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Research Article

Asymptotically Stable Solutions of a Generalized Fractional Quadratic Functional-Integral Equation of Erdélyi-Kober Type

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We study a generalized fractional quadratic functional-integral equation of Erdélyi-Kober type in the Banach space $BC(\mathbb{R}_+)$. We show that this equation has at least one asymptotically stable solution.

1. Introduction

Quadratic integral equations with nonsingular kernels have received a lot of attention because of their useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in kinetic theory of gases, in the theory of neutron transport, and in the traffic theory; see [1–8]. The existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels has been studied by several authors, for example, Argyros [9], Banaś et al. [10–12], Benchohra and Darwish [13, 14], Caballero et al. [15–17], Darwish et al. [18, 19], Leggett [20], and Stuart [21]. There is a great interest in studying singular quadratic integral equations by many authors, after the appearance of Darwish's paper [22], for example, Banaś and O'Regan [23], Banaś and Rzepka [24, 25], Darwish [26, 27], Darwish and Sadarangani [28], Darwish and Ntouyas [29], Darwish et al. [30], and Wang et al. [31, 32].

In this paper, we will study the quadratic functional-integral equation of fractional order

$$x(t) = a(t) + f\left(t, \frac{\beta g(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}} u(t, s, x(s))\, ds\right), \quad t \in \mathbb{R}_+, \tag{1}$$

where $\alpha \in (0, 1)$ and $\beta > 0$.

If $\beta = 1$ and $f(t, u) = u$, we obtain a quadratic Urysohn-Volterra integral equation of fractional order studied by Banas' and O'Regan in [23] while in the case where $\beta = 1$, $f(t, u) = u$, and $u(t, s, x) = v(t, x)$, we get a fractional quadratic integral equation of Hammerstein-Volterra type studied by Darwish in [22]. Moreover, in the case where $\beta = 1$, we obtain the quadratic functional-integral equation of fractional order studied by Darwish and Sadarangani in [28].

The aim of this paper is to prove the existence of solutions of (1) in the space of real functions, defined, continuous, and bounded on an unbounded interval. Moreover, we will obtain some asymptotic characterization of solutions of (1). Our proof depends on suitable combination of the technique of measures of noncompactness and the Schauder fixed point principle.

2. Notation and Auxiliary Facts

This section is devoted to collecting some definitions and results which will be needed further on. First, we recall from [33–35] that the Erdélyi-Kober fractional integral of a continuous function f is defined as

$$I_\beta^\gamma f(t) = \frac{\beta}{\Gamma(\gamma)} \int_0^t \frac{s^{\beta-1} f(s)}{(t^\beta - s^\beta)^{1-\gamma}} ds, \quad \beta > 0, \ 0 < \gamma < 1. \tag{2}$$

When $\beta = 1$, we obtain Riemann-Liouville fractional integral; that is,

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad 0 < \gamma < 1. \tag{3}$$

Now, let $(E, \|\cdot\|)$ be an infinite dimensional Banach space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a subset of E, then $\overline{X}$ and $\text{Conv}X$ denote the closure and convex closure of X, respectively. Moreover, we denote by $\mathscr{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathscr{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the definition of the concept of a measure of noncompactness [36].

Definition 1. A mapping $\mu : \mathscr{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

(1) The family $\ker \mu = \{X \in \mathscr{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathscr{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.

(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $0 \leq \lambda \leq 1$.

(5) If $X_n \in \mathscr{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\ker \mu$ described above is called the kernel of the measure of noncompactness μ. Let us observe that the intersection set X_∞ from (5) belongs to $\ker \mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for every, then we have that $\mu(X_\infty) = 0$.

In what follows we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded, and continuous on $\mathbb{R}_+$. This space is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}. \tag{4}$$

Next, we give the construction of the measure of noncompactness in $BC(\mathbb{R}_+)$ which will be used as main tool of the proof of our main result; see [37, 38] and references therein.

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+)$ and numbers $\varepsilon > 0$ and $T > 0$. For arbitrary function $x \in X$ let us denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, T]$; that is,

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T],\ |t - s| \leq \varepsilon\}. \tag{5}$$

Further, let us put

$$\begin{gathered} \omega^T(X, \varepsilon) = \sup\left\{\omega^T(x, \varepsilon) : x \in X\right\}, \\ \omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X, \varepsilon), \\ \omega_0^\infty(X) = \lim_{T\to\infty} \omega_0^T(X). \end{gathered} \tag{6}$$

Moreover, for a fixed number $t \in \mathbb{R}_+$ let us define

$$\begin{gathered} X(t) = \{x(t) : x \in X\}, \\ \text{diam}\, X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}, \\ c(X) = \limsup_{t\to\infty} \text{diam}\, X(t). \end{gathered} \tag{7}$$

Let us mention that the kernel $\ker \omega_0^\infty$ consists of all nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$. On the other hand, the kernel $\ker c$ is the family containing all nonempty and bounded sets X in the space $BC(\mathbb{R}_+)$ such that the thickness of the bundle formed by the graphs of functions belonging to X tends to zero at infinity.

