0^{u_2\eta} g(s,x(s),y(s))\,d_ps - \int_{u_2\eta}^{\eta}\frac{\eta-ps}{(1-u_2)\eta} g(s,x(s),y(s))\,d_ps\Bigg)\Bigg],
\end{aligned}$$

$$
\begin{aligned}
G_2(x,y)(t) &= -\lambda_2 \int_0^t y(s)\,d_p s + \int_0^t (t-ps) \\
&\cdot g(s,x(s),y(s))\,d_p s \\
&+ \frac{[t(1+\alpha_2\lambda_2)+\alpha_2](1+\alpha_1\lambda_1)}{A}\left[\Omega_1\left(\frac{\lambda_1}{(1-u_1)\xi}\right.\right. \\
&\cdot \int_{u_1\xi}^{\xi} x(s)\,d_q s - \int_0^{u_1\xi} f(s,x(s),y(s))\,d_q s \\
&- \int_{u_1\xi}^{\xi} \frac{(\xi-qs)}{(1-u_1)\xi} f(s,x(s),y(s))\,d_q s \\
&- \lambda_2 \int_0^T y(s)\,d_p s \\
&\left. + \int_0^T (T-ps)\,g(s,x(s),y(s))\,d_p s\right) \\
&- \lambda_1 \int_0^T x(s)\,d_q s + \int_0^T (T-qs) \\
&\cdot f(s,x(s),y(s))\,d_q s + \frac{\lambda_2}{(1-u_2)\eta} \\
&\cdot \int_{u_2\eta}^{\eta} y(s)\,d_p s - \int_0^{u_2\eta} g(s,x(s),y(s))\,d_p s \\
&\left. - \int_{u_2\eta}^{\eta} \frac{(\eta-ps)}{(1-u_2)\eta} g(s,x(s),y(s))\,d_p s\right].
\end{aligned} \tag{27}
$$

In addition, we set constants

$$
\begin{aligned}
M_1 &= \frac{T^2|A| + [u_1\xi(1+q) + (1-u_1q)\xi + T^2|\Omega_2|]\delta}{|A|(1+q)}, \\
M_2 &= \frac{[T^2 + |\Omega_2|u_2\eta(1+p) + (1-u_2p)|\Omega_2|\eta]\delta}{|A|(1+p)}, \\
M_3 &= \frac{|\lambda_1|T|A| + |\Omega_2||\lambda_1|T\delta + |\lambda_1|\delta}{|A|}, \\
M_4 &= \frac{|\lambda_2|T\delta + |\Omega_2||\lambda_2|\delta}{|A|}, \\
M_5 &= \frac{[T^2 + |\Omega_1|u_1\xi(1+q) + (1-u_1q)|\Omega_1|\xi]\rho}{|A|(1+q)}, \\
M_6 &= \frac{T^2|A| + [u_2\eta(1+p) + (1-u_2p)\eta + T^2|\Omega_1|]\rho}{|A|(1+p)}, \\
M_7 &= \frac{|\lambda_1|T\rho + |\Omega_1||\lambda_1|\rho}{|A|}, \\
M_8 &= \frac{|\lambda_2|T|A| + |\Omega_1||\lambda_2|T\rho + |\lambda_2|\rho}{|A|},
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
\delta &= T((1+|\alpha_1\lambda_1|) + |\alpha_1|)(1+|\alpha_2\lambda_2|), \\
\rho &= T((1+|\alpha_2\lambda_2|) + |\alpha_2|)(1+|\alpha_1\lambda_1|).
\end{aligned} \tag{29}
$$

Theorem 4. *Assume that $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist positive constants $m_i, n_i = 1,2$ such that, for all $t \in J$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$,*

$$
\begin{aligned}
&|f(t,x_1,y_1) - f(t,x_2,y_2)| \\
&\quad \le m_1|x_1-x_2| + m_2|y_1-y_2|, \\
&|g(t,x_1,y_1) - g(t,x_2,y_2)| \\
&\quad \le n_1|x_1-x_2| + n_2|y_1-y_2|.
\end{aligned} \tag{30}
$$

In addition, assume that

$$
\begin{aligned}
B_1 &< \frac{1}{2}, \\
U_1 &< \frac{1}{2},
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
B_1 &= (m_1+m_2)M_1 + (n_1+n_2)M_2 + M_3 + M_4, \\
U_1 &= (m_1+m_2)M_5 + (n_1+n_2)M_6 + M_7 + M_8.
\end{aligned} \tag{32}
$$

Then problem (1) has a unique solution on J.

Proof. Define $\sup_{t\in J}|f(t,0,0)| = N_1 < \infty$ and $\sup_{t\in J}|g(t,0,0)| = N_2 < \infty$ and choose a real number R such that

$$
R \ge \max\left\{\frac{M_1N_1 + M_2N_2}{1/2 - B_1}, \frac{M_5N_1 + M_6N_2}{1/2 - U_1}\right\}. \tag{33}
$$

Note that from (31) the constant $R > 0$.

