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Research Article

Sparsity-Homotopy Perturbation Inversion Method with Wavelets and Applications to Black-Scholes Model and Todaro Model

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Sparsity regularization method plays an important role in reconstructing parameters. Compared with traditional regularization methods, sparsity regularization method has the ability to obtain better performance for reconstructing sparse parameters. However, sparsity regularization method does not have the ability to reconstruct smooth parameters. For overcoming this difficulty, we combine a sparsity regularization method with a wavelet method in order to transform smooth parameters into sparse parameters. We use a sparsity-homotopy perturbation inversion method to improve the accuracy and stability and apply the proposed method to reconstruct parameters for a Black-Scholes option pricing model and a Todaro model. Numerical experiments show that the proposed method is convergent and stable.

1. Introduction

The reconstruction of parameters plays an important role in financial mathematics, such as the reconstructions of a volatility and a policy parameter [1, 2] and, in other fields, imaging enhancement techniques [3, 4], seismic signals [5, 6], and electrocardiogram signals (ECG) [7, 8]. With the development of economy and financial mathematics, the reconstructions of a volatility and a policy parameter have been widely used in many real applications. In general, the reconstruction of parameters is ill-posed. In other words, the small noisy level of measurement data can lead to the large error of reconstruction [9]. In order to overcome the ill-posedness, some regularization methods are developed. The most popular method is Tikhonov regularization method, which is composed of a fitting term and a penalty term in L_2 norm. The aims of those two terms are to match measurement data and to suppress noises, respectively.

The numerical methods for Tikhonov regularization method have been conducted, such as a well-known Landweber method [10], a Gauss-Newton method [11], a regularizing Newton-Kaczmarz method [12], and a multiscale smoothing method [13]. These methods have the ability to reconstruct smooth parameters in the case of sufficient measurement data. When measurement data are limited, smooth parameters are very difficult to be reconstructed. In the fields of economy and finance, reconstructed parameters are smooth and measurement data are limited. Hence, we need to use wavelet and curvelet transformations from smooth parameters to sparse parameters (i.e., the number of nonzero elements of parameter is very limited). Sparsity regularization methods are used to reconstruct sparse parameters. Compared with Tikhonov regularization method, sparsity regularization methods are not differentiable, and hence some specific techniques were developed to overcome this difficulty, such as Bregman iterations [14–17]. For reducing the computational time, a homotopy perturbation inversion method has been widely used in real applications, such as nonlinear systems, optimal control, and heat transfer equation [18–20].

In this paper, we combine a homotopy perturbation inversion method with a sparsity regularization method, in order to improve the accuracy and stability. After introducing the proposed method, we reconstruct two economic parameters based on a Black-Scholes option pricing model and a Todaro model. From the numerical experiments, a sparsity-homotopy perturbation inversion method with wavelets is convergent and stable.

2. Sparsity Regularization Method

The reconstruction of parameters is ill-posed, and hence we should employ the regularization method to reconstruct parameters. The Tikhonov regularization method can reconstruct smooth parameters while measurement data are sufficient. However, measurement data are limited in real applications. We combine a sparsity regularization method with a wavelet method, in order to improve the performance of reconstructing smooth parameters for limited measurement data.

2.1. Tikhonov Regularization Method. Many inverse problems can be formulated as a nonlinear operator equation

$$F(x) = y, \tag{1}$$

where F, x, and y denote a nonlinear operator, reconstructed parameter, and measurement data, respectively. We assume measurement data contaminated by noise

$$\left\| y^{\delta} - y \right\|_2 \leq \delta, \tag{2}$$

where y^{δ}, δ stand for the real measurement data and noisy level in the L_2 norm.

Ill-posedness means that the small noisy level included in measurement data may lead to the large error of reconstruction. We apply the regularization method in order to overcome ill-posedness. It is very important to balance a fitting term and a penalty term. Next, we introduce a Tikhonov regularization functional as follows:

$$J(x) = \left\| F(x) - y^{\delta} \right\|_2^2 + \alpha \|x\|_2^2, \tag{3}$$

where $\|F(x)-y^{\delta}\|_2$ is a fitting term, $\|x\|_2$ is a penalty term, and α is a regularized parameter balancing the fitting term and the penalty term. For reconstruction, we should minimize the functional (3):

$$x = \arg\min J(x). \tag{4}$$

The minimizer satisfies the Euler equation:

$$F'(x)^* \left(F(x) - y^{\delta} \right) + \alpha x = 0, \tag{5}$$

where $F'(x)$ is the Fréchet derivative. The Landweber method is widely used for solving (5). The Landweber method can be written as follows:

$$x_{n+1} = x_n - \left[F'(x_n)^* \left(F(x_n) - y^{\delta} \right) + \alpha x_n \right], \tag{6}$$

where n denotes the iteration number. Equation (6) is a well-known Landweber method that is stable; however, the convergent rate is slow and the accuracy is low.