Finally, with the help of the above quantities we can define a measure of noncompactness as

$$\mu(X) = \omega_0^\infty(X) + c(X). \tag{8}$$

The function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ [36, 37].

In the end of this section, we recall the definition of the asymptotic stability solutions which will be used in the proof of our main result. To this end we assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$. Let $Q : \Omega \to BC(\mathbb{R}_+)$ be a given operator. We consider the following operator equation:

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{9}$$

Definition 2. One says that solutions of (9) are asymptotically stable if there exists a ball $B(x_0, r)$ such that $\Omega \cap B(x_0, r) \neq \emptyset$ and such that for each $\varepsilon > 0$ there exists $T > 0$ such that for arbitrary solutions $x = x(t)$, $y = y(t)$ of this equation belonging to $\Omega \cap B(x_0, r)$ the inequality $|x(t) - y(t)| \leq \varepsilon$ is satisfied for any $t \geq T$.

3. The Existence and Asymptotic Stability of Solutions

In this section we will study (1) assuming that the following hypotheses are satisfied.

(h_1) $a : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function on $\mathbb{R}_+$.

(h_2) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t, 0)$ is bounded on $\mathbb{R}_+$ with $f^* = \sup\{|f(t, 0)| : t \in \mathbb{R}_+\}$. Moreover, there exists a continuous function $m(t) = m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \leq m(t)|x - y| \tag{10}$$

for all $x, y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(h_3) $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $n(t) = n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, x) - g(t, y)| \leq n(t)|x - y| \tag{11}$$

for all $x, y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(h_4) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, there exist a function $l(t) = l : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and nondecreasing on $\mathbb{R}_+$ with $\Phi(0) = 0$ such that

$$\left|u(t,s,x) - u(t,s,y)\right| \le l(t)\,\Phi\left(\left|x-y\right|\right) \tag{12}$$

for all $t, s \in \mathbb{R}_+$ such that $t \ge s$ and for all $x \in \mathbb{R}$.

For further purpose let us define the function $u^* : \mathbb{R}_+ \to \mathbb{R}_+$ by $u^*(t) = \max\{|u(t,s,0)| : 0 \le s \le t\}$.

(h_5) The functions $\phi, \psi, \xi, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\phi(t) = m(t)n(t)l(t)t^{\alpha\beta}$, $\psi(t) = m(t)n(t)u^*(t)t^{\alpha\beta}$, $\xi(t) = m(t)l(t)|g(t,0)|t^{\alpha\beta}$, and $\eta(t) = m(t)u^*(t)|g(t,0)|t^{\alpha\beta}$ are bounded on $\mathbb{R}_+$ and the functions ϕ and ξ vanish at infinity; that is, $\lim_{t\to\infty}\phi(t) = \lim_{t\to\infty}\xi(t) = 0$.

(h_6) There exists a positive solution r_0 of the inequality

$$\begin{aligned}&\left(\|a\| + f^*\right)\Gamma(\alpha+1)\\ &\quad + \left[\phi^* r\Phi(r) + \psi^* r + \xi^*\Phi(r) + \eta^*\right]\\ &\le r\Gamma(\alpha+1)\end{aligned} \tag{13}$$

and $\phi^*\Phi(r_0) + \psi^* < \Gamma(\alpha+1)$, where $\phi^* = \sup\{\phi(t) : t \in \mathbb{R}_+\}$, $\psi^* = \sup\{\psi(t) : t \in \mathbb{R}_+\}$, $\xi^* = \sup\{\xi(t) : t \in \mathbb{R}_+\}$, and $\eta^* = \sup\{\eta(t) : t \in \mathbb{R}_+\}$.

Now, we are in a position to state and prove our main result.