Firstly, we will show that $GB_R \subset B_R$, where $B_R = \{(x, y) \in \mathcal{X} \times \mathcal{X} : \|(x,y)\| \le R\}$. For $(x,y) \in B_R$, we have

$$\begin{aligned}
\|G_1(x,y)\| &\le \sup_{t\in J}\Bigg\{\Bigg|-\lambda_1\int_0^t x(s)\,d_qs+\int_0^t(t-qs)\,f(s,x(s),y(s))\,d_qs+\frac{[t(1+\alpha_1\lambda_1)+\alpha_1](1+\alpha_2\lambda_2)}{A}\\
&\cdot\Bigg[\frac{\lambda_1}{(1-u_1)\xi}\int_{u_1\xi}^{\xi}x(s)\,d_qs-\int_0^{u_1\xi}f(s,x(s),y(s))\,d_qs-\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}f(s,x(s),y(s))\,d_qs\\
&-\lambda_2\int_0^T y(s)\,d_ps+\int_0^T(T-ps)\,g(s,x(s),y(s))\,d_ps+\Omega_2\Bigg(-\lambda_1\int_0^T x(s)\,d_qs+\int_0^T(T-qs)\,f(s,x(s),y(s))\,d_qs\\
&+\frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta}y(s)\,d_ps-\int_0^{u_2\eta}g(s,x(s),y(s))\,d_ps-\int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}g(s,x(s),y(s))\,d_ps\Bigg)\Bigg]\Bigg|\Bigg\}\\
&\le(m_1\|x\|+m_2\|y\|+N_1)\left(\frac{T^2|A|+[u_1\xi(1+q)+(1-u_1q)\xi+T^2|\Omega_2|]\delta}{|A|(1+q)}\right)+(n_1\|x\|+n_2\|y\|+N_2)\\
&\cdot\left(\frac{[T^2+|\Omega_2|u_2\eta(1+p)+(1-u_2p)|\Omega_2|\eta]\delta}{|A|(1+p)}\right)+\frac{|\lambda_1|T|A|+|\Omega_2||\lambda_1|T\delta+|\lambda_1|\delta}{|A|}\|x\|+\frac{|\lambda_2|T\delta+|\Omega_2||\lambda_2|\delta}{|A|}\|y\|\\
&\le((m_1+m_2)M_1+(n_1+n_2)M_2+M_3+M_4)R+M_1N_1+M_2N_2=B_1R+M_1N_1+M_2N_2\le\frac{R}{2},
\end{aligned}\tag{34}$$

and in a similar way

$$\begin{aligned}
\|G_2(x,y)\| &\le(m_1\|x\|+m_2\|y\|+N_1)\\
&\cdot\left(\frac{[T^2+|\Omega_1|u_1\xi(1+q)+(1-u_1q)|\Omega_1|\xi]\rho}{|A|(1+q)}\right)\\
&+(n_1\|x\|+n_2\|y\|+N_2)\\
&\cdot\left(\frac{T^2|A|+[u_2\eta(1+p)+(1-u_2p)\eta+T^2|\Omega_1|]\rho}{|A|(1+p)}\right)\\
&+\frac{|\lambda_1|T\rho+|\Omega_1||\lambda_1|\rho}{|A|}\|x\|\\
&+\frac{|\lambda_2|T|A|+|\Omega_1||\lambda_2|T\rho+|\lambda_2|\rho}{|A|}\|y\|\\
&\le((m_1+m_2)M_5+(n_1+n_2)M_6+M_7+M_7)R\\
&+M_5N_1+M_6N_2=U_1R+M_1N_1+M_2N_2\le\frac{R}{2}.
\end{aligned}\tag{35}$$

This shows that $GB_R\subset B_R$.

Next, we will prove that the operator G is contractive. For any $(x_1,y_1),(x_2,y_2)\in\mathcal{X}\times\mathcal{X}$, and $t\in J$, we have

$$\begin{aligned}
\|G_1(x_2,y_2)-G_1(x_1,y_1)\| &\le|\lambda_1|\int_0^T|x_2(s)-x_1(s)|\,d_qs+\int_0^T(T-qs)\,|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|\,d_qs\\
&+\frac{\delta}{|A|}\Bigg[\frac{|\lambda_1|}{(1-u_1)\xi}\int_{u_1\xi}^{\xi}|x_2(s)-x_1(s)|\,d_qs+\int_0^{u_1\xi}|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|\,d_qs\\
&+\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|+|\lambda_2|\int_0^T|y_2(s)-y_1(s)|\,d_ps+\int_0^T(T-ps)\\
&\cdot|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|\,d_ps+|\Omega_2|\Bigg(|\lambda_1|\int_0^T|x_2(s)-x_1(s)|\,d_qs\\
&+\int_0^T(T-qs)\,|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|\,d_qs+\frac{|\lambda_2|}{(1-u_2)\eta}\int_{u_2\eta}^{\eta}|y_2(s)-y_1(s)|\,d_ps\\
&+\int_0^{u_2\eta}|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|\,d_ps\\
&+\int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|\,d_ps\Bigg)\Bigg]\le(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\\
&\cdot\left(\frac{T^2|A|+[u_1\xi(1+q)+(1-u_1q)\xi+T^2|\Omega_2|]\delta}{|A|(1+q)}\right)+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)
\end{aligned}$$