2.2. Sparsity Regularization Method with Wavelets. The Tikhonov regularization method can obtain better reconstruction for smoothing parameters; however, its performance for reconstructing sparse parameters is worse. Hence, the classical Tikhonov regularization method is modified as the following sparsity regularization method:

$$J(x) = \left\| F(x) - y^{\delta} \right\|_2^2 + \alpha \|x\|_0, \tag{7}$$

where the norm of $\|\cdot\|_0$ means the number of nonzeros in x. The minimization problem (7) has the ability to obtain the best reconstruction for sparse parameters. However, the penalty term of (7) is nondifferentiable and minimization problem (7) is a NP problem. For speeding up minimization problem (7), we rewrite (7) as

$$J(x) = \left\| F(x) - y^{\delta} \right\|_2^2 + \alpha \|x\|_1, \tag{8}$$

where the norm of $\|\cdot\|_1$ means $\int |x| d\sigma$. In functional (8), the L_1 norm replaces the L_0 norm in order to overcome a NP problem. This improvement can reduce the cost time of (7) dramatically.

Because the penalty term of (8) is nondifferentiable, we modify problem (8) as follows:

$$J(x) = \left\| F(x) - y^{\delta} \right\|_2^2 + \alpha \|x\|_{2,\varepsilon}, \tag{9}$$

where $\|x\|_{2,\varepsilon} = \sqrt{x^2 + \varepsilon^2}$ replaces $\|x\|_1$, and the auxiliary parameter ε is a positive real number. As the Tikhonov regularization method, we can apply the Landweber method to minimize functional (9):

$$x_{n+1} = x_n - \left[F'(x_n)^* \left(F(x_n) - y^{\delta} \right) + \alpha \frac{x_n}{\left(x_n^2 + \varepsilon^2 \right)^{1/2}} \right]. \tag{10}$$

For a smooth parameter, we combine a sparsity regularization method with a wavelet method. A wavelet transformation converts a smooth parameter into a sparse parameter. A wavelet transformation is written as

$$x = W x_w, \tag{11}$$

where W, x_w stand for an inverse transformation matrix and the sparse representation of x, respectively. Taking (11) into (9) and (10), we arrive at

$$J(x_w) = \left\| F(W x_w) - y^{\delta} \right\|_2^2 + \alpha \left\| x_w \right\|_{2,\varepsilon}, \tag{12}$$

$$x_{w,n+1} = x_{w,n} - \left[F'(W x_{w,n})^* \left(F(W x_{w,n}) - y^{\delta} \right) + \alpha \frac{x_{w,n}}{\left(x_{w,n}^2 + \varepsilon^2 \right)^{1/2}} \right]. \tag{13}$$

3. Sparsity-Homotopy Perturbations Inversion Method

Because the Landweber method (13) is very slow, it is not realistic to reconstruct large scale real applications. We use a homotopy perturbation inversion method to modify the classical Landweber method. For convenience, we omit W and replace x_w with x.

Setting a homotopy mapping

$$\begin{aligned} &H : F \times [0,1] \longrightarrow Y \\ &H(\tilde{x}, p) \\ &\quad = p\left[F'(\tilde{x})^{*}\left(F(\tilde{x}) - y^{\delta}\right) + \alpha \frac{\tilde{x}}{\left(\tilde{x}^2 + \varepsilon^2\right)^{1/2}}\right] \\ &\qquad + (1-p)(\tilde{x} - x_0) = 0, \quad p \in [0,1], \end{aligned} \tag{14}$$

where p is an embedding parameter and x_0 is an initial guess value. Hence,

$$\begin{aligned} &H(\tilde{x}, 0) = \tilde{x} - x_0 = 0, \\ &H(\tilde{x}, 1) = \left[F'(\tilde{x})^{*}\left(F(\tilde{x}) - y^{\delta}\right) + \alpha \frac{\tilde{x}}{\left(\tilde{x}^2 + \varepsilon^2\right)^{1/2}}\right] \\ &\qquad = 0. \end{aligned} \tag{15}$$