Theorem 3. *Let the hypotheses* (h_1)–(h_6) *be satisfied. Then* (1) *has at least one solution* $x \in BC(\mathbb{R}_+)$ *and all solutions of this equation belonging to the ball* B_{r_0} *are asymptotically stable.*

Proof. Denote by $\mathcal{F}$ the operator associated with the right-hand side of (1). Then, (1) takes the form

$$x = \mathcal{F}x, \tag{14}$$

where

$$\begin{aligned}\mathcal{F}x &= a + F\mathcal{H}x,\\ (\mathcal{H}x)(t) &= (Gx)(t)\cdot(\mathcal{U}x)(t),\\ (\mathcal{U}x)(t) &= \frac{\beta}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{\left(t^\beta - s^\beta\right)^{1-\alpha}}ds, \quad t \in \mathbb{R}_+.\end{aligned} \tag{15}$$

Here, F and G are the superposition operators, generated by the functions $f = f(t,x)$ and $g = g(t,x)$ involved in (1), defined by

$$(Fx)(t) = f(t, x(t)), \tag{16}$$

$$(Gx)(t) = g(t, x(t)), \tag{17}$$

respectively, where $x = x(t)$ is an arbitrary function defined on $\mathbb{R}_+$ (see [39]).

Solving (1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $BC(\mathbb{R}_+)$.

For convenience, we divide the proof into several steps.

Step 1 ($\mathcal{F}x$ is continuous on $\mathbb{R}_+$). To prove the continuity of the function $\mathcal{F}x$ on $\mathbb{R}_+$ it suffices to show that if $x \in BC(\mathbb{R}_+)$, then $\mathcal{U}x$ is continuous function on $\mathbb{R}_+$, thanks to (h_1), (h_2), and (h_3). For this purpose, take an arbitrary $x \in BC(\mathbb{R}_+)$ and fix $\varepsilon > 0$ and $T > 0$. Assume that $t_1, t_2 \in [0, T]$ are such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then we get

$$\begin{aligned}&\left|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)\right|\\
&= \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_2}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_1^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\le \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_2}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\quad + \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\quad + \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_1^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\le \frac{\beta}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{s^{\beta-1}\left|u(t_2,s,x(s))\right|}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\\
&\quad + \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}\left[\left|u(t_2,s,x(s)) - u(t_1,s,x(s))\right|\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\\
&\quad + \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}s^{\beta-1}\left|u(t_1,s,x(s))\right|\\
&\qquad \times\left[\left(t_1^\beta - s^\beta\right)^{\alpha-1} - \left(t_2^\beta - s^\beta\right)^{\alpha-1}\right]ds.\end{aligned} \tag{18}$$

Let us denote

$$\omega_d^T(u,\varepsilon) = \sup\{|u(t_2,s,y)-u(t_1,s,y)| : s,t_1,t_2 \in [0,T], \; t_1 \ge s,\; t_2 \ge s,\; |t_2-t_1| \le \varepsilon, \; y \in [-d,d];\; d \ge 0\}; \tag{19}$$

then we obtain

$$\begin{aligned}
&|(\mathscr{U}x)(t_2)-(\mathscr{U}x)(t_1)| \\
&\le \frac{\beta}{\Gamma(\alpha)} \times \int_{t_1}^{t_2} \frac{s^{\beta-1}[|u(t_2,s,x(s))-u(t_2,s,0)|+|u(t_2,s,0)|]}{(t_2^\beta - s^\beta)^{1-\alpha}} ds \\
&+ \frac{\beta}{\Gamma(\alpha)} \int_0^{t_1} \frac{s^{\beta-1}\omega_{\|x\|}^T(u,\varepsilon)}{(t_2^\beta - s^\beta)^{1-\alpha}} ds \\
&+ \frac{\beta}{\Gamma(\alpha)} \times \int_0^{t_1} s^{\beta-1}[|u(t_1,s,x(s))-u(t_1,s,0)|+|u(t_1,s,0)|] \times \left[(t_1^\beta - s^\beta)^{\alpha-1} - (t_2^\beta - s^\beta)^{\alpha-1}\right] ds \\
&\le \frac{\beta}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{s^{\beta-1}[l(t_2)\Phi(|x(s)|)+u^*(t_2)]}{(t_2^\beta - s^\beta)^{1-\alpha}} ds \\
&+ \frac{\omega_{\|x\|}^T(u,\varepsilon)}{\Gamma(\alpha+1)} \left[t_2^{\alpha\beta} - (t_2^\beta - t_1^\beta)^\alpha\right] \\
&+ \frac{\beta}{\Gamma(\alpha)} \int_0^{t_1} s^{\beta-1}[l(t_1)\Phi(|x(s)|)+u^*(t_1)] \times \left[(t_1^\beta - s^\beta)^{\alpha-1} - (t_2^\beta - s^\beta)^{\alpha-1}\right] ds \\
&\le \frac{[l(t_2)\Phi(\|x\|)+u^*(t_2)]}{\Gamma(\alpha+1)} (t_2^\beta - t_1^\beta)^\alpha \\
&+ \frac{\omega_{\|x\|}^T(u,\varepsilon)}{\Gamma(\alpha+1)} t_2^{\alpha\beta} \\
&+ \frac{l(t_1)\Phi(\|x\|)+u^*(t_1)}{\Gamma(\alpha+1)} \left[t_1^{\alpha\beta} - t_2^{\alpha\beta} + (t_2^\beta - t_1^\beta)^\alpha\right].
\end{aligned} \tag{20}$$