$$\cdot\left(\frac{\left[T^2+|\Omega_2|u_2\eta(1+p)+(1-u_2p)|\Omega_2|\eta\right]\delta}{|A|(1+p)}\right)+\frac{|\lambda_1|T|A|+|\Omega_2||\lambda_1|T\delta+|\lambda_1|\delta}{|A|}\|x_2-x_1\|$$
$$+\frac{|\lambda_2|T\delta+|\Omega_2||\lambda_2|\delta}{|A|}\|y_2-y_1\|\le(m_1M_1+n_1M_2+M_3)\|x_2-x_1\|+(m_2M_1+n_2M_2+M_4)\|y_2-y_1\|. \tag{36}$$

Thus,

$$\begin{aligned}&\|G_1(x_2,y_2)-G_1(x_1,y_1)\|\\&\quad\le B_1(\|x_2-x_1\|+\|y_2-y_1\|).\end{aligned} \tag{37}$$

Similarly,

$$\begin{aligned}&\|G_2(x_2,y_2)-G_2(x_1,y_1)\|\\&\quad\le U_1(\|x_2-x_1\|+\|y_2-y_1\|).\end{aligned} \tag{38}$$

It follows from (37) and (38) that

$$\begin{aligned}&\|G(x_2,y_2)-G(x_1,y_1)\|\\&\quad\le(B_1+U_1)(\|x_2-x_1\|+\|y_2-y_1\|).\end{aligned} \tag{39}$$

From (31), therefore, G is a contraction mapping. So, by Banach's fixed point theorem, the operator G has a unique fixed point, which is the unique solution of problem (1). This completes the proof. □

In the next result, we prove the existence of solutions for problem (1) by Leray-Schauder alternative.

Lemma 5 ((Leray-Schauder alternative) (see [24])). *Let G be a nonmed linear space and let $F : G \to G$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let*

$$\mathscr{E}(F)=\{x\in G: x=\kappa F(x),\ 0<\kappa<1\}. \tag{40}$$

Then either the set $\mathscr{E}(F)$ is unbounded or F has at least one fixed point.

For convenience, we set constants

$$\begin{aligned}E_1&=(M_1+M_5)L_1+(M_2+M_6)R_1+M_3+M_7,\\E_2&=(M_1+M_5)L_2+(M_2+M_6)R_2+M_4+M_8,\end{aligned} \tag{41}$$

$$E^*=\min\{1-E_1,1-E_2\}. \tag{42}$$

Theorem 6. *Assume that $f,g : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist real constants $L_i, R_i \ge 0$ $(i=1,2)$ and $L_0>0$, $R_0>0$, such that $\forall x_i\in\mathbb{R}$ $(i=1,2)$; we have*

$$\begin{aligned}|f(t,x_1,x_2)|&\le L_0+L_1|x_1|+L_2|x_2|,\\|g(t,x_1,x_2)|&\le R_0+R_1|x_1|+R_2|x_2|.\end{aligned} \tag{43}$$

If $E_1<1$, $E_2<1$, $M_3+M_4\neq 1$, and $M_7+M_8\neq 1$, then there exists at least one solution for problem (1) on J.

Proof. Now we show that the operator $G:\mathscr{X}\times\mathscr{X}\to\mathscr{X}\times\mathscr{X}$ is completely continuous. Let $B_r\subset\mathscr{X}\times\mathscr{X}$ where $B_r=\{(x,y)\in\mathscr{X}\times\mathscr{X}:\|(x,y)\|\le r\}$. Then there exist positive constants P_1 and P_2 such that

$$\begin{aligned}&|f(t,x(t),y(t))|\le P_1,\\&|g(t,x(t),y(t))|\le P_2,\\&\qquad\qquad\forall(x,y)\in B_r,\end{aligned} \tag{44}$$

and a positive real number r such that

$$r\ge\max\left\{\frac{P_1M_1+P_2M_2}{|1-(M_3+M_4)|},\frac{P_1M_5+P_2M_6}{|1-(M_7+M_8)|}\right\}. \tag{45}$$