We write $\tilde{x}$ as the power series of p,

$$\tilde{x} = x_0 + p^1 x_1 + p^2 x_2 + \cdots, \tag{16}$$

and obtain the approximation of functional (12):

$$x = \lim_{p \to 1} \tilde{x} = x_0 + x_1 + x_2 + \cdots. \tag{17}$$

We expand $F(x)$ in (14) as a Taylor series near x_0:

$$\begin{aligned} &H(\tilde{x}, p) = p\left[F'(x_0)^{*}\left(F(x_0) + F'(x_0)(\tilde{x} - x_0)\right.\right. \\ &\quad \left. + o(\tilde{x} - x_0) - y^{\delta}\right) + \alpha\left(\frac{x_0}{\left(x_0^2 + \varepsilon^2\right)^{1/2}}\right. \\ &\quad \left.\left. + \frac{\varepsilon^2}{\left(x_0^2 + \varepsilon^2\right)^{3/2}}(\tilde{x} - x_0) + o(\tilde{x} - x_0)\right)\right] + (1-p) \\ &\quad \cdot (\tilde{x} - x_0) = 0, \\ &p\left[F'(x_0)^{*}\left(F(x_0) + F'(x_0)\left(p^1 x_1 + p^2 x_2 + \cdots\right)\right.\right. \\ &\quad \left. - y^{\delta}\right) + \alpha\left(\frac{x_0}{\left(x_0^2 + \varepsilon^2\right)^{1/2}}\right. \\ &\quad \left.\left. + \frac{\varepsilon^2}{\left(x_0^2 + \varepsilon^2\right)^{3/2}}\left(p^1 x_1 + p^2 x_2 + \cdots\right)\right)\right] + (1-p) \\ &\quad \cdot \left(p^1 x_1 + p^2 x_2 + \cdots\right) = 0. \end{aligned} \tag{18}$$

Following the power of p, we can get

$$\begin{aligned} &p^1 : x_1 = -F'(x_0)^{*}\left(F(x_0) - y^{\delta}\right) - \alpha \frac{x_0}{\left(x_0^2 + \varepsilon^2\right)^{1/2}}, \\ &p^2 : x_2 = \left(I - F'(x_0)^{*} F(x_0) - \frac{\alpha\varepsilon^2}{\left(x_0^2 + \varepsilon^2\right)^{3/2}}\right) \\ &\qquad \cdot \left(-F'(x_0)^{*}\left(F(x_0) - y^{\delta}\right) - \alpha \frac{x_0}{\left(x_0^2 + \varepsilon^2\right)^{1/2}}\right), \end{aligned} \tag{19}$$

$$\begin{aligned} &x = x_0 + \left(-F'(x_0)^{*}\left(F(x_0) - y^{\delta}\right)\right. \\ &\quad \left. - \alpha \frac{x_0}{\left(x_0^2 + \varepsilon^2\right)^{1/2}}\right) + \left(I - F'(x_0)^{*} F(x_0)\right. \\ &\quad \left. - \frac{\alpha\varepsilon^2}{\left(x_0^2 + \varepsilon^2\right)^{3/2}}\right)\left(-F'(x_0)^{*}\left(F(x_0) - y^{\delta}\right)\right. \\ &\quad \left. - \alpha \frac{x_0}{\left(x_0^2 + \varepsilon^2\right)^{1/2}} + \cdots\right). \end{aligned} \tag{20}$$

Following formula (20), the parameter for the noisy measurement data is reconstructed by the first two terms:

$$\begin{aligned} &x_{n+1} = x_n + \left(2I - F'(x_n)^{*} F(x_n) - \frac{\alpha\varepsilon^2}{\left(x_n^2 + \varepsilon^2\right)^{3/2}}\right) \\ &\qquad \cdot \left(-F'(x_n)^{*}\left(F(x_n) - y^{\delta}\right) - \alpha \frac{x_n}{\left(x_n^2 + \varepsilon^2\right)^{1/2}}\right), \end{aligned} \tag{21}$$

where n is the iteration number.