Thus

$$\omega^T(\mathscr{U}x,\varepsilon) \le \frac{1}{\Gamma(\alpha+1)} \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(\|x\|)+\hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u,\varepsilon)\right\}, \tag{21}$$

where

$$\begin{aligned}
\hat{l}(T) &= \max\{l(t) : t \in [0,T]\}, \\
\hat{u}(T) &= \max\{u^*(t) : t \in [0,T]\}.
\end{aligned} \tag{22}$$

In view of the uniform continuity of the function u on $[0,T]\times[0,T]\times[-\|x\|,\|x\|]$ we have that $\omega_{\|x\|}^T(u,\varepsilon) \to 0$ as $\varepsilon \to 0$. From the above inequality we infer that the function $\mathscr{U}x$ is continuous on the interval $[0,T]$ for any $T > 0$. This yields the continuity of $\mathscr{U}x$ on $\mathbb{R}_+$ and, consequently, the function $\mathscr{F}x$ is continuous on $\mathbb{R}_+$.

Step 2 ($\mathscr{F}x$ is bounded on $\mathbb{R}_+$). In view of our hypotheses for arbitrary $x \in BC(\mathbb{R}_+)$ and for a fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
&|(\mathscr{F}x)(t)| \\
&\le \left| a(t) + f\left(t, \frac{\beta g(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds\right)\right| \\
&\le \|a\| + \frac{\beta}{\Gamma(\alpha)} m(t)[|g(t,x(t))-g(t,0)|+|g(t,0)|] \\
&\quad \times \int_0^t \frac{s^{\beta-1}[|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|]}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + |f(t,0)| \\
&\le \|a\| + f^* + \frac{\beta m(t)[n(t)\|x\| + |g(t,0)|]}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1}[l(t)\Phi(|x(s)|)+u^*(t)]}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\le \|a\| + f^* + \frac{m(t)[n(t)\|x\| + |g(t,0)|]}{\Gamma(\alpha+1)} \\
&\quad \times [l(t)\Phi(\|x\|)+u^*(t)] t^{\alpha\beta} \\
&= \|a\| + f^* \\
&\quad + \frac{1}{\Gamma(\alpha+1)} [\phi(t)\|x\|\Phi(\|x\|) \\
&\qquad + \psi(t)\|x\| + \xi(t)\Phi(\|x\|) + \eta(t)].
\end{aligned} \tag{23}$$

Hence, $\mathscr{F}x$ is bounded on $\mathbb{R}_+$, thanks to hypothesis (h_5).

Step 3 ($\mathcal{F}$ maps the ball B_{r_0} into itself). Steps 2 and 3 allow us to conclude that the operator $\mathcal{F}$ transforms $BC(\mathbb{R}_+)$ into itself. Moreover, from the last estimate we have

$$\begin{aligned}
&\|\mathcal{F}x\| \\
&\leq \|a\| + f^* \\
&\quad + \frac{1}{\Gamma(\alpha+1)}\left[\phi^* \|x\| \Phi(\|x\|) + \psi^* \|x\| + \xi^* \Phi(\|x\|) + \eta^*\right].
\end{aligned} \tag{24}$$

From the last estimate with hypothesis (h_6) we deduce that there exists $r_0 > 0$ such that the operator $\mathcal{F}$ maps B_{r_0} into itself.