For any $(x,y)\in B_r$, we have

$$\begin{aligned}\|G_1(x,y)\|&\le\sup_{t\in J}\Bigg|-\lambda_1\int_0^t x(s)\,d_qs+\int_0^t(t-qs)f(s,x(s),y(s))\,d_qs+\frac{[t(1+\alpha_1\lambda_1)+\alpha_1](1+\alpha_2\lambda_2)}{A}\Bigg[\frac{\lambda_1}{(1-u_1)\xi}\int_{u_1\xi}^{\xi}x(s)\,d_qs-\int_0^{u_1\xi}f(s,x(s),y(s))\,d_qs\\&\quad-\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}f(s,x(s),y(s))\,d_qs-\lambda_2\int_0^T y(s)\,d_ps+\int_0^T(T-ps)g(s,x(s),y(s))\,d_ps\\&\quad+\Omega_2\left(-\lambda_1\int_0^T x(s)\,d_qs+\int_0^T(T-qs)f(s,x(s),y(s))\,d_qs+\frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta}y(s)\,d_ps-\int_0^{u_2\eta}g(s,x(s),y(s))\,d_ps-\int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}g(s,x(s),y(s))\,d_ps\right)\Bigg]\Bigg|\\&\le\left(\frac{T^2|A|+[u_1\xi(1+q)+(1-u_1q)\xi+T^2|\Omega_2|]\delta}{|A|(1+q)}\right)P_1+\left(\frac{[T^2+|\Omega_2|u_2\eta(1+p)+(1-u_2p)|\Omega_2|\eta]\delta}{|A|(1+p)}\right)P_2+\left(\frac{|\lambda_1|T|A|+|\Omega_2||\lambda_1|T\delta+|\lambda_1|\delta}{|A|}\right.\\&\quad\left.+\frac{|\lambda_2|T\delta+|\Omega_2||\lambda_2|\delta}{|A|}\right)r=M_1P_1+M_2P_2+(M_3+M_4)r.\end{aligned} \tag{46}$$

In the same way, we deduce that

$$\begin{aligned}\|G_2(x,y)\| &\le \sup_{t\in J}\Bigg|-\lambda_2\int_0^t y(s)\,d_ps+\int_0^t (t-ps)\,g(s,x(s),y(s))\,d_ps\\ &+\frac{[t(1+\alpha_2\lambda_2)+\alpha_2](1+\alpha_1\lambda_1)}{A}\Bigg[\Omega_1\Bigg(\frac{\lambda_1}{(1-u_1)\xi}\int_{u_1\xi}^{\xi}x(s)\,d_qs-\int_0^{u_1\xi}f(s,x(s),y(s))\,d_qs-\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}f(s,x(s),y(s))\,d_qs-\lambda_2\int_0^T y(s)\,d_ps+\int_0^T(T-ps)\,g(s,x(s),y(s))\,d_ps\Bigg)\\ &-\lambda_1\int_0^T x(s)\,d_qs+\int_0^T(T-qs)\,f(s,x(s),y(s))\,d_qs+\frac{\lambda_2}{(1-u_2)\eta}\int_{u_2\eta}^{\eta}y(s)\,d_ps-\int_0^{u_2\eta}g(s,x(s),y(s))\,d_ps-\int_{u_2\eta}^{\eta}\frac{(\eta-ps)}{(1-u_2)\eta}g(s,x(s),y(s))\,d_ps\Bigg]\Bigg|\\ &\le\left(\frac{[T^2+|\Omega_1|u_1\xi(1+q)+(1-u_1q)|\Omega_1|\xi]\rho}{|A|(1+q)}\right)P_1+\left(\frac{T^2|A|+[u_2\eta(1+p)+(1-u_2p)\eta+T^2|\Omega_1|]\rho}{|A|(1+p)}\right)P_2+\left(\frac{|\lambda_1|T\rho+|\Omega_1||\lambda_1|\rho}{|A|}+\frac{|\lambda_2|T|A|+|\Omega_1||\lambda_2|T\rho+|\lambda_2|\rho}{|A|}\right)r=M_5P_1\\ &+M_6P_2+(M_7+M_8)r.\end{aligned}\tag{47}$$

Therefore, G is uniformly bounded.

Next, we show that G is equicontinuous. Let $t_1, t_2 \in [0,T]$ with $t_2 < t_1$. Setting $\sup_{t\in J}|f(t,x(t),y(t))| = \overline{f}$ and $\sup_{t\in J}|g(t,x(t),y(t))| = \overline{g}$ and for any $x, y \in B_r$, we get

$$\begin{aligned}|G_1(x,y)(t_1)-G_1(x,y)(t_2)| &\le r\Bigg[|\lambda_1||t_1-t_2|\\ &+\frac{1}{|A|}\left([|t_1-t_2|(1+|\alpha_1\lambda_1|)+\alpha_1](1+|\alpha_2\lambda_2|)\right)\\ &\cdot\left(|\lambda_1|+|\lambda_2|T+|\lambda_1||\Omega_2|T+|\lambda_2||\Omega_2|\right)\Bigg]\\ &+\overline{f}\Bigg[|t_1-t_2|t_2+\frac{|t_1-t_2|(t_1+t_2)}{1+q}\\ &+\frac{1}{|A|}\left([|t_1-t_2|(1+|\alpha_1\lambda_1|)+\alpha_1](1+|\alpha_2\lambda_2|)\right)\\ &\cdot\left(u_1\xi+(1-u_1q)\xi+\frac{T^2|\Omega_2|}{1+q}\right)\Bigg]\\ &+\overline{g}\Bigg[\frac{1}{|A|}\left([|t_1-t_2|(1+|\alpha_1\lambda_1|)+\alpha_1](1+|\alpha_2\lambda_2|)\right)\\ &\cdot\left(\frac{T^2}{1+p}+u_2\eta|\Omega_2|+(1-u_2p)|\Omega_2|\eta\right)\Bigg].\end{aligned}\tag{48}$$