When the parameter is reconstructed by the first term,

$$\begin{aligned} &x_{n+1} = x_n \\ &\qquad - \left[F'(x_n)^{*}\left(F(x_n) - y^{\delta}\right) + \alpha \frac{x_n}{\left(x^2 + \varepsilon^2\right)^{1/2}}\right]. \end{aligned} \tag{22}$$

Equation (22) is a well-known Landweber method. Equation (21) is a modified version of (22), which is called a homotopy perturbation inversion method. The convergent rate is faster and the accuracy is higher than a Landweber method. By using a wavelet transformation, we can obtain

$$\begin{aligned} x_{w,n+1} = x_{w,n} + &\left(2I - F'\left(Wx_{w,n}\right)^{*} F\left(Wx_{w,n}\right) \right. \\ &\left. - \frac{\alpha\varepsilon^{2}}{\left(x_{w,n}^{2}+\varepsilon^{2}\right)^{3/2}} \right) \\ &\cdot \left(-F'\left(Wx_{w,n}\right)^{*} \left(F\left(Wx_{w,n}\right) - y^{\delta}\right) \right. \\ &\left. - \alpha \frac{x_{w,n}}{\left(x_{w,n}^{2}+\varepsilon^{2}\right)^{1/2}} \right). \end{aligned} \tag{23}$$

4. Numerical Experiments

4.1. Reconstructing Sparse Volatility. The reconstruction of a volatility is investigated based on the Black-Scholes (B-S) option pricing model. We use a finite difference method to solve the forward problem. In order to improve the accuracy and convergent rate of a Landweber method, we design a homotopy perturbation inversion method to reconstruct a volatility. The B-S formula is widely used in the field of a derivative pricing, when the price changes of a derivative satisfy the standard geometric Brown motion. The boundary conditions of the different values vary with the different types of derivatives. When boundary conditions are given, a derivative pricing V_t can be obtained by solving the B-S formula of the derivative pricing model. When the derivative is taken as option, we define the forward problem as the determination of the option pricing. For simplicity, we take the European call option as example.

The relationship between the European call and put options shows that the reconstructed volatility should be the same by using the call options market quotes or the put options market quotes. For the European call option in the time interval $[0, T]$, let $V_t = V(S_t, t; \sigma, K, T)$ be the European call option pricing, and V satisfies

$$\begin{aligned} &\frac{\partial V_t}{\partial t} + (r-q) S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^{2}\left(S_t, t\right) S_t^{2} \frac{\partial^{2} V_t}{\partial S_t^{2}} - rV_t = 0, \\ &V_T = \left(S_T - K\right)^{+}, \end{aligned} \tag{24}$$

where S_t is stock prices with time, K is strike price, r is interest rate, q is dividend, T is availability period, t is time, and σ is volatility.

In the B-S formula, except that the volatility is a free variable, the other parameters and variables are decided by the prevailing market conditions or contracts. Hence, the volatility σ is a very important parameter. For an option, we can infer the volatility from the B-S formula.

The inverse problem is defined as follows: reconstructing the local volatility σ from the measurement data $V(S_*, t_*; \sigma, K_i, T_i) = V_i \ (i = 1, 2, \ldots, N)$ and $S = S_*$.

Table 1: Relative errors and computational time.

Noisy level	Relative error (I)	Relative error (II)	Time (I)	Time (II)
0.00%	0.21%	1.36%	77 s	112 s
0.50%	1.28%	3.06%	196 s	332 s
1.00%	3.25%	6.98%	528 s	1052 s

We define a nonlinear vector-valued function $F : \sigma \rightarrow V$, namely, $F(\sigma) = V$. For testing the performance of sparse regularization and Tikhonov regularization methods, we set $T = 1$ and set stock prices $S = 50$, interest rate $r = 0.05$, and strike price $K = 35, 40, 45$. We add 0%, 0.5%, and 1% Gaussian random noises to the measurement data in order to test the stability. The exact volatility is provided as follows:

$$\sigma = 0.15 \sin\left(\frac{\pi}{T} t\right) + 0.15 \cos\left(\frac{\pi}{T} t\right). \tag{25}$$

We transform the smooth volatility into the sparse volatility

$$\sigma = W\sigma_w, \tag{26}$$

where σ_w is a sparse representation. Functional (12) is rewritten as

$$J\left(\sigma_w\right) = \left\| F\left(W\sigma_w\right) - V^{\delta} \right\|_2^2 + \alpha \left\| \sigma_w \right\|_{2,\varepsilon}. \tag{27}$$

The results of the proposed method (I) and Tikhonov method (II) are listed in Table 1.

From the above reconstructions, we can see that the proposed method has better performance than the Tikhonov regularization method.

4.2. Reconstructing Policy Parameter for Todaro Model. Todaro model is the famous model to describe the number of rural migrants (namely, workers in urban areas from rural areas) in the labor economics. The Todaro model is also used in development economics and welfare economics to explain some of the issues concerning rural-urban migration. The main assumption of the model is that the migration decision is based on expected income differentials between rural and urban areas [21]. A Todaro model shows the relationship between the number of rural migrants with the income difference of urban and rural areas. A Todaro model can be written in the following form:

$$M = f(d), \tag{28}$$

where M, d denote the number of rural migrants and the income difference of urban and rural areas, respectively. Function f is an increasing function, that is, $f' > 0$. A modified Todaro model considers policy parameter χ:

$$M = f(\chi, d), \tag{29}$$

where χ describes the efficiency of government policy including household registration policy, social security policy, old-supporting policy, and others.