Step 4 (an estimate of $\mathcal{F}$ with respect to the quantity c). Let us take a nonempty set $X \subset B_{r_0}$. Then, for arbitrary $x, y \in X$ and for a fixed $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
&\left|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\right| \\
&\leq \frac{\beta m(t)}{\Gamma(\alpha)} \left| g(t, x(t)) \int_0^t \frac{s^{\beta-1} u(t, s, x(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds \right. \\
&\qquad \left. - g(t, y(t)) \int_0^t \frac{s^{\beta-1} u(t, s, y(s))}{(t^\beta - s^\beta)^{1-\alpha}} ds \right| \\
&\leq \frac{\beta m(t) \left|g(t, x(t)) - g(t, y(t))\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} |u(t, s, x(s))|}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) \left|g(t, y(t))\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} \left|u(t, s, x(s)) - u(t, s, y(s))\right|}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{\beta m(t) n(t) \left|x(t) - y(t)\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} \left[|u(t, s, x(s)) - u(t, s, 0)| + |u(t, s, 0)|\right]}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) \left[n(t) \left|y(t)\right| + \left|g(t, 0)\right|\right]}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} l(t) \Phi(\left|x(s) - y(s)\right|)}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{\beta m(t) n(t) \left|x(t) - y(t)\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} \left[l(t) \Phi(|x(s)|) + u^*(t)\right]}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) \left[n(t) \left|y(t)\right| + \left|g(t, 0)\right|\right]}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} l(t) \Phi(|x(s)| + \left|y(s)\right|)}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{\beta m(t) n(t) l(t) (|x(t)| + \left|y(t)\right|)}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} \Phi(|x(s)|)}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) n(t) u^*(t) \left|x(t) - y(t)\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) n(t) l(t) \left|y(t)\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} \Phi(|x(s)| + \left|y(s)\right|)}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) l(t) \left|g(t, 0)\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^t \frac{s^{\beta-1} \Phi(|x(s)| + \left|y(s)\right|)}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{2\beta m(t) n(t) l(t) r_0 \Phi(r_0)}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) n(t) u^*(t) \operatorname{diam} X(t)}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) n(t) l(t) r_0 \Phi(2r_0)}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\quad + \frac{\beta m(t) l(t) \left|g(t, 0)\right| \Phi(2r_0)}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\alpha}} ds \\
&\leq \frac{2\phi(t) r_0 \Phi(r_0)}{\Gamma(\alpha+1)} + \frac{\psi(t)}{\Gamma(\alpha+1)} \operatorname{diam} X(t) \\
&\quad + \frac{\phi(t) r_0 \Phi(2r_0)}{\Gamma(\alpha+1)} + \frac{\xi(t) \Phi(2r_0)}{\Gamma(\alpha+1)}.
\end{aligned} \tag{25}$$

Hence, we can easily deduce the following inequality:

$$\begin{aligned}
\operatorname{diam}(\mathcal{F}X)(t) &\leq \frac{2\phi(t) r_0 \Phi(r_0)}{\Gamma(\alpha+1)} + \frac{\psi(t)}{\Gamma(\alpha+1)} \operatorname{diam} X(t) \\
&\quad + \frac{\phi(t) r_0 \Phi(2r_0)}{\Gamma(\alpha+1)} + \frac{\xi(t) \Phi(2r_0)}{\Gamma(\alpha+1)}.
\end{aligned} \tag{26}$$

Now, taking into account hypothesis (h_5) we obtain

$$c(\mathcal{F}X) \le qc(X), \tag{27}$$

where $q = (\phi^* \Phi(r_0) + \psi^*)/\Gamma(\alpha+1) \ge \psi^*/\Gamma(\alpha+1)$. Obviously, in view of hypothesis (h_6) we have that $q < 1$.

Step 5 (an estimate of $\mathcal{F}$ with respect to the modulus of continuity ω_0^∞). Take arbitrary numbers $\varepsilon > 0$ and $T > 0$. Choose a function $x \in X$ and take $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then, taking into account our hypotheses and (21), we have