Similarly, we obtain

$$\begin{aligned}|G_2(x,y)(t_1)-G_2(x,y)(t_2)| &\le r\Bigg[|\lambda_2||t_1-t_2|\\ &+\frac{1}{|A|}\left([|t_1-t_2|(1+|\alpha_2\lambda_2|)+\alpha_2](1+|\alpha_1\lambda_1|)\right)\\ &\cdot\left(|\lambda_2|+|\lambda_1|T+|\lambda_2||\Omega_1|T+|\lambda_1||\Omega_1|\right)\Bigg]\\ &+\overline{f}\Bigg[\frac{1}{|A|}\left([|t_1-t_2|(1+|\alpha_2\lambda_2|)+\alpha_2]\right.\\ &\cdot\left.(1+|\alpha_1\lambda_1|)\right)\left(\frac{T^2}{1+q}+u_1\xi|\Omega_1|+(1-u_1q)|\Omega_1|\right.\\ &\left.\cdot\xi\right)\Bigg]+\overline{g}\Bigg[|t_1-t_2|t_2+\frac{|t_1-t_2|(t_1+t_2)}{1+p}\\ &+\frac{1}{|A|}\left([|t_1-t_2|(1+|\alpha_2\lambda_2|)+\alpha_2](1+|\alpha_1\lambda_1|)\right)\\ &\cdot\left(u_2\eta+(1-u_2p)\eta+\frac{T^2|\Omega_1|}{1+p}\right)\Bigg].\end{aligned}\tag{49}$$

Then G is equicontinuous. So G is relatively compact on B_r, and by the Arzelá-Ascoli theorem G is completely continuous on B_r.

Finally, it will be verified that the set $\mathscr{E} = \{(x,y) \in \mathscr{X}\times\mathscr{X} \mid (x,y) = \kappa G(x,y),\ 0 < \kappa < 1\}$ is bounded. Let $(x,y) \in \mathscr{E}$; then $(x,y) = \kappa G(x,y)$. For any $t \in J$, we have

$$\begin{aligned}x(t) &= \kappa G_1(x,y)(t),\\ y(t) &= \kappa G_2(x,y)(t).\end{aligned}\tag{50}$$

Therefore, we obtain

$$\begin{aligned}|x(t)| = |\kappa G_1(x,y)(t)| &\le |\lambda_1|\,\|x\|\int_0^T 1d_qs+(L_0\\ &+L_1\|x\|+L_2\|y\|)\int_0^T(T-qs)\,d_qs\\ &+\frac{\delta}{|A|}\Bigg[\frac{|\lambda_1|\,\|x\|}{(1-u_1)\xi}\int_{u_1\xi}^{\xi}1d_qs+(L_0+L_1\|x\|\\ &+L_2\|y\|)\int_0^{u_1\xi}1d_qs+(L_0+L_1\|x\|+L_2\|y\|)\\ &\cdot\int_{u_1\xi}^{\xi}\frac{(\xi-qs)}{(1-u_1)\xi}+|\lambda_2|\,\|y\|\int_0^T 1d_ps+(R_0+R_1\|x\|\\ &+R_2\|y\|)\int_0^T(T-ps)\,d_ps+|\Omega_2|\\ &\cdot\Bigg(\lambda_1T\|x\|\int_0^T 1d_qs\\ &+(L_0+L_1\|x\|+L_2\|y\|)\int_0^T(T-qs)\,d_qs\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|\lambda_2| \|y\|}{(1-u_2)\eta} \int_{u_2\eta}^{\eta} 1 d_p s \\
&+ (R_0 + R_1 \|x\| + R_2 \|y\|) \int_0^{u_2\eta} 1 d_p s \\
&+ (R_0 + R_1 \|x\| + R_2 \|y\|) \int_{u_2\eta}^{\eta} \frac{(\eta - ps)}{(1-u_2)\eta} d_p s \bigg) \bigg] \\
&= (L_0 + L_1 \|x\| + L_2 \|y\|) M_1 + (R_0 + R_1 \|x\| \\
&+ R_2 \|y\|) M_2 + \|x\| M_3 + \|y\| M_4, \\
|y(t)| &= |\kappa G_2(x,y)(t)| \le |\lambda_2| \|y\| \int_0^T 1 d_p s + (R_0 \\
&+ R_1 \|x\| + R_2 \|y\|) \int_0^T (T - qs) d_p s + \frac{\rho}{|A|} \bigg[|\Omega_1| \\
&\cdot \bigg(\frac{|\lambda_1| \|x\|}{(1-u_1)\xi} \int_{u_1\xi}^{\xi} 1 d_q s + |\lambda_2| \|y\| \int_0^T 1 d_p s \\
&+ (L_0 + L_1 \|x\| + L_2 \|y\|) \int_0^{u_1\xi} 1 d_q s \\
&+ (R_0 + R_1 \|x\| + R_2 \|y\|) \int_0^T (T - ps) d_p s \\
&+ (L_0 + L_1 \|x\| + L_2 \|y\|) \int_{u_1\xi}^{\xi} \frac{(\xi - qs)}{(1-u_1)\xi} d_q s \bigg) \\
&+ |\lambda_1| \|x\| \int_0^T 1 d_q s + (L_0 + L_1 \|x\| + L_2 \|y\|) \\
&\cdot \int_0^T (T - qs) d_q s + \frac{|\lambda_2| \|y\|}{(1-u_2)\eta} \int_{u_2\eta}^{\eta} 1 d_p s + (R_0 \\
&+ R_1 \|x\| + R_2 \|y\|) \int_0^{u_2\eta} 1 d_p s + (R_0 + R_1 \|x\| \\
&+ R_2 \|y\|) \int_{u_2\eta}^{\eta} \frac{(\eta - ps)}{(1-u_2)\eta} d_p s \bigg] = (L_0 + L_1 \|x\| \\
&+ L_2 \|y\|) M_5 + (R_0 + R_1 \|x\| + R_2 \|y\|) M_6 + \|x\| \\
&\cdot M_7 + \|y\| M_8.
\end{aligned} \tag{51}
$$