We consider I urban areas and J rural areas. d_{ij} $(1 \le i \le I, 1 \le j \le J)$ stands for the income difference between the ith urban area and the jth rural area, and $g_{i\tilde{i}}$ $(1 \le i \le I, 1 \le \tilde{i} \le I)$ stands for the income difference between the ith and the $\tilde{i}$th urban areas. The number of migrants into the ith urban area is denoted as M_i. For the ith urban area, policy parameter χ_i is split into two parts $\chi_i = \chi_i^r + \chi_i^u$, where χ_i^r, χ_i^u denote the efficiency of government policy to rural and urban areas, respectively. We assume that the number of workers from urban areas into rural areas is zero. In real applications, Function f has many different representations. In this section, we focus on the performance of the proposed method, and hence we take f as a linear function. The Todaro model is modified as

$$\begin{aligned}
&\chi_1^r d_{11} + \chi_1^r d_{12} + \cdots + \chi_1^r d_{1J} + \chi_1^u g_{11} + \chi_1^u g_{12} + \cdots \\
&\quad + \chi_1^u g_{1I} = M_1 \\
&\qquad\qquad\vdots \\
&\chi_i^r d_{i1} + \chi_i^r d_{i2} + \cdots + \chi_i^r d_{iJ} + \chi_i^u g_{i1} + \chi_i^u g_{i2} + \cdots \\
&\quad + \chi_i^u g_{iI} = M_i \\
&\qquad\qquad\vdots \\
&\chi_I^r d_{I1} + \chi_I^r d_{I2} + \cdots + \chi_I^r d_{IJ} + \chi_I^u g_{I1} + \chi_I^u g_{I2} + \cdots \\
&\quad + \chi_I^u g_{II} = M_I,
\end{aligned} \tag{30}$$

where the income differences d_{ij} and $g_{i\tilde{i}}$ and the number M_i of migrants into the ith urban area are known. The policy parameter χ is unknown.

Let

$$\begin{aligned}
D_k &= \sum_{j=1}^{J} d_{kj}, \\
G_k &= \sum_{i=1}^{I} g_{ki}, \qquad 1 \le k \le I, \\
A &= \begin{pmatrix} D_1 & G_1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_i & G_i & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & D_I & G_I \end{pmatrix} \\
M &= (M_1, \ldots, M_i, \ldots, M_I)^T, \\
X &= (\chi_1^r, \chi_1^u, \ldots, \chi_i^r, \chi_i^u, \ldots, \chi_I^r, \chi_I^u)^T.
\end{aligned} \tag{31}$$

Equation (30) is recast as

$$AX = M. \tag{32}$$

Table 2: Relative errors.

n	Relative error (I)	Relative error (II)
4	3.61%	5.82%
5	3.76%	6.75%
6	4.97%	8.99%

Note that (32) is underdetermined, and hence we apply the regularization method to solve this equation. We transform a smooth policy parameter into a sparse parameter $X = WX_w$, where X_w means a sparse policy parameter. The cost functional is as follows:

$$J(X_w) = \left\| AWX_w - y^\delta \right\|_2^2 + \alpha \left\| X_w \right\|_{2,\varepsilon}. \tag{33}$$

In numerical tests, we take $I = 2^{n-1}$, $J = 2^n$. We add a 1% Gaussian random noise to the measurement data in order to test the stability. We reconstruct three policy parameters corresponding to $n = 4, 5, 6$. The relative errors of the proposed method (I) and Tikhonov method (II) are listed in Table 2. Due to linearity and small scale, the difference of the computational times of the proposed method and Tikhonov method is small. Table 2 shows that the proposed method is feasible to reconstruct policy parameters.

5. Conclusions

We design a sparsity regularization method to reconstruct a volatility and a policy parameter. Because parameters are often smooth in the fields of economy and finance, we apply a wavelet transformation from a smooth parameter into a sparse parameter. The homotopy perturbation inversion method is used to minimize the cost functional, and the accuracy and convergent rate of reconstruction are improved. The numerical experiments show that the proposed method can be applied to parameter identification and initial value problem in heat transfer equations [22, 23].

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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