$$\begin{aligned}
&\left|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\right| \\
&\le \left|a(t_2) - a(t_1)\right| \\
&\quad + m(t_2)\left|(Gx)(t_2)(\mathcal{U}x)(t_2) - (Gx)(t_1)(\mathcal{U}x)(t_2)\right| \\
&\quad + m(t_2)\left|(Gx)(t_1)(\mathcal{U}x)(t_2) - (Gx)(t_1)(\mathcal{U}x)(t_1)\right| \\
&\quad + \left|f(t_2, (Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left. - f(t_1, (Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\le \omega^T(a, \varepsilon) \\
&\quad + \frac{\beta m(t_2)\left|g(t_2, x(t_2)) - g(t_1, x(t_1))\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^{t_2} \frac{s^{\beta-1}\left[\left|u(t_2, s, x(s)) - u(t_2, s, 0)\right| + \left|u(t_2, s, 0)\right|\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds \\
&\quad + \frac{m(t_2)\left[\left|g(t_1, x(t_1)) - g(t_1, 0)\right| + \left|g(t_1, 0)\right|\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)\right\} \\
&\quad + \left|f(t_2, (Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left. - f(t_1, (Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\le \omega^T(a, \varepsilon) + \frac{\beta m(t_2)\left[n(t_2)\left|x(t_2) - x(t_1)\right| + \omega_g^T(\varepsilon)\right]}{\Gamma(\alpha)} \\
&\quad \times \int_0^{t_2} \frac{s^{\beta-1}\left[l(t_2)\Phi(|x(s)|) + u^*(t_2)\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds \\
&\quad + \frac{m(t_2)\left[n(t_1)\left|x(t_1)\right| + \left|g(t_1, 0)\right|\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)\right\} \\
&\quad + \left|f(t_2, (Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left. - f(t_1, (Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\le \omega^T(a, \varepsilon) + \frac{t_2^{\alpha\beta}}{\Gamma(\alpha+1)} m(t_2) \\
&\quad \times \left[n(t_2)\omega^T(x, \varepsilon) + \omega_g^T(\varepsilon)\right]\left[l(t_2)\Phi(r_0) + u^*(t_2)\right] \\
&\quad + \frac{\widehat{m}(T)\left[n(t_1)r_0 + \widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)\right\} \\
&\quad + \left|f(t_2, (Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left. - f(t_1, (Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\le \omega^T(a, \varepsilon) + \frac{\left[\phi(t_2)\Phi(r_0) + \psi(t_2)\right]}{\Gamma(\alpha+1)}\omega^T(x, \varepsilon) \\
&\quad + \frac{T^{\alpha\beta}\omega_g^T(\varepsilon)}{\Gamma(\alpha+1)}\widehat{m}(T)\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] \\
&\quad + \frac{\widehat{m}(T)\left[\widehat{n}(T)r_0 + \widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)\right\} \\
&\quad + \left|f(t_2, (Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left. - f(t_1, (Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\le \omega^T(a, \varepsilon) + \frac{\left[\phi^*\Phi(r_0) + \psi^*\right]}{\Gamma(\alpha+1)}\omega^T(x, \varepsilon) \\
&\quad + \frac{T^{\alpha\beta}\omega_g^T(\varepsilon)}{\Gamma(\alpha+1)}\widehat{m}(T)\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] \\
&\quad + \frac{\widehat{m}(T)\left[\widehat{n}(T)r_0 + \widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega_{\|x\|}^T(u, \varepsilon)\right\} \\
&\quad + \left|f(t_2, (Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left. - f(t_1, (Gx)(t_1)(\mathcal{U}x)(t_1))\right|.
\end{aligned} \tag{28}$$

In the last estimates, we have denoted by

$$\begin{aligned}
\omega_g^T(\varepsilon) &= \sup\left\{\left|g(t_2, x) - g(t_1, x)\right| : t_1, t_2 \in [0, T],\right. \\
&\qquad \left.\left|t_2 - t_1\right| \le \varepsilon, x \in [-r_0, r_0]\right\}, \\
\widehat{n}(T) &= \max\{n(t) : t \in [0, T]\}, \\
\widehat{m}(T) &= \max\{m(t) : t \in [0, T]\}, \\
\widehat{g}(T) &= \max\left\{\left|g(t, 0)\right| : t \in [0, T]\right\}.
\end{aligned} \tag{29}$$

Hence,

$$\begin{aligned}
&\omega^T(\mathcal{F}x,\varepsilon)\\
&\le \omega^T(a,\varepsilon)+\frac{[\phi^*\Phi(r_0)+\psi^*]}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon)\\
&\quad+\frac{T^{\alpha\beta}\omega_g^T(\varepsilon)}{\Gamma(\alpha+1)}\widehat{m}(T)\left[\widehat{l}(T)\Phi(r_0)+\widehat{u}(T)\right]\\
&\quad+\frac{\widehat{m}(T)\left[\widehat{n}(T)r_0+\widehat{g}(T)\right]}{\Gamma(\alpha+1)}\\
&\quad\times\left\{2\varepsilon^{\alpha\beta}\left[\widehat{l}(T)\Phi(r_0)+\widehat{u}(T)\right]+T^{\alpha\beta}\omega_{\|x\|}^T(u,\varepsilon)\right\}\\
&\quad+\sup_{t_1,t_2\in[0,T],\ \|x\|\le r_0}\left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right.\\
&\qquad\left.-f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right|.
\end{aligned}\tag{30}$$

Since the function $f(t,y)$ is uniformly continuous on the set $[0,T]\times[-H,H]$, the function $g(t,x)$ is uniformly continuous on the set $[0,T]\times[-r_0,r_0]$ and the function $u(t,s,x)$ is uniformly continuous on the set $[0,T]\times[0,T]\times[-r_0,r_0]$, where

$$\begin{aligned}
H=\sup&\left\{\frac{\beta\left|g(t_1,x(t_1))\right|}{\Gamma(\alpha)}\right.\\
&\times\int_0^{t_1}\frac{s^{\beta-1}\left|u(t_1,s,x(s))\right|}{\left(t_1^\beta-s^\beta\right)^{1-\alpha}}ds:t_1\in[0,T],\\
&\left.\|x\|\le r_0\right\};
\end{aligned}\tag{31}$$

we have

$$\begin{aligned}
&\sup\left\{\left|f(t_2,y)-f(t_1,y)\right|:t_1,t_2\in[0,T],\ \left|t_2-t_1\right|\le\varepsilon,\right.\\
&\qquad\left.\left|y\right|\le H\right\}\longrightarrow 0\quad\text{as }\varepsilon\longrightarrow 0.
\end{aligned}\tag{32}$$