So, we have

$$
\begin{aligned}
\|x\| + \|y\| &\le (M_1 + M_5) L_0 + (M_2 + M_6) R_0 \\
&+ ((M_1 + M_5) L_1 + (M_2 + M_6) R_1 + M_3 + M_7) \\
&\cdot \|x\| \\
&+ ((M_1 + M_5) L_2 + (M_2 + M_6) R_2 + M_4 + M_8) \\
&\cdot \|y\|.
\end{aligned} \tag{52}
$$

Consequently,

$$
\|(x,y)\| \le \frac{(M_1 + M_5) L_0 + (M_2 + M_6) R_0}{E^*}, \tag{53}
$$

for any $t \in J$, where E^* is defined by (42), so that $\mathscr{E}$ is bounded. Thus, by Lemma 5, the operator G has at least one fixed point. Hence, problem (1) has at least one solution on J. The proof is completed. □

4. Examples

In this section, we present examples to illustrate our result.

Example 1. Consider the following system of Langevin quantum difference equations subject to the coupled nonlocal q-derivatives boundary conditions:

$$
\begin{aligned}
&D_{1/4} \left(D_{1/4} + \frac{1}{30} \right) x(t) \\
&\quad = \frac{|x| e^{-t}}{5(10+t)^2} + \frac{|y| \sin^2(t)}{(14-t)^3} \left(\frac{|y|}{|y|+1} \right), \\
&\qquad t \in [0,4] \\
&D_{2/9} \left(D_{2/9} + \frac{1}{35} \right) y(t) \\
&\quad = \frac{2|x|}{12(5t+10)^2} \left(\frac{|x|}{|x|+3} \right) + \frac{12|y| \cos^2(t)}{5(t+10)^3}, \\
&\qquad t \in [0,4] \\
&x(0) = \frac{2}{11} D_{1/4} x(0), \\
&y(4) = D_{3/11} x(1), \\
&y(0) = \frac{3}{16} D_{5/6} y(0), \\
&x(4) = D_{1/13} y(2).
\end{aligned} \tag{54}
$$

Here $q = 1/4$, $p = 2/9$, $\lambda_1 = 1/30$, $\lambda_2 = 1/35$, $\alpha_1 = 2/11$, $\alpha_2 = 3/16$, $z_1 = 1/4$, $z_2 = 5/6$, $u_1 = 3/11$, $u_2 = 1/13$, $T = 4$, $\xi = 1$, $\eta = 2$, $f(t,x,y) = |x|e^{-t}/5(10+t)^2 + |y|\sin^2(t)/(14-t)^3(|y|/|y|+1)$, and $g(t,x,y) = 2|x|/12(5t+10)^2(|x|/|x|+3) + 12|y|\cos^2(t)/5(t+10)^3$.

We have

$$
\begin{aligned}
&|f(t,x_1,y_1) - f(t,x_2,y_2)| \\
&\quad \le \frac{1}{500} |x_1 - x_2| + \frac{1}{1000} |y_1 - y_2|, \\
&|g(t,x_1,y_1) - g(t,x_2,y_2)| \\
&\quad \le \frac{1}{600} |x_1 - x_2| + \frac{1}{2400} |y_1 - y_2|, \\
&|\alpha_1 \lambda_1| \simeq 0.00606 \ne 1,
\end{aligned}
$$

$$\begin{aligned}
&|\alpha_2\lambda_2| \simeq 0.00536 \neq 1,\\
&|\Omega_1| = T + \frac{\alpha_1}{1+\alpha_1\lambda_1} \simeq 4.18072,\\
&|\Omega_2| = T + \frac{\alpha_2}{1+\alpha_2\lambda_2} \simeq 4.18650,\\
&|A| = |(1+\alpha_1\lambda_1)(1+\alpha_2\lambda_2)[1-\Omega_1\Omega_2]|\\
&\quad \simeq 16.69158,\\
&\delta = T((1+|\alpha_1\lambda_1|)+|\alpha_1|)(1+|\alpha_2\lambda_2|) \simeq 4.77698,\\
&\rho = T((1+|\alpha_2\lambda_2|)+|\alpha_2|)(1+|\alpha_1\lambda_1|) \simeq 4.8004.
\end{aligned} \tag{55}$$