It is easy to see that $H<\infty$ because $u(t,s,x)$ is bounded on $[0,T]\times[0,T]\times[-r_0,r_0]$, $g(t,x)$ is bounded on $[0,T]\times[-r_0,r_0]$, and $(\beta/\Gamma(\alpha))\int_0^{t_1}(s^{\beta-1}/(t_1^\beta-s^\beta)^{1-\alpha})ds\le T^{\alpha\beta}/\Gamma(\alpha+1)$.

Therefore, from the last estimate we derive the following one:

$$\omega_0^T(\mathcal{F}X)\le q\omega_0^T(X).\tag{33}$$

Hence we have

$$\omega_0^\infty(\mathcal{F}X)\le q\omega_0^\infty(X).\tag{34}$$

Step 6 ($\mathcal{F}$ is contraction with respect to the measure of noncompactness μ). From (27) and (34) and the definition of the measure of noncompactness μ given by formula (8), we obtain

$$\mu(\mathcal{F}X)\le q\mu(X).\tag{35}$$

Step 7. We construct a nonempty, bounded, closed, and convex set Y on which we will apply a fixed point theorem.

In the sequel let us put $B_{r_0}^1=\operatorname{Conv}\mathcal{F}(B_{r_0})$, $B_{r_0}^2=\operatorname{Conv}\mathcal{F}(B_{r_0}^1)$, and so on. In this way we have constructed a decreasing sequence of nonempty, bounded, closed, and convex subsets $(B_{r_0}^n)$ of B_{r_0} such that $\mathcal{F}(B_{r_0}^n)\subset B_{r_0}^n$ for $n=1,2,\ldots$. Hence, in view of (35) we obtain

$$\mu\left(B_{r_0}^n\right)\le q^n\mu\left(B_{r_0}\right),\quad\text{for any }n=1,2,3,\ldots.\tag{36}$$

This implies that $\lim_{n\to\infty}\mu(B_{r_0}^n)=0$. Hence, taking into account Definition 1 we infer that the set $Y=\bigcap_{n=1}^\infty B_{r_0}^n$ is nonempty, bounded, closed, and convex subset of B_{r_0}. Moreover, $Y\in\ker\mu$. Also, the operator $\mathcal{F}$ maps Y into itself.

Step 8 ($\mathcal{F}$ is continuous on the set Y). Let us fix a number $\varepsilon>0$ and take arbitrary functions $x,y\in Y$ such that $\|x-y\|\le\varepsilon$. Using the fact that $Y\in\ker\mu$ and keeping in mind the structure of sets belonging to $\ker\mu$ we can find a number $T>0$ such that for each $z\in Y$ and $t\ge T$ we have that $|z(t)|\le\varepsilon$. Since $\mathcal{F}$ maps Y into itself, we have that $\mathcal{F}x,\mathcal{F}y\in Y$. Thus, for $t\ge T$ we get

$$\left|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\right|\le\left|(\mathcal{F}x)(t)\right|+\left|(\mathcal{F}y)(t)\right|\le 2\varepsilon.\tag{37}$$

On the other hand, let us assume $t\in[0,T]$. Then we obtain

$$\begin{aligned}
&\left|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\right|\\
&\le\frac{\beta m(t)n(t)\left|x(t)-y(t)\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t\frac{s^{\beta-1}\left[l(t)\Phi(|x(s)|)+u^*(t)\right]}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)\left[n(t)\left|y(t)\right|+\left|g(t,0)\right|\right]}{\Gamma(\alpha)}\\
&\quad\times\int_0^t\frac{s^{\beta-1}l(t)\Phi\left(\left|x(s)-y(s)\right|\right)}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\le\frac{\left[m(t)n(t)l(t)\Phi(r_0)+m(t)n(t)u^*(t)\right]\varepsilon\beta}{\Gamma(\alpha)}\\
&\quad\times\int_0^t\frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\left[m(t)n(t)l(t)r_0+m(t)l(t)\left|g(t,0)\right|\right]\Phi(\varepsilon)\beta}{\Gamma(\alpha)}\\
&\quad\times\int_0^t\frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\le\frac{\phi(t)\Phi(r_0)+\psi(t)}{\Gamma(\alpha+1)}\varepsilon+\frac{\phi(t)r_0+\xi(t)}{\Gamma(\alpha+1)}\Phi(\varepsilon)\\
&\le\frac{\phi^*\Phi(r_0)+\psi^*}{\Gamma(\alpha+1)}\varepsilon+\frac{\phi^*r_0+\xi^*}{\Gamma(\alpha+1)}\Phi(\varepsilon).
\end{aligned}\tag{38}$$

Now, taking into account (37) and (38) and hypothesis (h_5) we conclude that the operator $\mathcal{F}$ is continuous on the set Y.