Then, the assumption of Theorem 4 is satisfied with $m_1 = 1/500$, $m_2 = 1/1000$, $n_1 = 1/600$, $n_2 = 1/2400$, $M_1 \simeq 28.42756$, $M_2 \simeq 5.85791$, $M_3 \simeq 0.30263$, $M_4 \simeq 0.06694$, $M_5 \simeq 4.90542$, $M_6 \simeq 29.33758$, $M_7 \simeq 0.07269$, $M_8 \simeq 0.25991$, and

$$\begin{aligned}
B_1 &= (m_1+m_2)M_1 + (n_1+n_2)M_2 + M_3 + M_4\\
&\simeq 0.46701 < \frac{1}{2},\\
U_1 &= (m_1+m_2)M_5 + (n_1+n_2)M_6 + M_7 + M_8\\
&\simeq 0.40844 < \frac{1}{2}.
\end{aligned} \tag{56}$$

Therefore, we get that

$$B_1 + U_1 \simeq 0.87545 < 1. \tag{57}$$

Hence, by Theorem 4, problem (54) has a unique solution on $[0,4]$.

Example 2. Consider the following system of Langevin quantum difference equations subject to the coupled nonlocal q-derivatives boundary conditions:

$$\begin{aligned}
&D_{1/3}\left(D_{1/3}+\frac{1}{32}\right)x(t)\\
&\quad = \frac{1}{2} + \frac{|x|e^{-t}}{8(15-t)^2} + \frac{4|y|\cos^2(t)}{6(10-t)^3}\left(\frac{|y|}{|y|+3}\right), \quad t\in[0,5]\\
&D_{1/4}\left(D_{1/4}+\frac{1}{36}\right)y(t)\\
&\quad = \frac{\sqrt{3}}{4} + \frac{2|x|}{241(3+t)}\left(\frac{|x|}{|x|+2}\right) + \frac{|y|\sin^2(t)}{47(19-t)}, \quad t\in[0,5]\\
&x(0) = -\frac{2}{3}D_{1/5}x(0),\\
&y(5) = D_{2/3}x(2),\\
&y(0) = \frac{3}{5}D_{1/4}y(0),\\
&x(5) = D_{2/5}y(3).
\end{aligned} \tag{58}$$

Here $q = 1/3$, $p = 1/4$, $\lambda_1 = 1/32$, $\lambda_2 = 1/36$, $\alpha_1 = -2/3$, $\alpha_2 = 3/5$, $z_1 = 1/5$, $z_2 = 1/4$, $u_1 = 2/3$, $u_2 = 2/5$, $T = 5$, $\xi = 2$, $\eta = 3$, $f(t,x,y) = 1/2 + |x|e^{-t}/8(15-t)^2 + (4|y|\cos^2(t)/6(10-t)^3)(|y|/|y+3|)$, and $g(t,x,y) = \sqrt{3}/4 + (2|x|/241(3+t)^3)(|x|/|x|+2) + |y|\sin^2(t)/47(19-t)$.

So that

$$\begin{aligned}
&|f(t,x_1,x_2)| \le \frac{1}{2} + \frac{1}{800}|x_1| + \frac{1}{750}|x_2|,\\
&|g(t,x_1,x_2)| \le \frac{\sqrt{3}}{4} + \frac{2}{723}|x_1| + \frac{1}{658}|x_2|,\\
&|\alpha_1\lambda_1| \simeq 0.020833 \neq 1,\\
&|\alpha_2\lambda_2| \simeq 0.035294 \neq 1,\\
&|\Omega_1| = T + \frac{\alpha_1}{1+\alpha_1\lambda_1} \simeq 5.653061,\\
&|\Omega_2| = T + \frac{\alpha_2}{1+\alpha_2\lambda_2} \simeq 5.579546,\\
&|A| = |(1+\alpha_1\lambda_1)(1+\alpha_2\lambda_2)[1-\Omega_1\Omega_2]|\\
&\quad \simeq 32.278176,\\
&\delta = T((1+|\alpha_1\lambda_1|)+|\alpha_1|)(1+|\alpha_2\lambda_2|)\\
&\quad \simeq 8.735291,\\
&\rho = T((1+|\alpha_2\lambda_2|)+|\alpha_2|)(1+|\alpha_1\lambda_1|)\\
&\quad \simeq 8.346811.
\end{aligned} \tag{59}$$

Then, the assumptions of Theorem 6 are satisfied with $L_0 = 1/2$, $L_1 = 1/800$, $L_2 = 1/750$, $R_0 = \sqrt{3}/4$, $R_1 = 2/723$, $R_2 = 1/658$, $M_1 \simeq 47.738433$, $M_2 \simeq 10.485993$, $M_3 \simeq 0.400639$, $M_4 \simeq 0.079530$, $M_5 \simeq 8.503122$, $M_6 \simeq 50.105354$, $M_7 \simeq 0.086087$, $M_8 \simeq 0.349103$, and

$$\begin{aligned}
E_1 &= (M_1+M_5)L_1 + (M_2+M_6)R_1 + M_3 + M_7\\
&\simeq 0.724639 < 1,\\
E_2 &= (M_1+M_5)L_2 + (M_2+M_6)R_2 + M_4 + M_8\\
&\simeq 0.595706 < 1.
\end{aligned} \tag{60}$$

Consequently all conditions in Theorem 6 are satisfied. Therefore, problem (1) has at least one solution on $[0,5]$.

Competing Interests

The authors declare that they have no competing interests.

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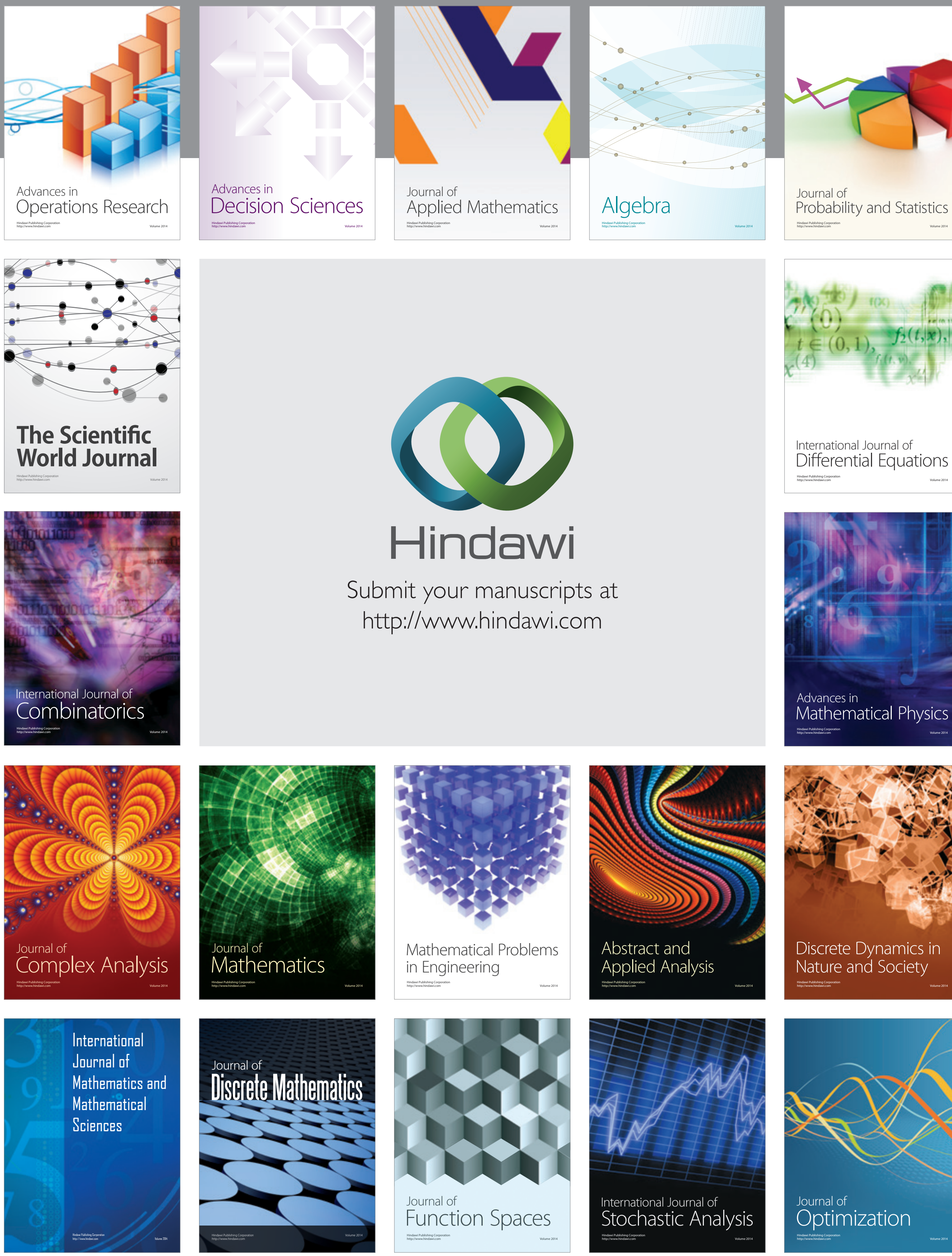
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