Step 9 (application of Schauder fixed point principle). Linking all above-obtained facts about the set Y and the operator $\mathcal{F} : Y \rightarrow Y$ and using the classical Schauder fixed point principle we deduce that the operator $\mathcal{F}$ has at least one fixed point x in the set Y. Obviously the function $x = x(t)$ is a solution of the quadratic integral equation (1). Moreover, since $Y \in \ker \mu$ we have that all solutions of (1) belonging to B_{r_0} are asymptotically stable in the sense of Definition 2. This completes the proof. □

4. Example

In this section, we present an example as an application of Theorem 3.

Consider the following integral equation of fractional order:

$$x(t) = te^{-t} + \frac{1}{1+t^3} + \arctan\left[\frac{1}{t^2+1} \cdot \frac{\sin(xt)}{2\Gamma(1/2)} \int_0^t \frac{\sqrt{1+\delta|x(s)|}}{\sqrt{s}\sqrt{\sqrt{t}-\sqrt{s}}} ds\right], \quad t \in \mathbb{R}_+. \tag{39}$$

Equation (39) is a special case of (1), where $\alpha = 1/2$, $\beta = 1/2$, δ is a positive constant, and

$$\begin{gathered} a(t) = te^{-t}, \\ f(t,x) = \frac{1}{1+t^3} + \arctan\left(\frac{1}{t^2+1} \cdot x\right), \\ g(t,x) = \sin(xt), \\ u(t,s,x) = \sqrt{1+\delta|x|}. \end{gathered} \tag{40}$$

It is easy to check that the assumptions of Theorem 3 are satisfied. In fact we have that the function $a(t) = te^{-t}$ is continuous and bounded on $\mathbb{R}_+$ and $\|a\| = 1/e$.

The function $f(t,x) = (1/(1+t^3)) + \arctan((1/(t^2+1)) \cdot x)$ satisfies assumption (h_2) with $m(t) = 1/(t^2+1)$ and $|f(t,0)| = f(t,0) = 1/(1+t^3)$, being $f^* = 1$.

Moreover, the function $g(t,x) = \sin(xt)$ satisfies assumption (h_3) with $n(t) = t$.

The function $u(t,s,x) = \sqrt{1+\delta|x|}$ satisfies assumption (h_4) with $l(t) = 1$, $\Phi(r) = \sqrt{\delta r}$, $u(t,s,0) = 1$, and $u^* = 1$.

Next, we are going to check that assumption (h_5) is satisfied. The functions ϕ, ψ, ξ, and η appearing in assumption (h_5) take the form

$$\begin{gathered} \phi(t) = \frac{t^{5/4}}{t^2+1}; \qquad \psi(t) = \frac{t^{5/4}}{t^2+1}; \\ \xi(t) = 0; \qquad \eta(t) = 0. \end{gathered} \tag{41}$$

It is easy to see that $\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \xi(t) = 0$.

Moreover we have $\phi^* = \psi^* = (3/8)\cdot(5/3)^{5/8}$, $\xi^* = \eta^* = 0$, and $\Gamma(3/2) = (1/2)\sqrt{\pi}$.

Therefore the inequality in assumption (h_6)

$$\|a\| + f^* + \frac{1}{\Gamma(\alpha+1)} \times \left[\phi^* r \Phi(r) + \psi^* r + \xi^* \Phi(r) + \eta^*\right] \le r \tag{42}$$

has the form

$$\frac{1}{e} + 1 + \frac{2}{\sqrt{\pi}}\left[\phi^* \sqrt{\delta} r^{3/2} + \psi^* r\right] \le r. \tag{43}$$

We can easily check that the number $r_0 = 7$ is a solution of the inequality (43) for $\delta \le 0,02$. Now, by Theorem 3, we infer that our equation has a solution in $B_{r_0} \subset BC(\mathbb{R}_+)$ and all solutions of (39) which belongs to B_{r_0} are asymptotically stable in the sense of the Definition 2.

Conflict of Interests

The authors declare that there is no conflict of interests in the submitted paper.

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